IA Probability Examples Sheet 1 **L08** W. T. G.

1. Let A, B and C be three events. Express in symbols the following events: (i) only A occurs; (ii) at least one event occurs; (iii) exactly one event occurs.

2. From a table of random digits, k are chosen. Let $0 \le r \le 9$. What is the probability that no digit exceeds r? What is the probability that r is the greatest digit drawn?

3. How many sequences $(A_1, \ldots, A_n)$ are there with the following properties:

(i) each A_i is a (possibly empty) subset of $\{1, 2, \ldots, n\}$;

(ii) $A_i \cap A_j = \emptyset$ whenever $i \neq j$;

(iii) $A_1 \cup \ldots \cup A_n = \{1, 2, \ldots, n\}$;

(iv) each A_i has size at most 2?

[Hint: think first about the sizes of the sets A_i and only later about what the actual sets are.]

4. How many of the numbers $1, 2, \ldots, 500$ are not divisible by 7, but are divisible by either 3 or 5?

5. A tennis championship is organized for 2^n players as a knock-out tournament with n rounds, the last round being the final. Two players are chosen at random. What are the probabilities that they meet (i) in the first round; (ii) in the final; (iii) in *some* round?

6. A sample of size r is taken from a population of size n, sampling without replacement. Calculate the probability that m given people will all be included in the sample (i) directly and (ii) by using the inclusion-exclusion formula. Hence show that

$$\binom{n-m}{r-m} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n-j}{r}.$$

7. (i) Let $0 < \alpha < 1$ be a rational number. Use Stirling's formula to obtain an estimate for the binomial coefficient $\binom{n}{m}$ when n is large and $m = \alpha n$ is an integer.

(ii) Suppose that $m \le n/3$. By considering ratios of successive binomial coefficients, prove that

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{m-1} \le \binom{n}{m}.$$

(iii) Suppose that $n = 3m$. Prove that the ratio of the two sides of the above inequality tends to 1 as n tends to infinity.

8. Two cards are taken at random from an ordinary pack of 52 cards. Find the probabilities of the following events: (i) both cards are aces (event A); (ii) the pair of cards includes an ace (event B); (iii) the pair of cards includes the ace of hearts (event C). Show that $\mathbb{P}(A|B) \neq \mathbb{P}(A|C)$.

9. Examination candidates are graded into four classes, known conventionally as I, II-1, II-2 and III. The probabilities of getting these classes are 1/8, 1/4, 3/8 and 1/4, respectively. A candidate who misreads the rubric—which happens with probability 2/3—generally does worse: the corresponding probabilities in this case are 1/10, 1/5, 2/5 and 3/10. What is the probability that a candidate who reads the rubric correctly gets a II-1? What is the probability that a candidate who gets a II-1 has read the rubric correctly?

10. Parliament contains a proportion p of Conservative members, who are incapable of changing their minds about anything, and a proportion $1-p$ of Labour members, who change their minds completely at random, with probability r, between successive votes on the same issue. A randomly chosen member is observed to have voted twice in succession in the same way. What is the probability that this member will vote in the same way next time?

11. By looking at the proof of Stirling's formula given in lectures, obtain the more precise conclusion that there are constants $0 < c < C$ such that

$$(1 + c/n)\sqrt{2\pi}e^{-n}n^{n+1/2} \leq n! \leq (1 + C/n)\sqrt{2\pi}e^{-n}n^{n+1/2}.$$

12. Let $A_1, \ldots, A_n$ be events, each of probability p. Suppose that $\bigcap_{i\in X} A_i$ has probability $p^{|X|}$ for every subset $X \subset \{1, 2, \ldots, n\}$. (In this case, the events are said to be *independent*.) Use the inclusion-exclusion formula to prove that $\mathbb{P}(A_1^c \cap \ldots \cap A_n^c) = (1-p)^n$. If k is even, deduce from the Bonferroni inequalities that

$$\sum_{j=0}^{k-1}(-1)^j\binom{n}{j}p^j \leq (1-p)^n \leq \sum_{j=0}^{k}(-1)^j\binom{n}{j}p^j.$$

If p is small, roughly how large does k have to be for the upper and lower bounds given above to be approximately equal? (You should think for yourself about how to make this question precise in a reasonable way.)

13. Suppose that n balls are placed at random into n boxes. Find the probability that there is exactly one empty box.

14. Mary tosses two coins and John tosses one coin. What is the probability that Mary gets strictly more heads than John? Answer the same question if Mary tosses three coins and John tosses two. Make a conjecture for the corresponding probability when Mary tosses $n+1$ coins and John tosses n. Now prove your conjecture.

15. A meeting about absent-mindedness is attended by n psychologists (a further m having forgotten to turn up). After the meeting, they all choose a coat at random. On their way home, each one has a probability p, independently of the others, of losing their coat. Show that the probability that nobody ends up at home with the correct coat is approximately $e^{-(1-p)}$.

16. A hand of thirteen cards is dealt from a well-shuffled pack of playing cards. What is the probability that it contains more kings than queens? How does the answer change if you are told that the hand contains at least one king? (It is fine to leave your answer as a complicated expression involving binomial coefficients.)

If you spot any mistakes on this sheet, I will be glad to know of them and can be contacted at wtg10@cam.ac.uk