Topics in Analysis, Examples Sheet 1. W. T. G. Michaelmas 2004

1. Let X be a non-compact subset of $\mathbb{R}^2$. Prove that there is a continuous unbounded function from X to $\mathbb{R}$.

2. Prove that a function f from a metric space X to a metric space Y is continuous if and only if $f^{-1}(A)$ is closed in X for every closed set $A \subset Y$.

3. Let K be a compact subset of $\mathbb{R}^2$ and let F be a closed subset of $\mathbb{R}^2$ such that $K \cap F = \emptyset$. Give two proofs that there exists $\delta > 0$ such that $d(x, y) \geqslant \delta$ for every $x \in K$ and every $y \in F$, one proof directly from the compactness of K and the other using sequential compactness instead.

4. Let I, J and K be three arcs, each making an angle of $2\pi/3$, whose union is the whole of the unit circle (as in lectures). Let A, B and C be *open* subsets of the closed unit disc D with $I \subset A$, $J \subset B$ and $K \subset C$. Suppose that $A \cup B \cup C$ is the whole of D. Deduce that $A \cap B \cap C$ is non-empty from the corresponding result about closed sets. [Hint: One way to do it is to start by defining $F \subset A$ to be the set of all x such that $d(x, A^c) \geqslant \max\{d(x, B^c), d(x, C^c)\}$, and similarly for sets $G \subset B$ and $H \subset C$.]

5. Let f be a continuous function from the *open* unit disc to itself. Must it have a fixed point? Does your answer change if f is a surjection?

6. Let X and Y be metric spaces, let K be a compact subset of X and let $f : X \to Y$ be continuous. Prove that $f(K)$ is closed.

7. Let f be a continuous function from a compact metric space X to itself, and suppose that $d(f(x), f(y)) = d(x, y)$ for every $x, y \in X$. Prove that f is a surjection. [Hint: if not, pick $x \notin f(X)$, show that there is a ball about x that misses $f(X)$ and consider the sequence $x, f(x), f(f(x)), \ldots$.]

8. Let x be a point in the closed unit disc D, let K be a compact subset of D not containing x and define a map $f_x : K \to \partial D$ as follows. Given $y \in K$, take the line from x to y and extend it (in the x-to-y direction) until it first hits the boundary. Call this point $f_x(y)$. Prove that f_x is a continuous function.

9. Let $g : D \to D$ be a continuous function and let $x \in D$ be a point such that $x \neq g(x)$. Let $h(x) = f_x(g(x))$, where f_x is defined as in question 8 (on some open set containing x). Prove that h is continuous at x.

10. Let $C[0,1]$ be the metric space consisting of all continuous functions $f : [0,1] \to \mathbb{R}$, with the distance $d(f,g)$ between two functions f and g defined to be the supremum of $|f(x) - g(x)|$ over all $x \in [0,1]$.

(i) Explain why this supremum is in fact a maximum.

(ii) Let X consist of all functions f in $C[0,1]$ that take values in $[0,1]$. Prove that X is not a sequentially compact subset of $C[0,1]$.

(iii) This shows that X is not compact. Prove the same result by exhibiting an open cover of X that has no finite subcover.

11. Prove that every compact metric space is complete. The *diameter* of a metric space X is the supremum of $d(x,y)$ over all points $x, y \in X$. Give an example of a metric space that is complete and has diameter 1 but is not compact.

12. Let A be a 3×3-matrix with positive entries. Use Brouwer's fixed-point theorem to prove that A has an eigenvector with positive entries. [Hint: use A to define a map from T to itself, where T is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.]