1. A random variable $X$ has mean $\mu$ and variance $\sigma^2$. For each real number $t$, let $V(t) = E((X - t)^2)$. Prove that $E(V(X)) = 2\sigma^2$.

2. At time 0, a blood culture starts with one red cell. At the end of one minute, the red cell dies and is replaced by one of the following combinations with the following probabilities: two red cells (probability $1/4$); one red and one white cell (probability $2/3$); two white cells (probability $1/12$). Each red cell lives for one minute and gives birth to offspring in the same way as the parent cell. Each white cell lives for one minute and dies without reproducing. Individual cells behave independently.

   (a) When the culture has been going for just over $n$ minutes, what is the probability that no white cells have yet appeared?

   (b) What is the probability that the entire culture eventually dies out?

3. A slot machine operates in such a way that at the first turn your probability of winning is $1/2$. Thereafter, your probability of winning is $1/2$ if you lost at the last turn and $p$ (which is less than $1/2$) if you won. If $u_n$ is the probability that you win at the $n$th turn, find a recurrence relation that connects $u_n$ and $u_{n-1}$ whenever $n \geq 2$. Define a value for $u_0$ so that this recurrence relation is still valid when $n = 1$. By solving the recurrence relation, prove that

   $$u_n = \frac{1 + (-1)^{n-1}(1/2 - p)^n}{3 - 2p}.$$
Let $G(z)$ be the generating function $\sum_{n=0}^{\infty} u_n z^n$. Prove that this sum converges whenever $|z| < 1/4$. By using the recurrence above, prove also that $zG(z)^2 = G(z) - 1$. Solve this quadratic to obtain a formula for $G(z)$ (explaining carefully your choice of sign). Calculate the first few terms of the binomial expansion of your answer and check that they give the right first few values of $u_n$.

8. Let $X$ be a random variable with density $f$ and let $g$ be an increasing function such that $g(x) \to \pm \infty$ as $x \to \pm \infty$. Find a formula for the density of the random variable $g(X)$. If this density is $h$, check that $\int_{-\infty}^{\infty} h(y) \, dy = \int_{-\infty}^{\infty} g(x) \, f(x) \, dx$.

9. Let $X_1, X_2, X_3, \ldots$ be independent exponential random variables with parameter $\lambda$. Let $Y = \max\{r : X_1 + X_2 + \ldots + X_r \leq 1\}$. Prove that $Y$ is Poisson with parameter $\lambda$.

10. Alice and Bob agree to meet at the Copper Kettle after their Saturday lectures. They arrive at times that are independent and uniformly distributed between midday and 1pm. Each is prepared to wait 10 minutes before leaving. Find the probability that they meet.

11. The radius of a circle has the exponential distribution with parameter $\lambda$. Determine the probability density function of the area of the circle.

12. Suppose that $X$ and $Y$ are independent, identically distributed random variables, each uniformly distributed on $[0, 1]$. Let $U = X + Y$ and $V = X/Y$. Are $U$ and $V$ independent?

13. Let $(X_n)_{n\geq0}$ be a branching process such that $X_0 = 1$ and $E X_1 = m$. Let $F(z) = E z^{X_1}$ be the p.g.f. of $X_1$. Let $Y_n = X_0 + X_1 + \ldots + X_n$ be the total number of individuals in the generations $0, 1, 2, \ldots, n$, and let $G_n(z) = E z^{Y_n}$ be its generating function. Prove that $G_{n+1}(z) = zF(G_n(z))$. Deduce that if $Y = \sum_{n=0}^{\infty} X_n$, then $G(z) = E z^Y$ satisfies the equation $G(z) = z F(G(z))$ when $0 \leq z < 1$. (Here we interpret $z^{\infty}$ as 0.) If $m < 1$, prove that $EY = (1 - m)^{-1}$.

14. Let $k$ be a positive integer and let $X \sim N(0, 1)$. Find a formula for $E X^k$. Find also a formula for $E e^{\lambda x}$.