NUMBERS AND SETS EXAMPLES SHEET 4. W. T. G.

1. Is there a field with exactly four elements? Is there one with six elements?

2. Let $\mathbb{F}$ be a field. Prove that $(-1)(-1) = 1$ in $\mathbb{F}$. [-1 is of course defined to be the additive inverse of the multiplicative identity. If you have any difficulty knowing what you are allowed to assume, then change your notation so that it no longer looks familiar.]

3. Let $\mathbb{F}$ be an ordered field. Prove that $1 > 0$. Prove also that if $0 < x < y$, then $0 < 1/y < 1/x$.

4. Let p be a prime. Prove that the field $\mathbb{Z}_p$ cannot be made into an ordered field. Prove also that $\mathbb{C}$ cannot be made into an ordered field. [That is, in both cases show that there is no relation on the field that satisfies the required axioms.]

5. Prove that there is exactly one way to make $\mathbb{Q}$ into an ordered field. [That is, prove that the usual ordering $<$ is the only relation that satisfies the required axioms. To do this, let R be another such relation and prove from the axioms that xRy if and only if $x < y$.]

6. (i) Let $\mathbb{Q}(\sqrt{2})$ be the subset of $\mathbb{R}$ consisting of all numbers of the form $a + b\sqrt{2}$, where a and b belong to $\mathbb{Q}$. Assuming that $\mathbb{R}$ is a field, prove that the usual operations of addition and multiplication turn $\mathbb{Q}(\sqrt{2})$ into a field as well.

(ii) Prove that there are exactly two ways to turn $\mathbb{Q}(\sqrt{2})$ into an ordered field.

7. Prove that $\sqrt{2} + \sqrt{3}$ is irrational. Prove also that $2^{1/3} + 2^{2/3}$ is irrational. [Hint: you may find it helpful to use Question 7 from Sheet 3. You may of course assume that the numbers in question exist.]

8. Let a and b be real numbers. Prove that $a + b$ must be irrational if a is rational and b is irrational. Is it possible for $a + b$ to be rational if both a and b are irrational?

9. Let r be a real number and let A be the set of all rational numbers less than r. Prove carefully that the least upper bound of A is r.

10. Prove carefully, using the least upper bound axiom, that there is a real number x such that $x^3 = 2$.

11. A real number $x = 0.x_1x_2x_3...$ is called *repetitive* if its decimal expansion contains arbitrarily long pairs of identical blocks - that is, if for every k there exist distinct m and n

such that $x_m = x_n$, $x_{m+1} = x_{n+1}, \ldots, x_{m+k} = x_{n+k}$. Prove that the square of a repetitive number is repetitive.

12. Find an injection from $\mathbb{R}^2$ to $\mathbb{R}$.

13. Prove that the set of all finite subsets of $\mathbb{N}$ is countable. What goes wrong if we try to use a diagonal argument to show that it is uncountable?

14. Prove that the set of all irrational numbers is uncountable.

15. Let A be an uncountable set of real numbers and let $f : A \to \mathbb{R}$ be an injection. Prove that there is an irrational number $x \in A$ such that $f(x)$ is also irrational. Deduce that there exists a pair of irrational numbers a, b such that a^b is rational. Find an actual example of such a pair. [For this question you should assume the familiar properties of the function $x \mapsto a^x$ even though we haven't done them formally.]

16. A function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n+1) \geqslant f(n)$ for every n and *decreasing* if $f(n+1) \leqslant f(n)$ for every n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

17. Let S be a *nested* collection of subsets of $\mathbb{N}$ - that is, for every $A, B \in S$ we have either $A \subset B$ or $B \subset A$. Is it possible for S to be uncountable?

18. Prove that $\sqrt[100]{\sqrt{3}+\sqrt{2}} + \sqrt[100]{\sqrt{3}-\sqrt{2}}$ is irrational.

19. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval - in other words, such that for every $a < b$ and every c there is some x with $a < x < b$ and $f(x) = c$.