1. (i) Deduce the following extension of Roth’s theorem from Ruzsa’s lemma (and Plünnecke’s inequality). For every $C$ there exists $n$ such that, if $A \subset \mathbb{Z}$, $|A| \geq n$ and $|A + A| \leq C|A|$, then $A$ contains an arithmetic progression of length three.

(ii) Using the Balog-Szemerédi theorem, deduce a further extension.

2. Let $A \subset \mathbb{Z}_N$. Define a $k$-cube in $A$ to be a mapping $\phi : \{0,1\}^k \rightarrow \mathbb{Z}_N$ of the form $\phi(\epsilon) = x_0 + \sum_{i=1}^k \epsilon_i x_i$, where $x_0, x_1, \ldots, x_k$ belong to $\mathbb{Z}_N$ and the image of $\phi$ lies in $A$. (For example, a 2-cube can be thought of as a quadruple $(a, a+b, a+c, a+b+c) \in A^4$.)

(i) Let $A$ have size $\delta N$. Prove that $A$ contains at least $\delta^2 N^{k+1} k$-cubes.

(ii) Suppose that $A$ is quadratically $\alpha$-uniform. Prove that $A$ contains at most $(\delta^8 + c(\alpha))N^4$ 3-cubes, where $c(\alpha)$ tends to zero as $\alpha$ tends to zero.

3. Obtain a decent lower bound (without appealing to the geometry of numbers) for the size of the Bohr neighbourhood $B(r_1, \ldots, r_k; \delta)$.

4. Let $A$ be a subset of $\mathbb{Z}_N$ of cardinality $\delta N$. Prove that $A + A + A$ contains an arithmetic progression (mod $N$) of length $N^c$, where $c$ depends on $\delta$ only. Give an example of a set $A$ such that the longest arithmetic progression in $A + A + A$ has length $N^{c'}$ for some $c'$ that tends to zero with $\delta$. (You ought to be able to get $c$ proportional to $\delta^2$ and $c'$ proportional to $\log(1/\delta)$. Closing the gap between these two bounds is an important open problem.)

5. (Hard - in fact, probably an unsolved problem.) Let $\epsilon > 0$, let $N$ be prime and let $A$ and $B$ be subsets of $\mathbb{Z}_N$ of size $[N/2]$. If $N$ is large enough (as a function of $\epsilon$), prove that there exists some $x \in \mathbb{Z}_N$ such that $A \cap (B + x)$ has cardinality between $(\frac{1}{4} - \epsilon)N$ and $(\frac{1}{4} + \epsilon)N$. What can go wrong if $N$ is not prime?

6. Let $f$ be an increasing function. Obtain an estimate for $\left| \sum_{x \leq n} f(x) e(\alpha x + \beta) \right|$ under the assumption that $(a, q) = 1$ and $|\alpha - a/q| \leq q^{-2}$.

7. By carrying out the following steps, prove that the subset $A = \{x : x^2 \leq N/10000\}$ of $\mathbb{Z}_N$ discussed in lectures satisfies $\max_{r \neq 0} |\hat{A}(r)| = O(\sqrt{N \log N})$.

(i) Prove that if $f(x) = \omega^{ax^2}$ then $|\hat{f}(r)| = \sqrt{N}$ for every $r$.

(ii) Set $M = [N/10000]$, let $I = [-M, M]$ and notice that $A(x) = I(x^2)$ for every $x$. Use the inversion formula to write an expression for $A(x)$ in terms of the Fourier coefficients of $I$. 

1
(iii) Finish off by using (i) and a simple estimate for the sizes of the Fourier coefficients of $I$.

8. Prove also that $A$ contains significantly more than $10^{-16}N^2$ arithmetic progressions (in $\mathbb{Z}_N$) of length four. (You should find 7(ii) helpful here as well.)

9. Let $A$ be a subset of $I = \{x \in \mathbb{Z} : |x| \leq n\}$ of cardinality $\delta N$. Prove that there is a constant $k$, depending on $\delta$ only, such that $kA - kA$ contains the whole of $I$. (I’m not quite sure how to do this question but I know it’s known.)

10. Show that there are constants $0 < \beta < \alpha^3 \leq 1$ such that, for infinitely many $N$, $\mathbb{Z}_N$ contains a set of size $\alpha N$ with at most $\beta N^2$ triples of the form $(a, a + d, a + 2d)$ (mod $N$).

11. Let $A \subset \mathbb{N}$. Suppose that

$$\forall \alpha \in \mathbb{R} \forall \epsilon > 0 \exists q \in A \|q\alpha\| \leq \epsilon.$$ 

Prove that

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall \alpha \in \mathbb{R} \exists q \leq N \|q\alpha\| \leq \epsilon.$$ 

(That is, obtain a result which is uniform over $\alpha$.)