Topics in Analysis, Examples Sheet 2. W. T. G. Michaelmas 2004

General remark. Unless it is clearly inappropriate, you may quote from IA and IB.

1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function, let n be a positive integer and for $0 \leqslant i \leqslant n$ let $x_i = i/n$. Define a function $g_n : [0,1] \to \mathbb{R}$ by setting $g(x_i) = f(x_i)$ for each i, and making g linear on all the intervals $[x_{i-1}, x_i]$. Show that the functions g_n converge uniformly to f.

2. Let $p(z)$ be the quadratic polynomial $z^2 - 4z + 3$ and let $f : [0,1] \to \mathbb{C}$ be the closed path $f(t) = p(2e^{2\pi i t})$. (Thus, the image of f is the image of the restriction of p to the circle of radius 2 and centre 0.)

(i) Calculate the winding number of f about 0 directly from the definition of winding number.

(ii) Can you give another proof that uses some of the results of the course?

3. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(t) = 0$ if $t \leqslant 1/2$ and $f(t) = 2t - 1$ if $1/2 \leqslant t \leqslant 1$. Find a polynomial p such that $|p(t) - f(t)| \leqslant 1/5$ for every $t \in [0,1]$.

4. It is clear that a function with a jump-discontinuity cannot be uniformly approximated by polynomials. However, that is not the only kind of discontinuity. True or false: no discontinuous function on $[0,1]$ can be uniformly approximated by polynomials?

5. Imitate the proof of the Weierstrass approximation theorem to prove that a continuous function $f : [0,1]^2 \to \mathbb{R}$ can be uniformly approximated by polynomials. [In the final version of this question I shall give an outline of how the proof should go, but it would be a good exercise to think about it without those hints first.]

6. Let $f : [0,1]^2 \to \mathbb{R}$ such that $f(x,y)$ is continuous in x for each fixed y and continuous in y for each fixed x. Does it follow that f is continuous?

7. Without looking at a book or at your old notes, prove that a continuous function from a compact metric space to $\mathbb{R}$ is uniformly continuous. [Write out the definition of uniform continuity, negate it, use the negated definition to generate a pair of sequences (x_n) and (y_n) and apply sequential compactness.]

8. Calculate the first five Chebyshev polynomials.

9. Calculate the first four Legendre polynomials. Do it both from the formula and by using orthogonality and check that your answers agree.

10. The Chebyshev polynomials form an orthogonal system with respect to a certain positive weight function w. That is, $\int_{-1}^{1} T_m(x) T_n(x) w(x)\, dx = 0$ whenever $m \neq n$. Work out what the weight function should be, and prove the orthogonality. [Hint: use an appropriate substitution for x!]