NUMBERS AND SETS EXAMPLES SHEET 1. W. T. G.

1. Let A, B and C be three sets. Give a proof that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ using the criterion for equality of sets.

2. The *symmetric difference* $A \bigtriangleup B$ of A and B is defined to be $(A \setminus B) \cup (B \setminus A)$. (That is, it is the set of elements that belong to one of A and B but not both.) Write out a truth table to show that the operation $\bigtriangleup$ is associative. Show that x belongs to $A \bigtriangleup (B \bigtriangleup C)$ if and only if x belongs to an odd number of the sets A, B and C and use this observation to give a second proof that $\bigtriangleup$ is associative.

3. Let A, B, C and D be sets. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. Is it necessarily true that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$?

4. Write down the negations of the following statements.

   (i) n is even or m is a multiple of 3.

   (ii) Every $x \in A$ is also an element of $B \cap C$.

   (iii) If it is not raining today then no pigs can fly.

5. Let f and g be functions and let $h = g \circ f$. If f and g are injections/surjections, prove that h is an injection/surjection.

6. Let f be a function from a set X to a set Y and let C and D be subsets of Y. Prove that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. Now let A and B be subsets of X. Is it necessarily true that $f(A \cap B) = f(A) \cap f(B)$?

7. How many functions are there from the set $\{1, 2, 3, 4, 5\}$ to the set $\{1, 2, 3\}$? How many of them are surjections?

8. Let n be odd. Prove that exactly half of the 2^n subsets of $\{1, 2, \ldots, n\}$ have even size. Now show that the same is true when n is even (and non-zero). [Hint: divide the sets into those that contain the element 1 and those that do not.]

9. Use the inclusion-exclusion principle to determine how many numbers in the set $\{1, 2, 3, \ldots, 500\}$ are divisible by none of $2, 3, 5$ or 7. (If we have not yet covered the inclusion-exclusion principle, then see if you can work out the answer anyway.)

10. Give an example of a relation that is symmetric and transitive but not reflexive, or else prove that no such relation exists.

11. Define a binary operation $*$ on $\mathbb{Z}^2$ by the formula $(a, b) * (c, d) = (ac, ad + bc)$. Prove that the operation $*$ is commutative and associative. Write down an expression for

$$(a_1, b_1) * (a_2, b_2) * \ldots * (a_k, b_k) \ .$$

12. Let $A_1, A_2, \ldots$ be sets such that for every positive integer n we have $A_1 \cap \ldots \cap A_n \neq \emptyset$. Is it possible that $A_1 \cap A_2 \cap \ldots = \emptyset$?

13. Let f be a function from the real numbers to the real numbers. Say that f is *strictly increasing* if $f(x) < f(y)$ whenever $x < y$. Show that if f is strictly increasing, then it is an injection. Must it also be a surjection? Suppose that f is a bijection and that $f(0) = 0$ and $f(1) = 1$. Does it follow that f is strictly increasing?

14. Find a bijection from the set of all rational numbers to the set of all non-zero rational numbers. Is there such a bijection that is strictly increasing?