1. For each \( n \), let \( f_n : [0, 1] \to \mathbb{R} \) be a function and suppose that the functions \( f_n \) converge uniformly to a function \( f \). Suppose also that \( f \) is bounded (above and below). Prove that for any positive integer \( m \) the functions \( g_n(t) = f_n(t)^m \) converge uniformly to \( g(t) = f(t)^m \).

2. Prove that there is no sequence of analytic functions \( f_n \) that converges uniformly on the unit circle to the function \( 1/z \) (which on the circle is the same as \( \bar{z} \)). Why does this not contradict Runge’s theorem?

3. Construct a sequence of polynomials that converges uniformly to \( 1/z \) on the semicircle consisting of all points of the unit circle that have real part greater than or equal to 0.

4. Work out continued-fraction expansions for \( 71/49 \) and \( \sqrt{3} \).

5. Define a sequence \( (x_n) \) by \( x_1 = 1, x_2 = 3 \) and \( x_n = x_{n-1} + x_{n-2} \) for \( n \geq 3 \). Prove that \( x_n/x_{n-1} \) converges to the golden ratio \( (1 + \sqrt{5})/2 \). Describe the continued-fraction expansion of \( x_n/x_{n-1} \).

6. Let \( x \) have continued-fraction expansion

\[
1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{z + \frac{1}{z + \ldots}}}}
\]

with the 1s and 2s continuing to alternate. Calculate the value of \( x \).

7. What rational with denominator less than 10 best approximates the number \( 71/49 \)?

8. Let \( C \) be a circle with centre 0 in the complex plane, traversed anticlockwise, and let \( K \) be a compact set disjoint from \( C \). Define a function on \( K \) by

\[
f(z) = \frac{1}{2\pi i} \int_C \frac{dw}{w - z}.
\]

By splitting up the path integral into small pieces and approximating the contribution from each piece, prove that \( f \) can be uniformly approximated on \( K \) by functions of the form \( f_n(z) = \sum_{i=1}^{N} a_i/(w_i - z) \), where the \( a_i \) and \( w_i \) are complex numbers with the \( w_i \) lying in \( C \).

9. Prove that \( \sqrt{3} + \sqrt{5} \) and \( e^2 \) are irrational.