**Part III Quasirandomness Examples Sheet 1.** W. T. G.

**1.** Prove that for every $a > 0$ there exists $c > 0$ with the following property. Whenever $A_1, \dots, A_m$ is a collection of subsets of $\{1, 2, \dots, n\}$ such that $\Big||A_i| - n/2\Big| \leqslant cn$ for every $i$ and $\Big||A_i \cap A_j| - n/4\Big| \leqslant cn$ for all but at most $cm^2$ pairs $(i, j)$ then it is also the case that $\Big||A_i \cap A_j \cap A_k| - n/8\Big| \leqslant an$ for all but at most $am^3$ triples $(i, j, k)$. [You may quote results from the course if that helps.]

**2.** Prove that for every $a > 0$ there exists $c > 0$ with the following property. Let $X$ and $Y$ be sets of sizes $m$ and $n$, respectively. Let $f : X \times Y \to [-1, 1]$ be a function such that $\sum_x \Big(\sum_y f(x, y)\Big)^2 \leqslant cmn^2$ and $\sum_y \Big(\sum_x f(x, y)\Big)^2 \leqslant cm^2 n$. Then for any functions $u, v : X \times Y \to [-1, 1]$ we have the inequality

$$\Big|\sum_{x,x' \in X} \sum_{y,y' \in Y} u(x, y) f(x, y') v(x', y)\Big| \leqslant am^2 n^2 \ .$$

Explain why this result can be used to shorten the proof in the notes that if a bipartite graph $G$ has few 4-cycles then the "sum over rectangles" of the corresponding function $f$ is small.

**3.** Let $G$ be a graph with vertex set $X$ of size $n$ and with $p\binom{n}{2}$ edges . Prove that the following two statements are equivalent, in the sense of the lectures.

(i) For any two sets $A, B \subset X$ the number of pairs $(x, y) \in A \times B$ such that $xy$ is an edge of $G$ differs from $p|A||B|$ by at most $cn^2$.

(ii) For any set $A \subset X$ the number of edges spanned by the set $A$ differs from $p\binom{|A|}{2}$ by at most $c'n^2$.

**4.** Let $u$, $v$, $w$ and $x$ be vectors in $\mathbb{R}^n$. Write out a concise expression for the composition of the linear maps (or product of the matrices) $u \otimes v$ and $w \otimes x$.

**5.** Let $A(x, y)$ be a real symmetric $n \times n$ matrix. Prove that if $u$ is a unit vector that maximizes the sum $\sum_{x,y} A(x, y)u(x)u(y)$ then $u$ is an eigenvector with the largest possible eigenvalue.

**6.** Let $A(x, y)$ be a real matrix, but not necessarily symmetric this time. Suppose that $u$ and $v$ are unit vectors such that $\sum_{x,y} A(x, y)u(x)v(y)$ is maximized. Show that $Av = \lambda u$ for some constant $\lambda$. Let $B(x, y) = A(x, y) - \lambda u(x)v(y)$. Prove that $\langle Bw, v\rangle = 0$ for every vector $w$. By using these facts, or otherwise, show that $A$ can be written as a sum $\sum_{i=1}^n \lambda_i u_i(x)v_i(y)$, with both the $u_i$ and $v_i$ forming orthonormal bases. Prove that the

$u_i$ and $v_j$ are eigenvectors of the matrices $A^TA$ and $AA^T$, respectively. What are the corresponding eigenvalues? [Warning: I could have made mistakes in the details of this question, but if any of the statements are false then find modified statements that are true and prove them. To a lesser extent, this applies to the whole sheet.]

**7.** In lectures we proved the homomorphisms version of the counting lemma. Prove the version that deals with isomorphic embeddings.

**8.** Prove that there is a function $f : \{1, 2, \ldots, n\}^2 \to \{-1, 1\}$ such that $\sum_{x,x'} \sum_{y,y'} f(x, y)f(x, y')f(x', y)f(x', y') \leqslant Cn^3$ for some absolute constant $C$. [Hint: choose $f$ randomly.]

**9.** Let $p$ be a prime of the form $4n + 3$ and let $\mathbb{Z}_p$ stand for the set of integers mod $p$. Define a function $f : \mathbb{Z}_p^2 \to \{-1, 1\}$ by setting $f(x, y) = 1$ if $y - x$ is a quadratic residue, and $f(x, y) = -1$ otherwise. Prove that $\sum_{x,x'} \sum_{y,y'} f(x, y)f(x, y')f(x', y)f(x', y') \leqslant Cn^3$ for some absolute constant $C$. [NB: the hint in the previous question was compulsory!]

**10.** Let $f : \{1, 2, \ldots, n\}^2 \to \{-1, 1\}$ be any function. Prove that $\sum_{x,x'} \sum_{y,y'} f(x, y)f(x, y')f(x', y)f(x', y') \geqslant n^3$.

**11.** Let $H$ be a graph with $k$ vertices, which we think of as fixed and small. Prove that for every $a > 0$ there exists $c > 0$ such that if at most $cn^k$ functions from $V(H)$ to $V(G)$ are homomorphisms, then it is possible to remove at most $an^2$ edges from $G$ in such a way that $G$ no longer contains any copies of $H$.

**12.** (A proof of the Erdős-Stone theorem.) Let $H$ be a fixed graph, as above. Write $ex(n, H)$ for the largest density of any graph with $n$ vertices that does not contain any copy of $H$. Turán's theorem implies that if $H$ is the complete graph on $k$ vertices (and if $n$ is a multiple of $k - 1$) then $ex(n, H) = 1 - 1/(k - 1)$. Suppose now that the chromatic number of $H$ is $r$. Prove that the limit of $ex(n, H)$ as $n$ tends to $\infty$ is $1 - 1/(r - 1)$. [Hint: Let $G$ be an extremal graph. Remove a few edges from $G$ in such a way that if the resulting graph contains a $K_r$ then it also contains several copies of $H$. Most of the work is contained in the previous question.]

**13.** Let $A$ be a subset of $\{1, 2, \ldots, n\}^2$ of cardinality at least $\delta n^2$. Prove that there exists some pair $(u, v) \in \mathbb{N}^2$ such that $A \cap \big((u, v) - A\big) = \{(a, b) \in A : (u - a, v - b) \in A\}$ has cardinality at least $c\delta n^2$ for some absolute constant $c > 0$. Deduce that the theorem about corners can be strengthened so that it guarantees that $A$ contains a triple $(x, y)$, $(x, y + d)$, $(x + d, y)$ with $d > 0$.

**14.** Let $A$ be a subset of $\mathbb{Z}_N$ and define a graph $G(A)$ with vertex set $\mathbb{Z}_N$ by joining $x$ to $y$ if and only if $x+y\in A$. By considering 4-cycles in $G$, prove that $G$ is $c$-quasirandom if and only if $A$ is $c$-uniform. It follows that the sum of the fourth powers of the eigenvalues of $G$ is the sum of the fourth powers of the Fourier coefficients of $A$. Can you prove this more directly?

**15.** Let $A,B,C,D$ be four subsets of $\mathbb{Z}$ of size $N$. Suppose that there are $\alpha N^3$ quadruples $(a,b,c,d)\in A\times B\times C\times D$ such that $a+b=c+d$. Prove that there are at least $\alpha N^3$ quadruples $(a,b,c,d)\in A^4$ with $a+b=c+d$.

**16.** Suppose that $A$ has size $N$ and contains at least $\alpha N^2$ arithmetic progressions of length 3. Prove that there are at least $\alpha N^3$ quadruples $(a,b,c,d)\in A^4$ with $a+b=c+d$.

**17.** Let $A$ be a subset of $\mathbb{Z}_N$ of size at most $(1/20)\log N$. Prove that there exists $r\neq 0$ such that $|\hat{A}(r)|\geqslant |A|/2$.

**18.** Let $N$ be a prime, let $P$ be an arithmetic progression mod $N$, and let $A$ be the set of all $x\in\mathbb{Z}_N$ such that $x^2\in P$. Let $|P|=\delta N$.

(i) Prove that for every $r\neq 0$ the Fourier coefficient $\hat{A}(r)$ has modulus at most $C\log N\sqrt{N}$ for some absolute constant $C$. [Hint: $A(x)=P(x^2)$. Now split up $P$ into a sum of trigonometric functions.]

(ii) Prove that the cardinality of $A$ is approximately equal to $\delta N$, and give an estimate for the error. [Again, write $A$ as above and use the alternative expression to estimate $\sum|A(x)|^2$.]

(iii) Show also that if $P$, $Q$ and $R$ are progressions mod $N$, then the number of pairs $(x,d)$ such that $x^2\in P$, $(x+d)^2\in Q$ and $(x+2d)^2\in R$ is approximately $|P||Q||R|N^{-1}$, and again give an estimate for the error.

(iv) Prove that the number of progressions of length 4 in $A$ is at least $c\delta^3N^2$, for some absolute constant $c>0$.

Remarks, corrections, comments to wtg10@dpmms.cam.ac.uk.