

The number of $K_{m,m}$ -free graphs

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Abstract

A graph is called H -free if it contains no copy of H . Denote by $f_n(H)$ the number of (labeled) H -free graphs on n vertices. Erdős conjectured that $f_n(H) \leq 2^{(1+o(1))\text{ex}(n,H)}$. This was first shown to be true for cliques; then, Erdős, Frankl, and Rödl proved it for all graphs H with $\chi(H) \geq 3$. For most bipartite H , the question is still wide open, and even the correct order of magnitude of $\log_2 f_n(H)$ is not known. We prove that $f_n(K_{m,m}) \leq 2^{O(n^{2-1/m})}$ for every m , extending the result of Kleitman and Winston and answering a question of Erdős. This bound is asymptotically sharp for $m \in \{2, 3\}$, and possibly for all other values of m , for which the order of $\text{ex}(n, K_{m,m})$ is conjectured to be $\Theta(n^{2-1/m})$. Our method also yields a bound on the number of $K_{m,m}$ -free graphs with fixed order and size, extending the result of Füredi. Using this bound, we prove a relaxed version of a conjecture due to Haxell, Kohayakawa, and Łuczak and show that almost all $K_{3,3}$ -free graphs of order n have more than $1/20 \cdot \text{ex}(n, K_{3,3})$ edges.

1 Introduction

Let H be an arbitrary graph. We say that a graph G is H -free, if G does not contain H as a (not necessarily induced) subgraph. Denote by $\mathcal{F}_n(H)$ the family of labeled H -free graphs with vertex set $\{1, \dots, n\}$, and let $f_n(H) = |\mathcal{F}_n(H)|$. Let $\text{ex}(n, H)$ denote the Turán number for H , i.e., the maximum number of edges (size) that an H -free graph on n vertices may have. The celebrated theorem of Turán [24] states that

$$\text{ex}(n, K_m) = \left(1 - \frac{1}{m-1}\right) \frac{n^2}{2} + O(n),$$

and the unique K_m -free graph with $\text{ex}(n, K_m)$ edges is the complete $(m-1)$ -partite graph with all parts as equal as possible. Generalizing this, Erdős and Stone [12] showed that the

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chromatic number of H determines the order of magnitude of $\text{ex}(n, H)$ provided that H is not bipartite, i.e.,

$$\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \frac{n^2}{2} + o(n^2). \quad (1)$$

Since every subgraph of an H -free graph is also H -free, $\mathcal{F}_n(H)$ contains at least $2^{\text{ex}(n, H)}$ graphs. Erdős, Kleitman, and Rothschild [11] proved that this crude lower bound is in fact tight for complete graphs, obtaining an asymptotic formula for $\log_2 f_n(K_m)$, namely

$$\text{ex}(n, K_m) \leq \log_2 f_n(K_m) \leq (1 + o(1)) \text{ex}(n, K_m). \quad (2)$$

Later, Kolaitis, Prömel, and Rothschild [19] obtained an asymptotic formula for $f_n(K_m)$ by proving that almost all K_m -free graphs are m -colorable. Erdős asked if (2) is also true when one replaces K_m by an arbitrary graph H . The question was resolved in the affirmative by Erdős, Frankl, and Rödl [10] in the case $\chi(H) \geq 3$. For a brief survey and some related results see, e.g., [1, 2, 3, 22].

The picture is very different when one drops the $\chi(H) \geq 3$ assumption. For the remainder of this discussion, assume that H is a bipartite graph that contains a cycle. For most such H , the problem of determining $f_n(H)$ remains wide open. Moreover, for a general bipartite H , not much is even known about the order of magnitude of $\text{ex}(n, H)$. Unlike the non-bipartite case, the trivial lower and upper bounds for $f_n(H)$, i.e.,

$$2^{\text{ex}(n, H)} \leq f_n(H) \leq \sum_{s=0}^{\text{ex}(n, H)} \binom{\binom{n}{2}}{s}, \quad (3)$$

do not even determine the order of magnitude of $\log_2 f_n(H)$. The only nontrivial bipartite graphs for which an estimate stronger than (3) is known are cycles. Kleitman and Winston [17] proved that $\log_2 f_n(C_4) \leq 2.16384 \cdot \text{ex}(n, C_4)$, and later Kleitman and Wilson [16] proved $\log_2 f_n(C_6) = \Theta(\text{ex}(n, C_6))$. Similar results on the number of graphs with large (even) girth, i.e., graphs with no short (even) cycles, were proved in [16, 18]. Our main result extends that of Kleitman and Winston from $K_{2,2}$ to all complete bipartite graphs with equal class sizes.

Definition 1. The binary entropy function $H: [0, 1] \rightarrow \mathbb{R}$ is defined as

$$H(x) = -x \log_2 x - (1 - x) \log_2(1 - x).$$

For every positive integer m with $m \geq 2$, let

$$C_m = \sup_{x \in (0, 1)} (x^{-1+1/m} H(x))$$

and observe that $C_m \in [m\gamma, (m+2)\gamma]$, where $\gamma = (\log_2 e)/e \approx 0.531$; for details, see Appendix A.

Theorem 2. *The number of labeled $K_{m,m}$ -free graphs on n vertices satisfies*

$$\log_2 f_n(K_{m,m}) \leq (1 + o(1)) \frac{m(m-1)^{1/m}}{2m-1} C_m \cdot n^{2-1/m}.$$

This is known to be asymptotically sharp if $m \leq 3$. For other values of m , Erdős conjectured (see [8]) that $\text{ex}(n, K_{m,m}) = \Theta(n^{2-1/m})$, i.e., that the $O(n^{2-1/m})$ upper bound on $\text{ex}(n, K_{m,m})$ proved by Kővári, Sós, and Turán [20] is optimal. If this conjecture is true, Theorem 2 would be sharp for all m .

An algebraic construction of Brown [7] proves that $\text{ex}(p^3, K_{3,3}) \geq (p^5 - p^4)/2$ for all primes p such that $p \equiv 3 \pmod{4}$. Füredi [14] showed that this construction is asymptotically optimal, i.e., $\text{ex}(n, K_{3,3}) = (1/2 + o(1))n^{5/3}$. Together with Theorem 2, this implies the following.

Corollary 3. *The number of labeled $K_{3,3}$ -free graphs of order n is bounded as follows:*

$$(1/2 + o(1))n^{5/3} \leq \log_2 f_n(K_{3,3}) \leq (1.64618\dots)n^{5/3}.$$

Let $f_{n,s}(H)$ denote the number of H -free graphs with exactly s edges. Our methods give an upper bound on $f_{n,s}(K_{m,m})$, which extends the result in [13].

Theorem 4. *There is an n_0 depending only on m such that for all n and s with $n \geq n_0$ and $s \geq n^{2-m/(m^2-m+1)}(\log n)^2$, the number of labeled $K_{m,m}$ -free graphs of order n and size s satisfies*

$$f_{n,s}(K_{m,m}) \leq \left(3m \frac{n^{2m-1}}{s^m}\right)^s.$$

Let H be a fixed non-bipartite graph. Then for every positive ε , almost all H -free graphs of order n have at least $(\frac{1}{2} - \varepsilon)\text{ex}(n, H)$ and at most $(\frac{1}{2} + \varepsilon)\text{ex}(n, H)$ edges. It is not known if a similar concentration around a half occurs when H is bipartite. Still, one should expect that the number of edges in a “typical” H -free graph is at least bounded away from the extremal values, 0 and $\text{ex}(n, H)$. Balogh, Bollobás, and Simonovits [1] formalized this intuition by conjecturing that for every bipartite graph H that contains a cycle, there is a positive constant c such that almost all H -free graphs of order n have at least $c \cdot \text{ex}(n, H)$ and at most $(1 - c) \cdot \text{ex}(n, H)$ edges. So far this has been proved only for C_4 [4, 13] and partially (only the lower bound) for C_6 [13, 16]. An immediate corollary of Theorem 4 proves the lower bound in the case $H = K_{3,3}$.

Corollary 5. *Almost all $K_{3,3}$ -free graphs of order n have more than $1/20 \cdot \text{ex}(n, K_{3,3})$ edges.*

Given graphs G and H , let us define $\text{ex}(G, H) = \max\{e(K) : H \not\subseteq K \subseteq G\}$, where $e(K)$ denotes the size of K . As $\text{ex}(n, H) = \text{ex}(K_n, H)$, where K_n denotes the complete graph on n vertices, the above definition is a natural generalization of the Turán number. If we fix an H and any graph sequence $(G_n)_n$, a simple averaging argument implies that

$$\liminf_{n \rightarrow \infty} \frac{\text{ex}(G_n, H)}{e(G_n)} \geq 1 - \frac{1}{\chi(H) - 1}. \quad (4)$$

Haxell, Kohayakawa, and Łuczak [15] conjectured that if $e(G_n) \rightarrow \infty$, the number of copies $N_G(H)$ of H in G_n is larger than $e(G_n)$, and these copies are “uniformly” distributed in G_n , one has equality in (4) with \liminf replaced by \lim .

Definition 6. A graph H is balanced if

$$\max_{H' \subseteq H} \frac{e(H') - 1}{v(H') - 2} = \frac{e(H) - 1}{v(H) - 2}.$$

Conjecture 7 ([15]). *Let H be a fixed balanced graph and let $G(n, p)$ denote the usual binomial random graph of order n with edge probability p . Suppose that $\mathbb{E}[N_{G(n, p)}(H)] \geq \omega p n^2$ for some ω such that $\omega(n) \rightarrow \infty$ and $n \rightarrow \infty$. Then with probability tending to 1 as $n \rightarrow \infty$,*

$$\text{ex}(G(n, p), H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) e(G(n, p)).$$

We prove the above conjecture for $H = K_{m, m}$ under an additional assumption on the growth rate of ω .

Theorem 8. *Fix a real number $\gamma \in (0, 1]$. There is a constant C such that, if $p(n) \geq C n^{-m/(m^2-m+1)} (\log n)^2$, then with probability tending to 1,*

$$\text{ex}(G(n, p), K_{m, m}) < \gamma \cdot e(G(n, p)).$$

In particular, if $p n^{m/(m^2-m+1)} \gg (\log n)^2$, then asymptotically almost surely

$$\text{ex}(G(n, p), K_{m, m}) = o(e(G(n, p))). \quad (5)$$

Remark 9. Note that in order to prove Conjecture 7, one would have to show that (5) is still true if we only assume that $p n^{2/(m+1)} \rightarrow \infty$. Still, unless $p n^{1/m} \rightarrow \infty$, and hence $\text{ex}(n, K_{m, m}) = o(\mathbb{E}[e(G(n, p))])$, the result proved by Theorem 8 is non-trivial.

In particular, proving Conjecture 7 for $H = K_{3, 3}$ would require showing that (5) holds with high probability whenever $p \gg n^{-1/2}$. Note that the assumptions on p in the statement of Theorem 8 fall only a little short of that threshold.

Corollary 10. *If $p = p(n) \gg n^{-3/7} (\log n)^2$, then a.a.s.*

$$\text{ex}(G(n, p), K_{3, 3}) = o(e(G(n, p))).$$

As it will become clear in the proof of Theorem 8, our method allows us to prove (5) in a stronger form. Namely, the little o in (5) can be replaced with an explicit function of n and p . In the case of $K_{2, 2}$ (and all even cycles), this is done in [18], where sharp estimates are obtained. For details, we refer the reader to [18].

Since our work was completed, Conlon and Gowers [9] and, independently, Schacht [23] have proved Conjecture 7 in its full generality. In particular, their results imply that Theorem 8 is still true if we only assume that $p n^{2/(m+1)} \rightarrow \infty$, but only with (5) in the weaker little o form.

For a graph G , we denote its vertex and edge sets by $V(G)$ and $E(G)$, respectively. The number of edges in G is $e(G)$. For a vertex $v \in V(G)$, we denote the set of its neighbors by $N_G(v)$. The degree of v in G , denoted $d_G(v)$ or $d(v)$, is the size of its neighborhood, i.e., $d_G(v) = d(v) = |N_G(v)|$. The minimum degree of G , denoted $\delta(G)$ is defined as $\delta(G) = \min_{v \in V(G)} d_G(v)$. For a set A of vertices of G , by $N_G^*(A)$ we will denote the set of common neighbors of all vertices in A . Given an arbitrary set X , the power set of X , i.e., the family of all subsets of X is denoted by $\mathcal{P}(X)$. For a non-negative integer k , the subfamily of $\mathcal{P}(X)$ containing all k -element subsets is denoted by $\binom{X}{k}$. Finally, the term k -set abbreviates the phrase k -element set. Also, throughout the paper \log will always denote the natural logarithm.

The paper is organized as follows. In Section 2 we formulate and prove a general counting lemma, which is one of the basic building blocks of the proof of Theorem 2. The proof of Theorem 2 is given in Section 3. Theorems 4 and 8 are proved in Sections 4 and 5, respectively. Finally, Section 6 contains a few concluding remarks.

2 Counting complete bipartite subgraphs

One of the most important ingredients in our proof of Theorem 2 is Lemma 14 – an estimate on the number of copies of the complete bipartite graph $K_{m-1,m}$ in a larger graph with bounded minimum degree. Lemma 14 is a straightforward corollary of a more general statement that we prove below. The proof of Lemma 11 relies on a classic double counting argument in the spirit of Kővári, Sós, and Turán [20].

Lemma 11. *Fix two integers s and t with $1 \leq s \leq t$ and a positive real ε such that $\varepsilon(1+\varepsilon)^t \leq 1$. Let G be an n -vertex graph with minimum degree at least d , and A be any set of a vertices of G , where $a \geq (1+\varepsilon)(t-1)\binom{n}{s}/\binom{d}{s}$. Then the number of copies of $K_{s,t}$ in G with the larger partite set completely contained in A , denoted $N_{s,t}(A)$, satisfies*

$$N_{s,t}(A) \geq \beta \cdot a^t,$$

where

$$\beta = \beta(s, t, n, d, \varepsilon) = \frac{\varepsilon^t}{t!} \binom{d}{s}^t / \binom{n}{s}^{t-1}.$$

Proof. Let U be an s -set of vertices of G and assume that $U = \{u_1, \dots, u_s\}$. Let $c(U)$ be the number of common neighbors of u_1, \dots, u_s in the set A , i.e.,

$$c(U) = |N_G^*(U) \cap A|.$$

Clearly,

$$\sum_U c(U) = \sum_{w \in A} \binom{d_G(w)}{s} \geq a \binom{\delta(G)}{s} \geq a \binom{d}{s}.$$

The number of copies of $K_{s,t}$ in G with the larger partite set contained in A satisfies

$$N_{s,t}(A) = \sum_U \binom{c(U)}{t} \geq \binom{n}{s} \binom{a \binom{d}{s} / \binom{n}{s}}{t},$$

where the above inequality follows from convexity of the function B_t defined by

$$B_t(x) = \begin{cases} 0 & \text{if } x \leq t-1, \\ \binom{x}{t} & \text{if } x > t-1, \end{cases}$$

and the assumption that $a \binom{d}{s} / \binom{n}{s} > t-1$. It follows that

$$\begin{aligned} N_{s,t}(A) &\geq \binom{n}{s} \cdot \frac{1}{t!} \prod_{i=0}^{t-1} \left(\frac{a \binom{d}{s}}{\binom{n}{s}} - i \right) = \binom{n}{s} \cdot \left(\frac{a \binom{d}{s}}{\binom{n}{s}} \right)^t \cdot \frac{1}{t!} \prod_{i=0}^{t-1} \left(1 - i \frac{\binom{n}{s}}{a \binom{d}{s}} \right) \\ &\geq \frac{a^t}{t!} \binom{d}{s}^t / \binom{n}{s}^{t-1} \cdot \prod_{i=0}^{t-1} \left(1 - \frac{i}{(1+\varepsilon)(t-1)} \right) \\ &\geq \frac{a^t}{t!} \binom{d}{s}^t / \binom{n}{s}^{t-1} \cdot \left(1 - \frac{1}{1+\varepsilon} \right)^{t-1} \geq \frac{\varepsilon^t}{t!} \binom{d}{s}^t / \binom{n}{s}^{t-1} \cdot a^t, \end{aligned}$$

where the last inequality follows from the fact that $\varepsilon(1+\varepsilon)^{t-1} \leq 1$. □

3 Proof of Theorem 2

Let G be a $K_{m,m}$ -free graph on n vertices and let v be a vertex of minimum degree in G . Furthermore, let $G' = G - \{v\}$ and let $d = d(v) - 1$. Clearly G' is $K_{m,m}$ -free and $\delta(G') \geq \delta(G) - 1 = d$. Arguing along these lines one can find an ordering v_1, \dots, v_n of all the vertices of G , such that if we denote the subgraph induced on $\{v_1, \dots, v_i\}$ by G_i , then

$$\delta(G_i) \geq d_{G_{i+1}}(v_{i+1}) - 1 \text{ for all } i \in \{1, \dots, n-1\}.$$

In other words, every n -vertex $K_{m,m}$ -free graph can be obtained from a single vertex by successively adjoining a vertex of degree $d + 1$ to a graph with minimum degree at least d , for some d (which can obviously change as the graph grows). The general idea in the proof is to show that the number of ways in which one can obtain a $K_{m,m}$ -free graph of order $i + 1$ from some i -vertex $K_{m,m}$ -free graph in the above process of adjoining vertices of minimum degree is $2^{O(i^{1-1/m})}$, and therefore the number of labeled $K_{m,m}$ -free graphs on n vertices satisfies

$$f_n(K_{m,m}) \leq n! \cdot \prod_{i=1}^{n-1} 2^{O(i^{1-1/m})} = 2^{O(n^{2-1/m})}.$$

For the remainder of the proof, fix some positive integer d and an n -vertex $K_{m,m}$ -free graph G with minimum degree at least d . In the sequel, we will give an $2^{O(n^{1-1/m})}$ bound on $f(G; d, m)$ – the number of ways to adjoin to G a vertex v of degree $d + 1$, so that the resulting graph is still $K_{m,m}$ -free. Clearly,

$$f(G; d, m) \leq \binom{n}{d+1} \leq n^{d+1} = 2^{(d+1)\log_2 n}, \quad (6)$$

and so if $d + 1 \leq n^{1-1/m} / \log_2 n$, then $f(G; d, m) \leq 2^{n^{1-1/m}}$. Therefore, from now on we can assume that d is “large”, i.e., $d > n^{1-1/m} / (2 \log n)$.

Since $\delta(G) \geq d \gg n^{1-1/(m-1)}$, G contains numerous and evenly distributed copies of $K_{m-1,m}$. More precisely, larger partite sets of copies of $K_{m-1,m}$ in G constitute a big proportion of m -subsets of every large enough $A \subseteq V(G)$. Obviously we cannot make v adjacent to all vertices in any such m -set, since that would create a copy of $K_{m,m}$ in the graph $G \cup \{v\}$. Hence, it is clear that making v adjacent to some of the vertices in G will forbid many other adjacencies. In fact, we will prove that choosing as few as $O((\log n)^{m^2+1})$ neighbors for v restricts the remaining choices (for neighbors of v) to a set of rather small size. Now we will formalize these intuitions.

Definition 12. Let $B = \{w_1, \dots, w_m\}$ be a set of m vertices of G and let $N_G^*(B)$ be the set of their common neighbors, i.e., $N_G^*(B) = \bigcap_{w \in B} N_G(w)$. We say that B is *dangerous* if $|N_G^*(B)| \geq m - 1$, i.e., if G contains a copy of $K_{m-1,m}$, in which B is the larger partite set. For a set $A \subseteq V(G)$, we denote the number of its dangerous m -subsets by $D_m(A)$. In other words,

$$D_m(A) = \left| \{B \subseteq A : |B| = m \text{ and } B \text{ is dangerous}\} \right|.$$

Observation 13. *Let $B \subseteq V(G)$ be a dangerous m -set. Then the adjoined vertex v can be connected to at most $m - 1$ vertices in B .*

Lemma 14. Fix some positive ε satisfying $\varepsilon(1 + \varepsilon)^m \leq 1$ and let A be any set of a vertices in G , where $a \geq (1 + \varepsilon)(m - 1) \binom{n}{m-1} / \binom{d}{m-1}$. If $d \geq d_0$, where d_0 is a constant depending only on m , then the number of dangerous m -sets in A satisfies

$$D_m(A) \geq \alpha \cdot a^m,$$

where

$$\alpha = \alpha(m, d, \varepsilon) = \frac{\varepsilon^m}{(m!)^2} \cdot \frac{d^{m(m-1)}}{n^{(m-1)^2}}. \quad (7)$$

Proof. Since G is $K_{m,m}$ -free, every dangerous m -set is the larger partite set of exactly one copy of $K_{m-1,m}$ in G , and therefore, by Lemma 11,

$$D_m(A) = N_{m-1,m}(A) \geq \beta(m-1, m, n, d, \varepsilon) \cdot a^m,$$

where $\beta(m-1, m, n, d, \varepsilon)$ is as defined in the statement of Lemma 11. It suffices to prove that $\beta \geq \alpha$. First let us observe that

$$\lim_{d \rightarrow \infty} (1 - m/d)^{m-1} = 1,$$

and hence there is a d_0 (depending only on m) such that if $d \geq d_0$, then

$$m \cdot (d - m)^{m(m-1)} \geq d^{m(m-1)}.$$

It follows that if $d \geq d_0$, then

$$\begin{aligned} \beta &= \frac{\varepsilon^m}{m!} \binom{d}{m-1} / \binom{n}{m-1}^{m-1} \geq \frac{\varepsilon^m}{m!} \cdot \left(\frac{(d-m)^{m-1}}{(m-1)!} \right)^m \cdot \left(\frac{(m-1)!}{n^{m-1}} \right)^{m-1} \\ &\geq \frac{\varepsilon^m}{m!} \cdot \frac{d^{m(m-1)}}{m(m-1)!n^{(m-1)^2}} = \alpha. \end{aligned}$$

□

Fix some function ε such that $\lim_{n \rightarrow \infty} \varepsilon(n) = 0$ and $\varepsilon(n) \gg (\log n)^{-1}$, and let $t_0 = (\log n)/\alpha$. The key step in the proof is to show that there is a map

$$\psi : \binom{V(G)}{(m-1)t_0} \rightarrow \mathcal{P}(V(G))$$

satisfying $|\psi(X)| \leq (1 + 2\varepsilon)(m-1)(n/d)^{m-1}$ such that the following holds.

Claim 15. Let G' be a $K_{m,m}$ -free graph obtained from G by adjoining a vertex v of degree $d+1$. Then there is an $X \subseteq N_{G'}(v)$ of size $(m-1)t_0$ such that $N_{G'}(v) \subseteq \psi(X)$.

Before we start proving Claim 15, let us first show how it implies an upper bound on the number of ways to connect a vertex v of degree $d+1$ to our graph G .

Corollary 16. With our assumptions on G , d , and ε ,

$$\log_2 f(G; d, m) \leq ((1 + 2\varepsilon)(m-1))^{1/m} C_m \cdot n^{1-1/m} + o(n^{1-1/m}), \quad (8)$$

where C_m is as defined in Definition 1.

Proof. By Claim 15, for every G' counted by $f(G; d, m)$, we can find some $X \subseteq N_{G'}(v) \subseteq V(G)$ of size $(m-1)t_0$, such that $N_{G'}(v) \subseteq \psi(X)$. Since for a fixed X , $\psi(X)$ depends only on G and not on G' , we have

$$f(G; d, m) \leq \sum_X \binom{|\psi(X)|}{d+1} \leq \binom{n}{(m-1)t_0} \cdot \max_X \binom{|\psi(X)|}{d+1}. \quad (9)$$

Since we assumed that $d > n^{1-1/m}/(2 \log n)$, we have

$$t_0 = \frac{\log n}{\alpha} = \frac{\log n \cdot (m!)^2 n^{(m-1)^2}}{\varepsilon^m d^{m(m-1)}} \leq (m!)^2 \cdot (2 \log n)^{m^2+1}. \quad (10)$$

Using (10), we can bound the first term in (9) as follows:

$$\binom{n}{(m-1)t_0} \leq n^{(m-1)t_0} \leq 2^{(\log_2 n) \cdot (m-1)(m!)^2 (2 \log n)^{m^2+1}} \ll 2^{n^{1-1/m}}. \quad (11)$$

Bounding the second term in (9) requires a little more work. First we note that

$$\binom{|\psi(X)|}{d+1} \leq n \cdot \binom{|\psi(X)|}{d} \leq n \cdot \binom{(1+2\varepsilon)(m-1)(n/d)^{m-1}}{d},$$

and then, using the well-known estimate relating binomial coefficients with the binary entropy function H (see, e.g., [21, Lemma 9]),

$$\frac{1}{n+1} \cdot 2^{nH(k/n)} \leq \binom{n}{k} \leq 2^{nH(k/n)},$$

we further estimate

$$\log_2 \binom{|\psi(X)|}{d+1} \leq \log_2 n + (1+2\varepsilon)(m-1)(n/d)^{m-1} \cdot H \left(\frac{d^m}{(1+2\varepsilon)(m-1)n^{m-1}} \right). \quad (12)$$

Let $x = d^m / ((1+2\varepsilon)(m-1)n^{m-1})$ and note that $x \in (0, 1)$. Rewriting (12) yields

$$\log_2 \binom{|\psi(X)|}{d+1} \leq \log_2 n + ((1+2\varepsilon)(m-1))^{1/m} \cdot \frac{H(x)}{x^{1-1/m}} \cdot n^{1-1/m}. \quad (13)$$

Recall that $C_m = \sup_{x \in (0,1)} (x^{-1+1/m} H(x))$. Clearly, (11) and (13) imply (8). \square

In order to complete the proof, we show the existence of a map ψ satisfying Claim 15. Recall that d is an integer and G is a fixed $K_{m,m}$ -free graph of order n with minimum degree at least d . We are going to describe an algorithm \mathcal{A} that works as follows:

- **INPUT:** A set $N \subseteq V(G)$ of size $d+1$, such that joining a new vertex v to all vertices in N yields a $K_{m,m}$ -free graph of order $n+1$.
- **OUTPUT:** A pair of sets (A, X) , such that A contains $N - X$ and has size at most $(1+\varepsilon)(m-1) \binom{n}{m-1} / \binom{d}{m-1}$, and X is a subset of N with exactly $(m-1)t_0$ elements.

Most importantly, A will depend solely on X , i.e., if for some two inputs our algorithm \mathcal{A} outputs the same set X , it also produces the same A . Hence putting $\psi(X) = A \cup X$ for every output (A, X) of \mathcal{A} uniquely defines an appropriate map ψ , as by the assumption $d > n^{1-1/m}/(2 \log n)$ and (10),

$$\begin{aligned} |\psi(X)| &\leq (m-1)t_0 + (1+\varepsilon)(m-1) \binom{n}{m-1} / \binom{d}{m-1} \\ &\leq (m-1) \cdot (m!)^2 (2 \log n)^{m^2+1} + (1+\varepsilon)(m-1)n^{m-1}/(d-m)^{m-1} \\ &\leq (1+2\varepsilon)(m-1)(n/d)^{m-1} \end{aligned}$$

whenever $n \geq n_0(m)$.

We now describe the algorithm \mathcal{A} :

1. Set $A_0 = V(G)$ and $X_0 = \emptyset$.
2. For $t = 0, \dots, t_0 - 1$, do the following:
 - (a) Set $A_t^0 = A_t$ and $S_t^0 = \emptyset$.
 - (b) For $i = 0, \dots, m-2$, do the following:
 - i. List all the vertices in A_t^i as $w_{t,i}^1, \dots, w_{t,i}^{|A_t^i|}$ in a unique way so that for each j , the vertex $w_{t,i}^{j+1}$ is the vertex with the minimum label among all vertices in $A_t^i - \{w_{t,i}^1, \dots, w_{t,i}^j\}$ belonging to the maximum number of dangerous sets B that contain S_t^i and the remaining $m-i$ vertices of B all come from the set $A_t^i - \{w_{t,i}^1, \dots, w_{t,i}^j\}$.
 - ii. Let $j(t, i)$ be the smallest j such that $w_{t,i}^j \in N$.
 - iii. Set $A_t^{i+1} = A_t^i - \{w_{t,i}^1, \dots, w_{t,i}^{j(t,i)}\}$ and $S_t^{i+1} = S_t^i \cup \{w_{t,i}^{j(t,i)}\}$.
 - (c) Let F_t be the set of all vertices $w \in A_t^{m-1}$ such that $\{w\} \cup S_t^{m-1}$ is a dangerous set. Set $A_{t+1} = A_t^{m-1} - F_t$ and $X_{t+1} = X_t \cup S_t^{m-1}$.
3. Set $A = A_{t_0}$ and $X = X_{t_0}$. Return (A, X) .

To make the analysis of \mathcal{A} a somewhat clearer, let us have one more definition. For fixed $t \in \{0, \dots, t_0 - 1\}$ and $i \in \{0, \dots, m-1\}$, let us say that an $(m-i)$ -set $C \subseteq A_t^i$ is (t, i) -dangerous if the m -set $C \cup \{w_{t,i}^{j(t,0)}, \dots, w_{t,i}^{j(t,i-1)}\}$ is dangerous. For a subset $A' \subseteq A_t^i$, define

$$D_t^i(A') = |\{C \subseteq A' : |C| = m-i \text{ and } C \text{ is } (t, i)\text{-dangerous}\}|.$$

Suppose we run the algorithm \mathcal{A} on some input N . An easy induction on t and i proves the following statement.

Claim 17. *If $0 \leq t < t_0$ and $0 \leq i < m$, then the following assertions are satisfied:*

- $S_t^i \subseteq N$,
- $N - X_t - S_t^i \subseteq A_t^i$,
- F_t is disjoint from N , and

- $|X_t| = (m - 1)t$.

It follows that $X \subseteq N$, $|X| = (m - 1)t_0$, and $N - X \subseteq A$.

Since, given a fixed graph G , the sequence $(j(t, i))_{t, i}$ uniquely determines both X and A , it should be clear that \mathcal{A} cannot output two pairs (X, A) and (X, A') with $A \neq A'$. As we have already mentioned, this allows us to define $\psi(X) = A \cup X$, where (X, A) ranges over all possible outputs of \mathcal{A} . In order to complete the proof of Claim 15, it remains to prove the following claim.

Claim 18. *Suppose we run the algorithm \mathcal{A} on some input N . Then*

$$|A| + |X| \leq (1 + 2\varepsilon)(m - 1)(n/d)^{m-1}. \quad (14)$$

The key step in proving Claim 18 is the following estimate.

Lemma 19. *If $0 \leq t < t_0$ and $0 \leq i < m$, then the following holds. Suppose that $D_t^i(A_t^i) \geq \gamma|A_t^i|^{m-i}$ for some $\gamma \in (0, 1]$. Then*

$$|F_t| + \sum_{k=i}^{m-2} j(t, k) \geq \gamma|A_t^i|. \quad (15)$$

Proof. For a fixed t , we prove the Lemma by reverse induction on i . Since $|F_t| = D_t^{m-1}(A_t^{m-1})$, inequality (15) is vacuously true if $i = m - 1$. Suppose that $i < m - 1$ and (15) holds for $i + 1$. For the sake of brevity, let $a = |A_t^i|$. Each of $w_{t, i}^1, \dots, w_{t, i}^{j(t, i)-1}$ belongs to at most a^{m-i-1} $(m - i)$ -subsets of A_t^i , and hence

$$\begin{aligned} D_t^i(A_t^i - \{w_{t, i}^1, \dots, w_{t, i}^{j(t, i)-1}\}) &\geq D_t^i(A_t^i) - (j(t, i) - 1) \cdot a^{m-i-1} \\ &\geq \gamma a^{m-i} - (j(t, i) - 1) \cdot a^{m-i-1}. \end{aligned} \quad (16)$$

If $j(t, i) \geq \gamma a$, then (15) holds, so we may suppose that the reverse inequality is true, and therefore the rightmost term in (16) is positive. Since we have selected $w_{t, i}^{j(t, i)}$ to maximize $D_t^{i+1}(A_t^i - \{w_{t, i}^1, \dots, w_{t, i}^{j(t, i)-1}, w\})$ over all $w \in A_t^i - \{w_{t, i}^1, \dots, w_{t, i}^{j(t, i)-1}\}$,

$$\begin{aligned} D_t^{i+1}(A_t^{i+1}) &\geq \frac{m - i}{a - j(t, i) + 1} \cdot D_t^i(A_t^i - \{w_{t, i}^1, \dots, w_{t, i}^{j(t, i)-1}\}) \\ &\geq \frac{m - i}{a - j(t, i) + 1} \cdot (\gamma a^{m-i} - (j(t, i) - 1) \cdot a^{m-i-1}) \\ &\geq \frac{\gamma a - j(t, i) + 1}{a - j(t, i) + 1} \cdot a^{m-i-1} \geq \frac{\gamma a - j(t, i)}{a - j(t, i)} \cdot |A_t^{i+1}|^{m-(i+1)}, \end{aligned} \quad (17)$$

where the last inequality holds since $|A_t^{i+1}| \leq |A_t^i| = a$ and $\gamma \leq 1$. Hence, by the inductive assumption, with $\gamma' = (\gamma a - j(t, i))/(a - j(t, i))$,

$$|F_t| + \sum_{k=i+1}^{m-2} j(t, k) \geq \frac{\gamma a - j(t, i)}{a - j(t, i)} \cdot |A_t^{i+1}| = \gamma a - j(t, i).$$

□

Recall the definition of α from Lemma 14. The following statement is a straightforward corollary of Lemma 19.

Corollary 20. *If $|A_t| \geq (1 + \varepsilon)(m - 1) \binom{n}{m-1} / \binom{d}{m-1}$, then $|A_{t+1}| \leq (1 - \alpha)|A_t|$.*

Proof. Recall that $A_{t+1} = A_t^{m-1} - F_t$ and hence

$$|A_{t+1}| = |A_t^0| - \sum_{i=0}^{m-2} (|A_t^i| - |A_t^{i+1}|) - |F_t| = |A_t| - \sum_{i=0}^{m-2} j(t, i) - |F_t|. \quad (18)$$

The assumed lower bound on $|A_t|$ guarantees that Lemma 14 can be applied and hence

$$D_t^0(A_t^0) = D_m(A_t) \geq \alpha|A_t|^m.$$

By (18) and Lemma 19, where we set $\gamma = \alpha$ and $i = 0$, we get

$$|A_{t+1}| \leq |A_t| - \alpha|A_t^0| = (1 - \alpha)|A_t|.$$

□

Proof of Claim 18. Note that by Corollary 20,

$$|A_{t_0}| \leq \max \left\{ (1 - \alpha)^{t_0} |A_0|, (1 + \varepsilon)(m - 1) \binom{n}{m-1} / \binom{d}{m-1} \right\}, \quad (19)$$

and recall that $t_0 = (\log n)/\alpha$. Therefore

$$(1 - \alpha)^{t_0} |A_0| \leq \exp(-\alpha t_0) \cdot |V(G)| = \exp(-\log n) \cdot n = 1.$$

This implies that the second term in the maximum in (19) is larger than the first, and so

$$\begin{aligned} |A_{t_0}| &\leq (1 + \varepsilon)(m - 1) \binom{n}{m-1} / \binom{d}{m-1} \leq (1 + \varepsilon)(m - 1) \frac{n^{m-1}}{(d - m)^{m-1}} \\ &\leq (1 + 2\varepsilon)(m - 1)(n/d)^{m-1}, \end{aligned}$$

provided that $n \geq n_0(m)$; recall that $d > n^{1-1/m}/(2 \log n)$. □

To complete the proof of Theorem 2, observe that, since G is $K_{m,m}$ -free, $\delta(G) \leq c_m n^{1-1/m}$ for some absolute constant c_m . By (6) and Corollary 16, the number of ways to adjoin to G a vertex of degree $d + 1 \leq \delta(G) + 1$, so that the resulting graph is $K_{m,m}$ -free, is

$$\begin{aligned} f(G; m) &= \sum_{d \leq \delta(G)} f(G; d, m) \leq \sum_{d+1 \leq \frac{n^{1-1/m}}{\log_2 n}} f(G; d, m) + \sum_{d > \frac{n^{1-1/m}}{2 \log n}} f(G; d, m) \\ &\leq \frac{n^{1-1/m}}{\log_2 n} \cdot 2^{n^{1-1/m}} + c_m n^{1-1/m} \cdot 2^{(1+o(1))(m-1)^{1/m} C_m \cdot n^{1-1/m}} \\ &\leq 2^{(1+o(1))(m-1)^{1/m} C_m \cdot n^{1-1/m}}. \end{aligned}$$

Hence,

$$\begin{aligned} \log_2 f_n(K_{m,m}) &\leq \log_2(n!) + (1 + o(1))(m - 1)^{1/m} C_m \cdot \sum_{k=1}^n k^{1-1/m} \\ &\leq (1 + o(1)) \cdot \frac{m(m - 1)^{1/m}}{2m - 1} C_m \cdot n^{2-1/m}. \end{aligned}$$

□

4 Proof of Theorem 4

For the sake of brevity, let $\mu = m/(m^2 - m + 1)$. As it was remarked at the beginning of the proof of Theorem 2, every n -vertex graph G can be constructed from an isolated vertex v_1 by successively connecting a vertex v_{i+1} to some d_i vertices in $G[\{v_1, \dots, v_i\}]$ in such a way that

$$d_i = \delta(G[\{v_1, \dots, v_{i+1}\}]) \leq \delta(G[\{v_1, \dots, v_i\}]) + 1$$

for all $i \in \{1, \dots, n-1\}$. Call the sequence $(d_i)_{i=1}^{n-1}$ a *degeneracy sequence* of G and note that $e(G) = \sum_{i=1}^{n-1} d_i$.

Recall from the proof of Theorem 2, that $f(G; d, m)$ is the number of ways one can adjoin to a $K_{m,m}$ -free graph G with $\delta(G) \geq d$ a new vertex of degree $d+1$, so that the graph remains $K_{m,m}$ -free. Clearly, all subgraphs of a $K_{m,m}$ -free graph are also $K_{m,m}$ -free, and hence, if we let

$$f(i; d, m) = \sup \{f(G; d, m) : G \text{ is a } K_{m,m}\text{-free graph of order } i \text{ with } \delta(G) \geq d\},$$

then

$$f_{n,s}(K_{m,m}) \leq n! \cdot \sum_{(d_i)} \prod_{i=1}^{n-1} f(i; d_i - 1, m) \quad (20)$$

where the above sum is taken over all degeneracy sequences $(d_i)_{i=1}^{n-1}$ with sum s .

If $d \leq n^{1-\mu}(\log n)^{2/3}$ and $n \geq n_0$, then we give a rather crude bound:

$$f(i; d, m) \leq \binom{i}{d+1} \leq n \binom{n}{d} \leq n \left(\frac{en}{d}\right)^d \leq \exp(n^{1-\mu}(\log n)^{5/3}). \quad (21)$$

Suppose now that $d > n^{1-\mu}(\log n)^{2/3}$, and let $\alpha(m, d, 1/(2m-2))$ be as in Lemma 14. Since

$$t_0 = \frac{\log n}{\alpha} = \frac{\log n \cdot (m!)^2 n^{(m-1)^2}}{(2m-2)^{-m} d^{m(m-1)}} \leq m^{4m} \cdot n^{1-\mu} (\log n)^{1-\frac{2}{3}m(m-1)} \ll n^{1-\mu} \leq d,$$

Claim 15 can be applied, and reasoning along the lines of Corollary 16, see (9), we show that for large enough n ,

$$\begin{aligned} f(i; d, m) &\leq i^{(m-1)t_0} \cdot \binom{m(i/d)^{m-1}}{d} \leq n^{n^{1-\mu}} \cdot \left(\frac{emn^{m-1}}{d^m}\right)^d \\ &\leq \exp\left(n^{1-\mu} \log n + d \log \frac{emn^{m-1}}{d^m}\right). \end{aligned} \quad (22)$$

Finally, fix some degeneracy sequence $(d_i)_{i=1}^{n-1}$ with sum s , let $I = \{i : d_i > n^{1-\mu}(\log n)^{2/3}\}$, and let $s' = \sum_{i \in I} (d_i - 1)$. Combining inequalities (21) and (22) yields

$$\prod_{i=1}^{n-1} f(i; d_i - 1, m) \leq \exp\left(n^{2-\mu}(\log n)^{5/3} + \sum_{i \in I} (d_i - 1) \log \frac{emn^{m-1}}{(d_i - 1)^m}\right). \quad (23)$$

The function $[0, \infty) \ni x \mapsto x \log x \in \mathbb{R}$ is convex, and so Jensen's inequality gives

$$\sum_{i \in I} (d_i - 1) \log(d_i - 1) \geq |I| \cdot (s'/|I|) \log(s'/|I|) \geq s' \cdot \log(s'/n).$$

This yields

$$\sum_{i \in I} (d_i - 1) \log \frac{emn^{m-1}}{(d_i - 1)^m} \leq s' \log(emn^{m-1}) - ms' \log(s'/n) = s' \log \frac{emn^{2m-1}}{s'^m}. \quad (24)$$

Since $\frac{d}{dx}(x \log(y/x)) = \log(y/x) - 1$, $s - s' = n - 1 + \sum_{i \notin I} (d_i - 1) \leq n + n^{2-\mu}(\log n)^{2/3}$, and $s \gg n^{2-\mu}(\log n)^{5/3}$, we get the estimate

$$\left| s' \log \frac{emn^{2m-1}}{s'^m} - s \log \frac{emn^{2m-1}}{s^m} \right| = O((s - s') \log n) = o(s),$$

which combined with (23) and (24) gives

$$\prod_{i=1}^{n-1} f(i; d_i, m) \leq \exp \left(n^{2-\mu}(\log n)^{5/3} + s \log \frac{emn^{2m-1}}{s^m} + o(s) \right). \quad (25)$$

Since

$$s \gg n^{2-\mu}(\log n)^{5/3}, \quad e < 3, \quad s \leq \text{ex}(n, K_{m,m}) \leq n^{2-1/m},$$

and there are at most $n!$ degeneracy sequences, combining (20) with (25) yields

$$f_{n,s}(K_{m,m}) \leq \left(\frac{3mn^{2m-1}}{s^m} \right)^s,$$

whenever n is large enough. □

5 Proof of Theorem 8

The proof is a rather straightforward application of Theorem 4 and the first moment method. We let $C = C(\gamma) = 3/\gamma$ and $s = (\gamma/3)pn^2 \geq n^{2-m/(m^2-m+1)} \log^2 n$. Recall that for any fixed positive ε , the random graph $G(n, p)$ asymptotically almost surely has at least $(1/2 - \varepsilon)pn^2$ edges. Hence,

$$s < \gamma \cdot e(G(n, p)) \quad (26)$$

holds asymptotically almost surely. Conditioning on (26), the event

$$\text{ex}(G(n, p), K_{m,m}) \geq \gamma \cdot e(G(n, p)) \quad (27)$$

implies that $G(n, p)$ contains a $K_{m,m}$ -free subgraph with s edges. But the expected number of copies of such a graph in $G(n, p)$ is

$$\begin{aligned} f_{n,s}(K_{m,m})p^s &\leq \left(3m \frac{n^{2m-1}}{s^m} p \right)^s = \left(\frac{3^{m+1}m}{\gamma^m} \cdot \frac{p}{np^m} \right)^s \\ &\leq \left(\frac{3^{m+1}m}{\gamma^m} \cdot \frac{1}{n^{1/(m^2-m+1)}} \right)^s = o(1). \end{aligned}$$

We conclude that

$$P(\text{ex}(G(n, p), K_{m,m}) \geq \gamma \cdot e(G(n, p))) = o(1). \quad \square$$

6 Concluding remarks

Unfortunately, the technique used in the proof of Theorem 2 fails to yield an $2^{O(n^{2-1/s})}$ bound on the number of $K_{s,t}$ -free graphs when we assume that $2 \leq s < t$. If we were to directly transfer the ideas from the proof of Theorem 2 to this new setting, we would similarly try to bound the number of ways to adjoin a vertex of degree $d + 1$ to an n -vertex $K_{s,t}$ -free graph G with minimum degree $\delta(G) \geq d$, so that the new graph is still $K_{s,t}$ -free. The case when $d + 1 \leq n^{1-1/s}/(\log_2 n)$ can be dealt with easily; the main problem is to give an $2^{O(n^{1-1/s})}$ bound in the case $d \geq n^{1-1/s}/(2 \log n)$. One can again introduce the notion of a dangerous set, which now is the larger partite set in a copy of $K_{s-1,t}$ in G (the other possibility, i.e., looking for copies of $K_{s,t-1}$, can be ruled out quite easily – under our assumptions on d , the double counting argument used in Lemma 11 cannot even prove existence of a single copy of $K_{s,t-1}$ in G ; this should not come at a surprise, as we know that $\text{ex}(n, K_{s-1,t}) \ll n^{2-1/s}$ and most likely $\text{ex}(n, K_{s,t-1}) = \Theta(n^{2-1/s})$). Using Lemma 11, we prove that every set of a vertices of G contains at least $\alpha \cdot a^t \approx d^{(s-1)t}/n^{(s-1)(t-1)} \cdot a^t$ dangerous sets, provided that $a \geq t \binom{n}{s-1} / \binom{d}{s-1}$. Then with the help of an algorithm very similar to \mathcal{A} , one could try to reprove versions of Claim 15 and Corollary 16, which would imply the desired upper bound. Here lies the difficulty. The set $X \subseteq N_{G'}(v)$ would have to be of size about $(t-1) \cdot (\log n)/\alpha$, and one can see that this is optimal, since one iteration of \mathcal{A} adds $(t-1)$ elements to X , shrinks the set A by multiplicative factor $1 - \alpha$, and in the end we clearly want $|A| = o(n)$. A simple computation shows that now $|X| \gg (t-1)d^{t-s} \geq (t-1)d \geq |N_{G'}(v)|$, which is impossible.

Since our work was completed, we have managed to overcome these difficulties and generalize Theorems 2 and 4 to all complete bipartite graphs. In [5], we construct a new, much more sophisticated algorithm for encoding neighborhoods of vertices in $K_{s,t}$ -free graphs with large minimum degree. One of the main new ideas is that this algorithm encodes a super-constant number of neighbors in a single iteration, which allows to shrink the set A by a multiplicative factor significantly smaller than $1 - \alpha$. For details, we refer the reader to [5].

Let H be a bipartite graph obtained from the complete bipartite graph $K_{m,m}$ by growing a tree out of each vertex so that all the trees are pairwise vertex-disjoint. Since in a graph G with large minimum degree, one can find a copy of any fixed-size tree T , even requiring of T to be rooted at a specified vertex and of the vertex set of T to avoid a specified small subset of the set of vertices of G , it is straightforward to reprove Lemma 11 with $K_{m,m}$ replaced with H . Consequently, one can reprove Lemma 14 with appropriately defined dangerous sets. Following the proof of Theorem 2 from there on gives

$$\log_2 f_n(H) \leq (1 + o(1)) \frac{m(m-1)^{1/m}}{2m-1} C_m \cdot n^{2-1/m}.$$

Finally, in [1] it is said that any bound on the number of $K_{3,3}$ -free graphs of small size that is similar to the one we obtained as Corollary 5 seems to be the only missing ingredient needed to prove Conjecture 31 from [1] with $a_0 = a_1 = 3$. The conjecture says that given integers a_0, \dots, a_p with $a_0 \leq \dots \leq a_p$, the vertex set of almost every $K(a_0, \dots, a_p)$ -free graph G of order n admits a partition (U_1, \dots, U_p) where $G[U_1]$ is $K(a_0, a_1)$ -free, and if $i > 1$, then the graph $G[U_i]$ has maximum degree less than a_1 .

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A Estimating the constant C_m

Let m be a positive integer with $m \geq 2$ and let $f_m(x) = x^{-1+1/m}H(x)$ for all $x \in (0, 1)$, where H is the binary entropy function. Recall from Definition 1 that we defined $C_m = \sup_{x \in (0,1)} f_m(x)$. Observe that the function f_m is non-negative, continuous on $(0, 1)$, and

$$\lim_{x \rightarrow 0} f_m(x) = \lim_{x \rightarrow 1} f_m(x) = 0.$$

Hence, there exists an $x_m \in (0, 1)$ such that $f'_m(x_m) = 0$ and $C_m = \sup_{x \in (0,1)} f_m(x) = f_m(x_m)$. Solving $f'_m(x_m) = 0$ yields

$$C_m = f_m(x_m) = \frac{m}{m-1} \cdot x_m^{1/m} H'(x_m) = \frac{m}{m-1} \cdot x_m^{1/m} \log_2 \frac{1-x_m}{x_m}.$$

It follows that

$$\begin{aligned} C_m &\leq \sup_{x \in (0,1)} \left(\frac{m}{m-1} \cdot x^{1/m} \log_2 \frac{1-x}{x} \right) \leq \frac{m}{m-1} \cdot \sup_{x \in (0,1)} \left(x^{1/m} \log_2 \frac{1}{x} \right) \\ &= \frac{m}{m-1} \cdot \sup_{z \in (0,1)} \left(z \log_2 \frac{1}{z^m} \right) = \frac{m^2}{m-1} \cdot \sup_{z \in (0,1)} \left(z \log_2 \frac{1}{z} \right) = \frac{m^2}{m-1} \cdot \frac{\log_2 e}{e}. \end{aligned} \tag{28}$$

On the other hand, since

$$H(x) = x \log_2 \frac{1}{x} + (1-x) \log_2 \frac{1}{1-x} \geq x \log_2 \frac{1}{x},$$

then

$$C_m \geq \sup_{x \in (0,1)} \left(x^{1/m} \log_2 \frac{1}{x} \right) = m \cdot \frac{\log_2 e}{e}. \quad (29)$$

Putting (28) and (29) together yields the desired bounds on C_m :

$$m \cdot \frac{\log_2 e}{e} \leq C_m \leq (m+2) \cdot \frac{\log_2 e}{e}.$$