Topics in Analysis Examples Sheet 4 W. T. G.

1. (i) For each n let F_n be the nth Fibonacci number, with $F_1 = F_2 = 1$. Prove the identities $F_{2n+1} = F_n^2 + F_{n+1}^2$ and $F_{2n} = F_n(F_{n-1} + F_{n+1})$. Let x_n stand for the ratio F_{n+1}/F_n. Use the above identities to express x_{2n} in terms of x_n.

(ii) If we set $y_k = x_{2^k}$, then we have expressed each y_k as a simple rational function of y_{k-1}. Explain why the sequence (y_k) converges very rapidly to the golden ratio. How does this relate to another method that produces rapid convergence?

2. What are the eigenvalues of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$? What are the connections between this, Fibonacci numbers and continued fractions?

3. Assuming the continued-fraction expansion of $\tan(x)$ given in lectures, derive an expansion for $\tanh(x)$. Deduce that

$$\frac{e+1}{e-1} = 2 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \dots}}}$$

4. Let X and Y be topological spaces, let $x \in X$ and let $f : X \to Y$. Say that f is *continuous at* x if for every neighbourhood N of $f(x)$ there exists a neighbourhood M of x such that $f(M) \subset N$. Prove that f is continuous if and only if it is continuous at every $x \in X$.

5. Let $X = \{0\} \cup \{n^{-1} : n \in \mathbb{N}\}$ and let X have the subspace topology inherited from $\mathbb{R}$. Which subsets of X are open? Which are closed? Which are compact?

6. The open sets in the *cofinite topology* on an infinite set X are the empty set and all subsets Y of X such that $X \setminus Y$ is finite. Write down and prove necessary and sufficient conditions for a function $f : X \to X$ to be continuous under this topology.

7. Let X be a Hausdorff topological space and let $x_1, \dots, x_n$ be n distinct points in X. Prove that there exist pairwise disjoint open sets $U_1, \dots, U_n$ with $x_i \in U_i$ for every i.

8. Let X be a complete metric space and let Y be another metric space that is homeomorphic to X. Must Y be complete?

9. Let $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ with the obvious topology coming from $\mathbb{C}$. Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $x - y$ is an integer. Prove that $\mathbb{T}$ is homeomorphic to $\mathbb{R}/\sim$ with the quotient topology.

10. Prove that every metric space is normal. Find an example of a Hausdorff topological space that is not normal.

11. Let X be a subset of $\mathbb{R}$. A point x in X is called *isolated* if there exists some $\delta > 0$ such that $|x - y| \geqslant \delta$ for every other $y \in X$. X is called *perfect* if it is closed and has no isolated points. Prove that a non-empty perfect set must be uncountable. (Hint: consider X as a metric space in its own right, and don't forget the Baire category theorem.)

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that for every $x > 0$ we have $f(nx) \to 0$ as $n \to \infty$. Prove that $f(x) \to 0$ as $x \to \infty$.