Grounded Lipschitz functions on trees are typically flat

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Abstract

A grounded M-Lipschitz function on a rooted d-ary tree is an integer-valued map on the vertices that changes by at most M along edges and attains the value zero on the leaves. We study the behavior of such functions, specifically, their typical value at the root v_0 of the tree. We prove that the probability that the value of a uniformly chosen random function at v_0 is more than $M+t$ is doubly-exponentially small in t. We also show a similar bound for continuous (real-valued) grounded Lipschitz functions.

1 Introduction

This note studies the typical behavior of grounded Lipschitz functions on trees. We consider the rooted d-ary tree, $d \geq 2$, of depth k, that is, the tree whose each non-leaf vertex, including the root, has d children. We denote this tree by $T(d,k)$ and its root by v_0. We consider two models for Lipschitz functions on such a tree:

(i) We call a real-valued function f on the vertices of a graph *Lipschitz* if $|f(u) - f(v)| \leq 1$ for every pair u, v of adjacent vertices. Let $L_\infty(d,k)$ be the family of all such Lipschitz functions on $T(d,k)$ that take the value zero on all leaves. Observe that $L_\infty(d,k)$ is naturally endowed with Lebesgue measure. Indeed, letting V be the set of internal (non-leaf) vertices of $T(d,k)$, one may naturally view $L_\infty(d,k)$ as a convex polytope $P \subseteq \mathbb{R}^V$ (of full dimension).

(ii) For an integer $M \geq 1$, we call an integer-valued function f on the vertices of a graph M-*Lipschitz* if $|f(u) - f(v)| \leq M$ for every two adjacent vertices u and v. Let $L_M(d,k)$ be the set of all M-Lipschitz functions on $T(d,k)$ that take the value zero on all leaves. Observe that $L_M(d,k)$ is a finite set. For example, $L_M(d,1)$ has $2M+1$ elements, one element for each of the possible values of $f(v_0)$ between $-M$ and M.

In this work we study the distribution of the value at the root vertex v_0 for a uniformly chosen function in $L_\infty(d,k)$ and $L_M(d,k)$. Our main result is that this value is very tightly concentrated around 0 for every $M \geq 1$ and $d \geq 2$.

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Theorem 1.1. *Let $d \geq 2$ and $k \geq 1$. If f is chosen uniformly at random from $L_\infty(d,k)$, then for every $x > 0$,*

$$\Pr(f(v_0) \geq 1 + x) \leq 2^{d+2} \cdot (9/10)^{d^{\lceil x \rceil - 1}}.$$

Moreover, if $d \geq 3$, then the constant $9/10$ above may be replaced by $(3/4)^d$.

This theorem follows as a corollary from the result on M-Lipschitz functions described next, by taking a limit as M tends to infinity.

Theorem 1.2. *Let $d \geq 2$ and $M, k \geq 1$. If f is chosen uniformly at random from $L_M(d,k)$, then for every integer $t \geq 1$,*

$$\Pr(f(v_0) = M + t) \leq (9/10)^{d^{\lfloor (t-1)/M \rfloor}} \cdot \Pr(f(v_0) = 0). \tag{1.1}$$

Moreover, if $d \geq 3$, then the constant $9/10$ above may be replaced by $(3/4)^d$. In addition,

$$\Pr(f(v_0) = 0) \leq \frac{1}{1 + 2^{1-d} M}.$$

Of course, by symmetry, the theorems imply a corresponding bound on the lower tail of the distribution of $f(v_0)$. We prove Theorem 1.2 by induction on k. The argument has three steps. The first step establishes that $p(t) := \Pr(f(v_0) = t)$ is unimodal in t with maximum at $t = 0$. The second step shows that $p(t)$ decays at least exponentially in t/M, i.e., that $p(t + M) \leq 9p(t)/10$ for $t \geq 1$. In the third step, inequality (1.1) is derived by induction on t.

2 Motivation, related work and discussion

The study of Lipschitz functions on graphs may be seen as a special case of the study of random surfaces in statistical mechanics. In that context, one usually considers a real-valued function on the vertices of a finite graph G, often a box in the integer lattice $\mathbb{Z}^d$, which is sampled from a non-uniform distribution, with the density (with respect to Lebesgue measure) of a function f proportional to

$$\exp\left(- \sum_{\{x,y\} \in E(G)} V(f(x) - f(y)) \right).$$

for some *potential* function V, assumed symmetric. The study of such random surfaces is especially well developed when the potential V is twice continuously differentiable with $c \leq V'' \leq C$ for some $C, c > 0$. However, the understanding of such surfaces is quite lacking when this condition fails [9].

Continuous Lipschitz functions naturally embed in the above setup when the potential V is the so-called *hammock* potential

$$V(x) := \begin{cases} 0 & |x| \leq 1 \\ \infty & |x| > 1 \end{cases}.$$

The hammock potential is not twice continuously differentiable and it is the case that when G is a box in $\mathbb{Z}^d$, there is no non-trivial bound on the variance of f at a given vertex, or on the expected range of values taken by f. The problem of finding such non-trivial bounds was mentioned already by Brascamp, Lieb, and Lebowitz [3]. Similar questions may be asked for M-Lipschitz functions on boxes in $\mathbb{Z}^d$ and similarly no non-trivial facts are known, except in the case $M = 1$ when it is known that a typical 1-Lipschitz function is quite 'flat' when the dimension d is sufficiently large [6]. Given the apparent difficulty of studying Lipschitz functions on the integer lattice, it is natural to study this model on more general graphs, in particular on trees, and this is the main motivation for our work.

In order to define a uniform distribution over Lipschitz functions on a finite connected graph, one needs to put certain boundary conditions. For instance, one can fix the value of the function at one given vertex to be zero. When the graph is a tree, it is natural to take this vertex to be the root. However, such a boundary condition on a tree leads to the model of a branching random walk, since the differences of the function along edges of the tree are independent. Such a model is closer in spirit to an ordinary random walk than to a random surface on a highly-connected graph such as $\mathbb{Z}^d$ for $d \geq 2$. In contrast, by choosing the boundary condition differently, one may obtain a model sharing more of the features of random surfaces on highly-connected graphs. The most natural alternative seems to be to fix the values of the function at all leaves of the tree to be zero, as we do in this paper.

We briefly survey some related work. Benjamini, Häggström, and Mossel [1] initiated the study of a related model, the model of *random graph homomorphisms*. An integer-valued function f on the vertices of a graph is called a graph homomorphism (or a homomorphism height function) if $|f(u) - f(v)| = 1$ for every pair u, v of adjacent vertices. Graph homomorphisms are similar to 1-Lipschitz functions (see the Yadin bijection described in [6] for a precise connection). Results for this model were obtained for: tree-like graphs [1], the hypercube [4, 5], and finite boxes in the integer lattice $\mathbb{Z}^d$ for large d [6]. A lower bound on the typical range of random graph homomorphisms on general graphs was established in [2].

In [1], a result analogous to our Theorem 1.2 was proved for random graph homomorphisms, using a similar, though somewhat simpler, method. Both approaches rely on a recursive formula relating the number of functions on trees with different depths. However, our analysis is made more complicated by the fact that certain symmetries present in the formula for graph homomorphisms used in [1] are no longer available for the M-Lipschitz model. This necessitates a different analysis employing unimodality and requiring a more careful consideration of the exact relation of d and M.

The behavior of M-Lipschitz functions on trees and on regular expander graphs was studied in [8], where results similar to Theorem 1.2 were obtained under an additional assumption that M is small with respect to the degree of the graph and its expansion properties. For trees, the assumption in [8] is that

$$M \leq cd/\log d \tag{2.1}$$

for some small absolute constant c. This assumption on M, while allowing the study of the model on a wider class of graphs, precluded the study of real-valued Lipschitz functions, or M-Lipschitz functions for large M, using the methods of that work.

We end the introduction with a discussion of the role of the assumption (2.1). In a forthcoming paper [7] we extend the methods and results of [8], showing that under the assumption (2.1), more is true of a uniformly chosen function from $L_M(d, k)$. In fact, at a vertex which is at even distance from the leaves, the function will take the value 0 with high probability, namely, with probability at least $1 - 2\exp(-cd/M)$ for some absolute constant $c > 0$. That is, the function value is concentrated on a single number. Note that this implies that for a vertex at odd distance from the leaves, with a similarly high probability, the function value at all of its neighbors is zero, and that conditioned on this event, the value at the vertex is uniform on $\{-M, \ldots, M\}$.

We expect that such a strong concentration of the values of the random function fails when $M \gg d$, or in the model of real-valued Lipschitz functions. More precisely, let us fix M, d such that $M \gg d$ and denote by $\mathcal{L}_k$ the distribution of the value at the root for a tree of depth k. We believe that $\mathcal{L}_k$ is no longer concentrated on a single value when k is even. Moreover, we suspect that $\mathcal{L}_k$ has a limit as k tends to infinity (so that there is asymptotically no distinction between even and odd depths). It would be interesting to establish such a transition phenomenon between the cases $M \ll d$ and $M \gg d$.

3 The proofs

In this section, we prove Theorems 1.1 and 1.2. First, we establish a series of lemmas that culminate in the proof of Theorem 1.2. At the end, we derive from it Theorem 1.1.

Fix integers $d \geq 2$ and $M \geq 1$. For integers t and $k \geq 1$, we let

$$G(t, k) = |\{f \in L_M(d, k)\colon f(v_0) = t\}|.$$

Several times in our proofs, we will use the fact that $G(t, k) = G(-t, k)$ for every t and k, which follows by symmetry.

3.1 A recursive formula

Since for every $k > 1$, the children of the root of $T(d, k)$ can be regarded as roots of isomorphic copies of $T(d, k-1)$, we have

$$G(t, k) = \sum_{t-M \leq t_1, \ldots, t_d \leq t+M} \prod_{s=1}^{d} G(t_s, k-1) = \left(\sum_{i=-M}^{M} G(t+i, k-1) \right)^d. \tag{3.1}$$

3.2 Unimodality

The following claim establishes unimodality.

Claim 3.1. *If $k \geq 1$ and $t \geq 0$, then $G(t+1, k) \leq G(t, k)$.*

Proof. We prove the claim by induction on k. If $k = 1$, then $G(t, k) = 1$ if $0 \leq t \leq M$ and $G(t, k) = 0$ if $t > M$. Assume that $k > 1$ and $t \geq 0$. By (3.1), it suffices to show that

$$G(t + M + 1, k-1) \leq G(t - M, k-1). \tag{3.2}$$

To see this, we consider two cases. First, if $t - M \geq 0$, then (3.2) follows directly from the inductive assumption. Otherwise, we note that $t + M + 1 \geq M - t > 0$ and hence by symmetry and induction,

$$G(t + M + 1, k-1) \leq G(M - t, k-1) = G(t - M, k-1). \qquad \square$$

3.3 Exponential decay of $G(t,k)$

The following claim establishes exponential decay of $G(t,k)$. We start with the case $d \geq 3$. The case $d = 2$ is more elaborate and we handle it separately later on.

Lemma 3.2. *If $d \geq 3$ and $k, t \geq 1$, then $G(t+M,k) \leq (3/4)^d \cdot G(t,k)$.*

Proof. Fix some $d \geq 3$. We prove the lemma by induction on k. Suppose that $t \geq 1$. If $k = 1$, then $G(t+M,k) = 0$ and the claimed inequality holds vacuously. Assume that $k > 1$. To simplify the notation, we let

$$\alpha = (3/4)^d \qquad \text{and} \qquad G(s) = G(s, k-1) \quad \text{for each } s \in \mathbb{Z}.$$

Moreover, for a set $S \subseteq \mathbb{Z}$, we let

$$G(S) = \sum_{s \in S} G(s).$$

If $t > M$, then by the inductive assumption, $G(t+M+i) \leq \alpha G(t+i)$ for every $-M \leq i \leq M$, so (3.1) implies that

$$G(t+M,k) = G(\{t,\ldots,t+2M\})^d \leq (\alpha \cdot G(\{t-M,\ldots,t+M\}))^d = \alpha^d \cdot G(t,k).$$

Assume that $1 \leq t \leq M$, let

$$A = G(\{0,\ldots,t-1\}), \quad B = G(\{1,\ldots,M-t\}), \quad \text{and} \quad C = G(\{t,\ldots,t+M\}),$$

and observe that, by (3.1) and symmetry,

$$G(t,k) = (A+B+C)^d.$$

On the other hand, since

$$\{t+M+1,\ldots,t+2M\} = (\{1,\ldots,t\}+2M) \cup (\{t+1,\ldots,M\}+M), \tag{3.3}$$

then (3.1), the inductive assumption, and Claim 3.1 imply that

$$G(t+M,k) \leq \left(\alpha^2 G(\{1,\ldots,t\}) + \alpha G(\{t+1,\ldots,M\}) + C\right)^d \leq (\alpha^2 A + \alpha B + C)^d.$$

Moreover, Claim 3.1 implies

$$(M+1)(A+B) \geq MC \quad \text{and} \quad MA \geq (A+B).$$

It therefore follows that

$$\begin{aligned}
\frac{G(t+M,k)}{G(t,k)} &\leq \left(\frac{\alpha^2 A + \alpha B + C}{A+B+C}\right)^d \leq \left(\frac{\alpha^2 A + \alpha B + (1+1/M)(A+B)}{A+B+(1+1/M)(A+B)}\right)^d \\
&= \left(\frac{(1+\alpha+1/M)(A+B) - (\alpha-\alpha^2)A}{(2+1/M)(A+B)}\right)^d \\
&\leq \left(\frac{1+\alpha+(1-\alpha+\alpha^2)/M}{2+1/M}\right)^d \leq \left(\max\left\{\frac{1+\alpha}{2}, \frac{2+\alpha^2}{3}\right\}\right)^d \leq \alpha,
\end{aligned}$$

where the final inequality holds by our assumption that $d \geq 3$. □

Remark 3.3. It follows from the proof of Lemma 3.2 that even when $d = 2$, the statement of the lemma still holds as long as we replace the constant $(3/4)^d$ with some constant $\alpha < 1$ that satisfies

$$\left(\frac{1+\alpha+(1-\alpha+\alpha^2)/M}{2+1/M}\right)^d \le \alpha. \tag{3.4}$$

Unfortunately, the smallest solution to (3.4) tends to 1 as $M \to \infty$. We thus need a more careful analysis to handle the case $d = 2$. Our proof in the case $d = 2$ shall require a mild lower bound on M. We therefore note that if $M \le 10$ and $\alpha = 9/10$, then (3.4) is satisfied.

Lemma 3.4. *Suppose that $d = 2$. For all $k, t \ge 1$, we have $G(t + M, k) \le (9/10) \cdot G(t, k)$.*

Proof. By Remark 3.3, we can safely assume that $M \ge 11$. Choose the following parameters:

$$\alpha = 9/10, \quad \mu = 1/4, \quad \beta = 1/3 \quad \text{and} \quad m = \lceil \mu M \rceil.$$

We are going to prove the following stronger statement by induction on k:

$$G(t+M,k) \le \begin{cases} \alpha \cdot G(t,k) & \text{if } t \in \{1,\dots,m-1\}, \\ \alpha^2 \cdot G(t,k) & \text{if } t \ge m. \end{cases} \tag{3.5}$$

If $k = 1$, then (3.5) holds vacuously as $G(t + M, k) = 0$ for every $t \ge 1$. Assume that $k > 1$ and fix some $t \ge 1$. To simplify notation, for every $s \in \mathbb{Z}$, we let

$$G(s) = G(s, k-1)$$

and for a set $S \subseteq \mathbb{Z}$,

$$G(S) = \sum_{s \in S} G(s).$$

If $t > M$, then by the inductive assumption and (3.1), we have

$$G(t+M,k) = G(\{t,\dots,t+2M\})^2 \le (\alpha \cdot G(\{t-M,\dots,t+M\}))^2 = \alpha^2 \cdot G(t,k).$$

Assume that $1 \le t \le M$, let

$$A = G(\{0,\dots,t-1\}), \quad B = G(\{1,\dots,M-t\}) \quad \text{and} \quad C = G(\{t,\dots,t+M\}),$$

and observe that by (3.1) and symmetry,

$$G(t,k) = (A+B+C)^2.$$

We split the proof into two cases, depending on the value of t.

Case 1: $\mathbf{t \ge m}$. Identity (3.1), the inductive assumption, and Claim 3.1 imply that (recall (3.3))

$$\begin{aligned} G(t+M,k)^{1/2} &\le \alpha^2 \cdot \alpha \cdot G(\{1,\dots,t\}) + \alpha^2 \cdot G(\{t+1,\dots,M\}) + C \\ &\le \left(\frac{t}{M} \cdot \alpha^3 + \frac{M-t}{M} \cdot \alpha^2\right) \cdot (A+B) + C \\ &\le (\mu\alpha^3 + (1-\mu)\alpha^2) \cdot (A+B) + C \end{aligned}$$

and Claim 3.1 implies that

$$(M+1)(A+B) \geq MC. \tag{3.6}$$

It hence follows that

$$\begin{aligned}\frac{G(t+M,k)}{G(t,k)} &\leq \left(\frac{(\mu\alpha^3 + (1-\mu)\alpha^2)\cdot(A+B)+C}{A+B+C}\right)^2 \\ &\leq \left(\frac{1+\mu\alpha^3+(1-\mu)\alpha^2+1/M}{2+1/M}\right)^2 < \alpha^2,\end{aligned}$$

where in the last inequality we used the assumption that $M \geq 11$.

Case 2: t < m. Identity (3.1), the inductive assumption, and Claim 3.1 imply that (recall (3.3))

$$G(t+M,k)^{1/2} \leq \alpha^3 \cdot G(\{1,\ldots,t\}) + \alpha \cdot G(\{t+1,\ldots,m-1\}) + \alpha^2 \cdot G(\{m,\ldots,M\}) + C. \tag{3.7}$$

We now further split into two cases, depending on whether or not the following inequality is satisfied:

$$G(\{m,\ldots,M\}) \geq \beta \cdot G(\{1,\ldots,M\}). \tag{3.8}$$

If (3.8) holds, then by (3.7) and Claim 3.1,

$$\begin{aligned}G(t+M,k)^{1/2} &\leq (\beta\alpha^2 + (1-\beta)\alpha)\cdot G(\{1,\ldots,M\}) + C \\ &\leq (\beta\alpha^2 + (1-\beta)\alpha)\cdot(A+B) + C.\end{aligned}$$

Consequently, by (3.6),

$$\begin{aligned}\frac{G(t+M,k)}{G(t,k)} &\leq \left(\frac{(\beta\alpha^2 + (1-\beta)\alpha)\cdot(A+B)+C}{A+B+C}\right)^2 \\ &\leq \left(\frac{1+\beta\alpha^2+(1-\beta)\alpha+1/M}{2+1/M}\right)^2 < \alpha,\end{aligned}$$

where in the last inequality we again used the assumption that $M \geq 11$.

If (3.8) does not hold, then we let

$$D = G(\{t,\ldots,m-1\}) \quad \text{and} \quad E = G(\{1,\ldots,M\}).$$

Identity (3.1), Claim 3.1, and the converse of (3.8) imply

$$\begin{aligned}G(t+M,k)^{1/2} = D + G(\{m,\ldots,2M+t\}) &\leq D + \frac{2M+t-m+1}{M-m+1}\cdot G(\{m,\ldots,M\}) \\ &< D + \frac{2M}{M-m+1}\cdot\beta\cdot E \leq D + \frac{2}{1-\mu}\cdot\beta\cdot E.\end{aligned}$$

On the other hand, again by symmetry and Claim 3.1, we have that

$$G(t,k)^{1/2} = G(\{t-M,\ldots,t+M\}) \geq D+E.$$

So, as $D \leq E$,

$$\frac{G(t+M,k)}{G(t,k)} \leq \left(1 - \frac{1-\mu-2\beta}{1-\mu}\cdot\frac{E}{D+E}\right)^2 \leq \left(1-\frac{1-\mu-2\beta}{2-2\mu}\right)^2 < \alpha. \qquad \square$$

3.4 Doubly exponential decrease

The exponential decay established in Lemmas 3.2 and 3.4 easily implies the following.

Corollary 3.5. *If $k, t \geq 1$, then*

$$G(t+M,k) \leq \alpha^{d^{\lfloor (t-1)/M \rfloor}} G(t,k), \tag{3.9}$$

where $\alpha = 9/10$ if $d = 2$ and $\alpha = (3/4)^d$ if $d \geq 3$.

Proof. We prove the statement by induction on k. If $k = 1$, then $G(t+M,k) = 0$. Assume that $k > 1$. When $1 \leq t \leq M$, we have $\lfloor (t-1)/M \rfloor = 0$, and (3.9) follows directly from Lemmas 3.2 and 3.4. If $t \geq M+1$, then by the inductive assumption and (3.1) we have

$$\begin{aligned} G(t+M,k) &= \big(G(t,k-1) + \ldots + G(t+2M,k-1)\big)^d \\ &\leq \big(\alpha^{d^{\lfloor (t-M-1)/M \rfloor}} (G(t-M,k-1) + \ldots + G(t+M,k-1))\big)^d \\ &= \alpha^{d^{1+\lfloor (t-M-1)/M \rfloor}} G(t,k) = \alpha^{d^{\lfloor (t-1)/M \rfloor}} G(t,k). \end{aligned}$$

□

3.5 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.2. The first part of the theorem follows immediately from Corollary 3.5. The upper bound on $\Pr(f(v_0) = 0)$ follows from the inequality $G(M,k) \geq 2^{-d} G(0,k)$, which we prove below, together with symmetry and Claim 3.1. If $k = 1$, we have $G(M,k) = G(0,k) = 1$. If $k > 1$, identity (3.1) and symmetry imply that

$$\left(\frac{G(M,k)}{G(0,k)}\right)^{1/d} \geq \frac{1}{2}.$$

□

Proof of Theorem 1.1. Let f be a uniformly chosen random element of $L_\infty(d,k)$ and let $x > 0$. The claimed bound on the probability that $f(v_0)$ exceeds $1+x$ follows fairly easily from Theorem 1.2. To see this, let, for every positive integer M, f_M be a uniformly chosen random element of $L_M(d,k)$. A moment of thought reveals that the sequence f_M/M converges to f in distribution.

Indeed, letting V be the set of internal (non-leaf) vertices of $T(d,k)$, one may naturally view $L_\infty(d,k)$ as a convex polytope $P \subseteq \mathbb{R}^V$ (of full dimension). Let μ and μ_M be the distributions of f and f_M/M, respectively. Observe that $\mu = \lambda/\mathrm{vol}(P)$, where λ is the $|V|$-dimensional Lebesgue measure, and that μ_M is the uniform measure on the (finite) set $P \cap (\frac{1}{M}\mathbb{Z})^V$. Since P is compact (as clearly $P \subseteq [-k,k]^V$), every continuous function $g \colon P \to \mathbb{R}$ is uniformly continuous and therefore,

$$\lim_{M \to \infty} \int_P g \, d\mu_M = \int_P g \, d\mu.$$

Now, Theorem 1.2 implies that for $M \geq 1/x$, letting α be as in the statement of Corollary 3.5,

$$\begin{aligned} \Pr\left(\frac{f_M(v_0)}{M} \geq 1+x\right) &= \sum_{t \geq xM} \Pr(f_M(v_0) = M+t) \leq \Pr(f_M(v_0) = 0) \cdot \sum_{s=\lfloor x-1/M \rfloor}^{\infty} M\alpha^{d^s} \\ &\leq \frac{M}{1+2^{1-d}M} \cdot \alpha^{d^{\lfloor x-1/M \rfloor}} \cdot \left(1 + \sum_{r=1}^{\infty} \alpha^{d^r/2}\right) \leq \frac{8M}{1+2^{1-d}M} \cdot \alpha^{d^{\lfloor x-1/M \rfloor}}, \end{aligned}$$

where in the last inequality we used the fact that $\alpha^{d^r/2} \le (9/10)^{2^{r-1}}$, and hence

$$\Pr(f(v_0) \ge 1 + x) = \lim_{M\to\infty} \Pr\left(\frac{f_M(v_0)}{M} \ge 1 + x\right) \le 2^{d+2} \cdot \alpha^{d^{\lceil x \rceil - 1}}. \qquad \square$$

References

[1] I. Benjamini, O. Häggström, and E. Mossel, *On random graph homomorphisms into* $\mathbb{Z}$, J. Combin. Theory Ser. B **78** (2000), 86–114.

[2] I. Benjamini, A. Yadin, and A. Yehudayoff, *Random graph-homomorphisms and logarithmic degree*, Electron. J. Probab. **12** (2007), 926–950.

[3] H. J. Brascamp, E. H. Lieb, and J. L. Lebowitz, *The statistical mechanics of anharmonic lattices*, Proceedings of the 40th Session of the International Statistical Institute (Warsaw, 1975), Vol. 1. Invited papers, vol. 46, 1975, pp. 393–404.

[4] D. Galvin, *On homomorphisms from the Hamming cube to* $\mathbb{Z}$, Israel J. Math. **138** (2003), 189–213.

[5] J. Kahn, *Range of cube-indexed random walk*, Israel J. Math. **124** (2001), 189–201.

[6] R. Peled, *High-dimensional Lipschitz functions are typically flat*, arXiv:1005.4636v1 [math-ph].

[7] R. Peled, W. Samotij, and A. Yehudayoff, *H-coloring expander graphs*, in preparation.

[8] ———, *Lipschitz functions on expanders are typically flat*, to appear in Combin. Probab. Comput.

[9] Y. Velenik, *Localization and delocalization of random interfaces*, Probab. Surv. **3** (2006), 112–169.