Topics in Analysis - Extra Examples Sheet. W. T. G.

Note. These problems are meant to help you to refamiliarize yourself with the content of the course, and to provide material for revision supervisions. They should not be taken as an indication of what the actual exam questions will be like. In particular, there are no bookwork questions on this sheet, and some of the problems, while not intended to be particularly difficult, are designed to make you think a bit harder than it would be reasonable to expect in an exam.

1. Using the Brouwer fixed-point theorem directly, prove that there is a complex number z such that

$$z^8 + 3z^6 - 2z^5 + z^3 - z^2 + 10z = 1 \ .$$

Using other results from the course (which you should state), but not the fundamental theorem of algebra, prove that there is a complex number w such that

$$w^8 + 3w^6 - 2w^5 + 10w^3 - w^2 + w = 1 \ .$$

2. Take an $n \times n$ grid of points and colour all its vertices red, blue, green or yellow. Do this in such a way that all the points along the bottom are red, all the ones on the right are blue, all the ones along the top are green and all the ones on the left are yellow. (At the four corners of the grid there will be a conflict: choose one of the two possible colours in such a way that the corners are all of different colours.) Prove that for at least one of the $(n-1)^2$ little squares in the grid, at least three colours are used. Find an example where no such square uses all four colours.

3. Construct a sequence of polynomials (P_n) such that $P_n(x)$ converges uniformly to $|x|$ in the interval $[-1, 1]$.

4. (i) Let $\mathcal{F}$ be the set of all functions defined on $[-1, 1]$ that can be uniformly approximated by polynomials. Using the result of the previous question, prove that if f and g belong to $\mathcal{F}$, then so do $f \vee g$ and $f \wedge g$. (These are defined by $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$.)

(ii) Use (i) to prove that any continuous piecewise linear function defined on $[-1, 1]$ belongs to $\mathcal{F}$. (A piecewise linear function is one that is allowed to change direction at finitely many points and is linear between them.)

(iii) Prove that any continuous function on $[-1,1]$ can be uniformly approximated by piecewise linear functions and deduce that $\mathcal{F} = C[-1,1]$. (If you get to this point, then you have given another proof of Weierstrass's approximation theorem.)

5. Without using the formula for the nth Legendre polynomial P_n, prove that it is an even function if n is even and an odd function if n is odd.

6. Construct a sequence of polynomials P_n such that, for every positive real number a and every positive real number $b < a$, $P_n(z)$ converges uniformly to z^{-1} on the disc $\{z \in \mathbb{C} : |z-a| < b\}$. (If you don't want to give an exact formula for P_n, then at least explain in principle how it can be done.)

7. (i) Suppose that x is a real number and it has a simple continued fraction expansion of the form

$$1 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Prove that there is a sequence of fractions (p_n/q_n) such that

$$|x - \frac{p_n}{q_n}| \leqslant \frac{1}{a_n q_n^2} \ .$$

(ii) If the coefficients a_n are unbounded, deduce that x cannot be a quadratic irrational. (It is possible to prove this by showing that the continued fraction for a quadratic irrational is eventually periodic. That is not the proof that is being suggested here. Note that if you know the continued fraction expansion of e then you can deduce that e is not the root of a quadratic polynomial with integer coefficients.)

8. (i) Let X be a non-compact topological space, and let x be a point not belonging to X. Define a topology on $X \cup \{x\}$ by taking all open sets $U \subset X$ together with all sets of the form $U \cup \{x\}$ such that U is open in X and $X \setminus U$ is compact. Prove that $X \cup \{x\}$ is compact under this topology. (It is called the *one-point compactification* of X.)

(ii) Let X be the complex plane $\mathbb{C}$. Prove that the one-point compactification of X is homeomorphic to the sphere $\{v \in \mathbb{R}^3 : \|v\|^2 = 1\}$.

9. Let X be an infinite-dimensional normed space. Prove that the linear span of any finite collection of vectors is nowhere dense (with respect to the metric $d(x,y) = \|x-y\|$. Deduce that if X is complete, then it is not the linear span of a countable collection of vectors. (The linear span of a subset $Y \subset X$ is the set of all *finite* linear combinations $\sum_{i=1}^n a_i y_i$ such that each y_i belongs to Y.)