Topics in Analysis Examples Sheet 3 W. T. G.

1. For each n, let $f_n : [0,1] \to \mathbb{R}$ be a function and suppose that the functions f_n converge uniformly to a function f. Suppose also that f is bounded (above and below). Prove that for any positive integer m the functions $g_n(t) = f_n(t)^m$ converge uniformly to $g(t) = f(t)^m$.

2. Prove that there is no sequence of analytic functions f_n that converges uniformly on the unit circle to the function $1/z$ (which on the circle is the same as $\bar{z}$). Why does this not contradict Runge's theorem?

3. Construct a sequence of polynomials that converges uniformly to $1/z$ on the semicircle consisting of all points of the unit circle that have real part greater than or equal to 0.

4. Work out continued-fraction expansions for $71/49$ and $\sqrt{3}$.

5. Define a sequence (x_n) by $x_1 = 1$, $x_2 = 3$ and $x_n = x_{n-1} + x_{n-2}$ for $n \geqslant 3$. Prove that x_n/x_{n-1} converges to the golden ratio $(1+\sqrt{5})/2$. Describe the continued-fraction expansion of x_n/x_{n-1}.

6. Let x have continued-fraction expansion

$$1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}$$

with the 1s and 2s continuing to alternate. Calculate the value of x.

7. What rational with denominator less than 10 best approximates the number $71/49$?

8. Let C be a circle with centre 0 in the complex plane, traversed anticlockwise, and let K be a compact set disjoint from C. Define a function on K by

$$f(z) = \frac{1}{2\pi i} \int_C \frac{dw}{w - z} \ .$$

By splitting up the path integral into small pieces and approximating the contribution from each piece, prove that f can be uniformly approximated on K by functions of the form $f_n(z) = \sum_{i=1}^N a_i/(w_i - z)$, where the a_i and w_i are complex numbers with the w_i lying in C.

9. Prove that $\sqrt{3} + \sqrt{5}$ and e^2 are irrational.