1. A random variable X has mean μ and variance σ^2. For each real number t, let $V(t) = \mathbf{E}(X-t)^2$. Prove that $\mathbf{E}V(X) = 2\sigma^2$.

2. At time 0, a blood culture starts with one red cell. At the end of one minute, the red cell dies and is replaced by one of the following combinations with the following probabilities: two red cells (probability 1/4); one red and one white cell (probability 2/3); two white cells (probability 1/12). Each red cell lives for one minute and gives birth to offspring in the same way as the parent cell. Each white cell lives for one minute and dies without reproducing. Individual cells behave independently.

(a) When the culture has been going for just over n minutes, what is the probability that no white cells have yet appeared?

(b) What is the probability that the entire culture eventually dies out?

3. A slot machine operates in such a way that at the first turn your probability of winning is 1/2. Thereafter, your probability of winning is 1/2 if you lost at the last turn and p (which is less than 1/2) if you won. If u_n is the probability that you win at the nth turn, find a recurrence relation that connects u_n and u_{n-1} whenever $n \geq 2$. Define a value for u_0 so that this recurrence relation is still valid when $n = 1$. By solving the recurrence relation, prove that

$$u_n = \frac{1 + (-1)^{n-1}(\frac{1}{2} - p)^n}{3 - 2p}.$$

4. A gambler plays the following game. He starts with r pounds, and is trying to end up with a pounds. At each go he chooses an integer s between 1 and the minimum of r and $a - r$ and then tosses a fair coin. If the coin comes up heads, then he wins s pounds, and if it comes up tails then he loses s pounds. The game finishes if he runs out of money (in which case he loses) or reaches a pounds (in which case he wins). Prove that whatever strategy the gambler adopts (that is, however he chooses each stake based on what has happened up to that point), the probability that the game finishes is 1 and the probability that the gambler wins is r/a.

5. A fair coin is tossed n times. Let u_n be the probability that the sequence of tosses never has two consecutive heads. Show that $u_n = \frac{1}{2}u_{n-1} + \frac{1}{4}u_{n-2}$. Find u_n, and check that your value of u_3 is correct.

6. A coin is repeatedly tossed, and at each toss comes up heads with probability p, the outcomes being independent. What is the expected number of tosses until the end of the first run of k heads in a row?

7. Let u_n be the number of walks of length $2n$ that start and end at the origin, move a distance 1 at each step, and remain non-negative at all times. (We interpret u_0 as 1.) By considering the last time that such a walk visits the origin *before* time n, prove that

$$u_n = u_0u_{n-1} + u_1u_{n-2} + u_2u_{n-3} + \ldots + u_{n-1}u_0.$$

Let $G(z)$ be the generating function $\sum_{n=0}^{\infty} u_n z^n$. Prove that this sum converges whenever $|z| < 1/4$. By using the recurrence above, prove also that $zG(z)^2 = G(z) - 1$. Solve this quadratic to obtain a formula for $G(z)$ (explaining carefully your choice of sign). Calculate the first few terms of the binomial expansion of your answer and check that they give the right first few values of u_n.

8. Let X be a random variable with density f and let g be an increasing function such that $g(x) \to \pm\infty$ as $x \to \pm\infty$. Find a formula for the density of the random variable $g(X)$. If this density is h, check that $\int_{-\infty}^{\infty} yh(y)\,dy = \int_{-\infty}^{\infty} g(x)f(x)\,dx$.

9. Let $X_1, X_2, X_3, \ldots$ be independent exponential random variables with parameter λ. Let $Y = \max\{r : X_1 + X_2 + \ldots + X_r \leq 1\}$. Prove that Y is Poisson with parameter λ.

10. Alice and Bob agree to meet at the Copper Kettle after their Saturday lectures. They arrive at times that are independent and uniformly distributed between midday and 1pm. Each is prepared to wait 10 minutes before leaving. Find the probability that they meet.

11. The radius of a circle has the exponential distribution with parameter λ. Determine the probability density function of the area of the circle.

12. Suppose that X and Y are independent, identically distributed random variables, each uniformly distributed on $[0, 1]$. Let $U = X+Y$ and $V = X/Y$. Are U and V independent?

13. Let $(X_n)_{n\geq 0}$ be a branching process such that $X_0 = 1$ and $\mathbf{E}X_1 = m$. Let $F(z) = \mathbf{E}z^{X_1}$ be the p.g.f. of X_1. Let $Y_n = X_0 + X_1 + \ldots + X_n$ be the total number of individuals in the generations $0, 1, 2, \ldots, n$, and let $G_n(z) = \mathbf{E}z^{Y_n}$ be its generating function. Prove that $G_{n+1}(z) = zF(G_n(z))$. Deduce that if $Y = \sum_{n=0}^{\infty} X_n$, then $G(z) = \mathbf{E}z^Y$ satisfies the equation $G(z) = zF(G(z)$ when $0 \leq z < 1$. (Here we interpret z^∞ as 0.) If $m < 1$, prove that $\mathbf{E}Y = (1-m)^{-1}$.

14. Let k be a positive integer and let $X \sim N(0, 1)$. Find a formula for $\mathbf{E}X^k$. Find also a formula for $\mathbf{E}e^{\lambda x}$.