IA Probability Examples Sheet 2, Lent 2009. W. T. G.

1. A coin with probability p of heads is tossed n times. Let E be the event "a head is obtained on the first toss2 and let F_k be the event "exactly k heads are obtained". For which pairs of integers (n, k) are the events E and F_k independent?

2. The events $A_1, \ldots, A_n$ are independent. Show that the events $A_1^c, \ldots, A_n^c$ are independent.

3. A sequence of n independent trials is performed, with each trial having a probability p of success. Show that the probability that the total number of successes is even is $\big(1 + (1-2p)^n\big)/2$.

4. Two darts players, A and B, throw alternately at a board and the first to score a bull's eye wins the contest. The outcomes of different throws are independent, and on each throw A has probability p_A of scoring a bull's eye, while B has a probability p_B. If A goes first, then what is the probability that A wins the contest?

5. The number of mispsrints on a page has a Poisson distribution with parameter λ, and the numbers on different pages are independent. What is the probability that the second misprint will occur on page r?

6. Suppose that X and Y are independent random variables with the Poisson distribution, with parameters λ and μ, respectively. Prove that the conditional distribution of X, given that $X + Y = n$, is binomial with parameters n and $\lambda/(\lambda + \mu)$.

7. Suppose that $X_1, \ldots, X_n$ are independent, identically distributed random variables with mean μ and variance σ^2. Find the mean of the random variable $S^2 = n^{-1} \sum_{i=1}^n (X_i - \overline{X})^2$, where $\overline{X}$ is the random variable $n^{-1} \sum_{i=1}^n X_i$. ($\overline{X}$ and S^2 are called the *sample mean* and *sample variance*, respectively.)

8. In a sequence of independent trials, the probability of a success at the ith trial is p_i. Show that the mean and variance of the total number of successes are $n\overline{p}$ and $n\overline{p}(1 - \overline{p}) - \sum_i (p_i - \overline{p})^2$, where $\overline{p} = \sum_i p_i/n$. For a given mean, when is the variance maximized?

9. Let K be a random variable with $\mathbf{P}(K = r)$ equal to $1/8$, for integers r between 0 and 7. Let $\theta = K\pi/4$ and let $X = \cos\theta$ and $Y = \sin\theta$.

Prove that the covariance of X and Y is zero, but that X and Y are not independent.

10. Elmo's bowl of spaghetti contains n strands. He selects two ends at random and joins them together. He does this until there are no ends left. What is the expected number of loops of spaghetti in the bowl?

11. Julia collects figures from cornflakes packets. Each packet contains one figure, and n distinct figures are needed to make a complete set. What is the expected number of packets that Julia will need to buy in order to collect a complete set?

12. Let $X_1, X_2, \ldots$ be independent identically distributed positive random variables with $\mathbf{E}X_1 = \mu < \infty$ and $\mathbf{E}(X_1^{-1}) < \infty$. Let $S_n = \sum_{i=1}^n X_i$. Show that $\mathbf{E}(S_m/S_n) = m/n$ when $m \leq n$ and $1 + (m-n)\mu\mathbf{E}(S_n^{-1})$ when $m \geq n$.

13. For each non-negative integer n, the probability that a football team will score n goals in a match is $p^n(1-p)$, independently of the number of goals scored by the other team. What is the probability of a score draw if teams with probabilities p_1 and p_2 meet? If $p_1 = p_2 = p$, what value of p gives the highest probability of a score draw, and what is this probability? [A score draw means a draw where both teams score at least one goal.]

14. A sample space Ω contains 2^n points, and $\mathbf{P}$ is some probability distribution on Ω. Let $A_1, \ldots, A_m$ be events, and suppose that no A_i is equal to $\emptyset$ or Ω. Prove that if the A_i are independent then $m \leq n$. If $\mathbf{P}$ is the uniform distribution on Ω, how many events is it possible to find such that each event has probability $1/2$ and any *two* of those events are independent?

15. You are playing a match against an opponent in which at each point either you serve or your opponent does. If you serve then you win the point with probability p_1; if your opponent serves then you win the point with probability p_2. Consider two possible conventions for serving:

(i) serves alternate;

(ii) the player serving continues to serve until he or she loses a point.

You serve first and the first player to reach n points wins the match. Show that your probability of winning the match does not depend on the serving convention adopted.