Weight spectral sequence and non-abelian Lubin-Tate theory

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This is a continuation of my talk in the same workshop two years ago about my joint work with R. Taylor on the compatibilty of local and global Langlands correspondences. In order to compute the local monodoromy of the Galois representation attached to conjugate self-dual cuspidal automorphic representation of GL_n over CM field, we studied the semistable reduction of certain unitary Shimura varieties with Iwahori level structure ([TY]). There, the weight spectral sequence ([RZ], [S]) corresponding to the cuspidal automorphic reprsentation was shown to degenerate at E_1-terms by somewhat mysterious vanishing of dimensions expressed as binomial coefficients. This was done by forgetting the action of local Hecke algebra (affine Iwahori Hecke algebra), because we did not need it to deduce the degeneration of the weight spectral sequence. In this talk we determine this action completely, using a general intersection-theoretic formula to compute the action of algebraic correspondences on weight spectral sequences. This leads to an observation that the computation was entirely of local nature – it suggests that the same method will compute purely locally the Hecke action on the cohomology of Lubin-Tate spaces with Iwahori level structure, partially recovering the results of Boyer obtained by global methods ([B]).

First we explain our formula on the action of algebraic correspondences on weight spectral sequences. Let K be a complete discrete valuation field with a finite residue field k and the ring of integers $\mathcal{O}_K$. Let X be a proper strictly semistable scheme of relative dimension $n-1$ over $\mathcal{O}_K$. Then its special fiber $Y := X \times_{\mathcal{O}_K} k$ is written as $Y = \bigcup_{i\in\Delta} Y_i$ with $\Delta := \{1, ..., t\}$ and Y_i proper smooth over k, where Y_i and Y_j intersect transversally for $i \neq j$. Let $Y_I := \bigcap_{i\in I} Y_i$ for $I \subset \Delta$, which is proper smooth over k of dimension $n - |I|$ if not empty, and $Y^{(m)} := \coprod_{|I|=m} Y_I$ for $1 \leq m \leq n$. For a prime $\ell \neq \mathrm{char} k$, the weight spectral sequence reads

$$E_1^{i,j} := \bigoplus_{s\geq\max(0,-i)} H^{j-2s}\big(Y^{(i+2s+1)} \times_k \overline{k},\ \overline{\mathbb{Q}}_\ell(-s)\big) \Longrightarrow H^{i+j}(X \times_K \overline{K},\ \overline{\mathbb{Q}}_\ell).$$

Now let Γ be an algebraic correspondence on X (namely an n-dimensional cycle on $X \times_{\mathcal{O}_K} X$) such that two projection maps $\Gamma \to X$ are both finite. We are interested in the action $[\Gamma_K]^* := \mathrm{pr}_{1*} \circ ([\Gamma_K]\cup) \circ \mathrm{pr}_2^*$ of $\Gamma_K := \Gamma \times_{\mathcal{O}_K} K$ on $H^*(X \times_K \overline{K}, \overline{\mathbb{Q}}_\ell)$. Let $Y_{I,J} := Y_I \times_k Y_J$ for $I, J \subset \Delta$, and write $Y_{i,j} := Y_{\{i\},\{j\}}$. Let $(X \times_{\mathcal{O}_K} X)_{\mathrm{sm}}$ be the smooth locus of the morphism $X \times_{\mathcal{O}_K} X \longrightarrow \mathrm{Spec}\mathcal{O}_K$, and let $Y^0_{i,j} := Y_{i,j} \cap (X \times_{\mathcal{O}_K} X)_{\mathrm{sm}}$. Then $Y^0_{i,j}$ is a Cartier divisor of $(X \times_{\mathcal{O}_K} X)_{\mathrm{sm}}$.

Theorem A. *There is a unique collection $\{\Gamma_{I,J}\}$ of cycles $\Gamma_{I,J}$ on $Y_{I,J}$ for all pairs (I, J) with $|I| = |J|$, satisfying the following two conditions.*

(i) *$\Gamma_{i,j}$ is the closure of the cycle $\Gamma^0_{i,j} := Y^0_{i,j} \cdot \Gamma|_{(X\times_{\mathcal{O}_K} X)_{\mathrm{sm}}}$ in $Y_{i,j}$.*

(ii) *When* $|I| = |J| + 1 = m$ *and* $I = \{i_1, ..., i_m\}$, $J = \{j_1, \dots, j_{m-1}\}$ *are in increasing order, there is an equality:*

$$\sum_{h=1}^{m} (-1)^h \, Y_{I,J} \cdot \Gamma_{I \setminus \{i_h\}, J} = \sum_{j \in \Delta \setminus J} (-1)^{h(j)} \, \Gamma_{I, J \cup \{j\}}$$

of $(n-m)$*-dimensional cycles on* $Y_{I,J}$*, where* $1 \le h(j) \le m$ *is determined by* $j_{h(j)-1} < j < j_{h(j)}$ *(set* $j_m := \infty$*).*

Then setting $\Gamma^{(m)} := \coprod_{|I|=|J|=m} \Gamma_{I,J}$ *as an* $(n-m)$*-dimensional cycle on* $Y^{(m)} \times_k Y^{(m)}$ *for* $1 \le m \le n$*, the action* $\oplus[\Gamma^{(i+2s+1)}]^*$ *on* $E_1^{i,j}$ *is compatible with the action* $[\Gamma_K]^*$ *on* $H^{i+j}(X \times_K \overline{K}, \overline{\mathbb{Q}}_\ell)$*.*

For the proof of this theorem, we build on the construction of [S], except that we eliminate the semistable resolution of $X \times_{\mathcal{O}_K} X$ from the description of the cycles $\Gamma_{I,J}$, in order to apply the formula to the Shimura varieties where the cycles have concrete moduli interpretation. For this we also need the cycles, not only cycle classes.

Now we introduce a class of Shimura varieties containg those studied in [HT]. Let F be a CM field, with complex conjugation c, of the form $F = EF^+$, where $F^+ \subset F$ is the fixed field of c and $E/\mathbb{Q}$ is imaginary quadratic. Let B be a simple algebra with center F and $\dim_F B = n^2$, with a positive involution $*$ with $*|_F = c$ and an alternating form $\langle , \rangle : B \times B \to \mathbb{Q}$ such that $\langle bx, y \rangle = \langle x, b^* y \rangle$ for $\forall b \in B$. Let G be the $\mathbb{Q}$-similitude group of $(B, \langle , \rangle)$. Then $G_0 := \mathrm{Ker}(G \to \mathbb{Q}^\times)$ is the restriction of scalars from a unitary group over F^+. We choose $\langle , \rangle$ so that $G_0(\mathbb{R}) \cong U(1, n-1) \times U(0, n)^{d-1}$, where $d := [F^+ : \mathbb{Q}]$. For each open compact subgroup $U \subset G(\mathbb{A}^\infty)$ small enough (where $\mathbb{A}^\infty := \widehat{\mathbb{Z}} \otimes \mathbb{Q}$), we define the Shimura variety X_U/F as a moduli of isogeny classes of quadruples $(A, \lambda, i, \eta U)$ of an abelian variety A of dimension dn^2, a polarization $\lambda : A \to A^\vee$, a ring homomorphism $i : B \to \mathrm{End}(A) \otimes \mathbb{Q}$, satisfying the Kottwitz conditon on Lie$A$ corresponding to G (see [K]) and $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$ for $\forall b \in B$, and a right U-orbit ηU of $B \otimes \mathbb{A}^\infty$-isomorphisms $\eta : B \otimes \mathbb{A}^\infty \longrightarrow VA := \left(\varprojlim_N A[N] \right) \otimes \mathbb{Q}$ which sends $\langle , \rangle$ to the λ-Weil pairing. Then X_U/F is a quasi-projective smooth variety of dimension $n-1$, which is projective if $d > 1$ or if B is a division algebra. We choose a place v of F lying over a prime p which splits in E. Then $G(\mathbb{Q}_p)$ is a product of $GL_n(F_v)$ and other factors, so set $G(\mathbb{A}^\infty) = G(\mathbb{A}^{\infty,v}) \times GL_n(F_v)$. We set $U = U^v \times \mathrm{Iw}_n$ where $U^v \subset G(\mathbb{A}^{\infty,v})$ and Iw_n is the open compact subgroup of $GL_n(\mathcal{O}_v)$ consisting of matrices which reduce to upper triangluar matrices modulo v. For $U_0 := U^v \times GL_n(\mathcal{O}_v)$ the X_{U_0} extends to a smooth scheme over $\mathcal{O}_v$ with a universal abelian scheme $\mathcal{A}/X_{U_0}$. Then $\mathcal{G} := \mathrm{diag}(1, 0, \dots, 0)\mathcal{A}[v^\infty]$ is a 1-dimensional Barsotti-Tate $\mathcal{O}_v$-module of $\mathcal{O}_v$-height n, i.e. $\mathcal{G}[v]$ is a finite flat group scheme of degree $|k(v)|^n$, and the moduli of chain of n isogenies each of degree $|k(v)|$ factoring $\mathcal{G} \to \mathcal{G}/\mathcal{G}[v]$ gives a regular strictly semistable model of X_U over $\mathcal{O}_v$, which is finite flat over X_{U_0}.

Now the special fiber $Y := X_U \otimes_{\mathcal{O}_v} k(v)$ is written as $Y = \bigcup_{1\le i\le n} Y_i$, where Y_i, smooth over $k(v)$, is the locus where the i-th isogeny in the chain induces zero map on the Lie algebra. This moduli interpretation allows us to apply Theorem A to X_U, when it is proper, and the Hecke correspondences generating the local Hecke algebra $\mathcal{H}_n := \overline{\mathbb{Q}}_\ell\big[\mathrm{Iw}_n\backslash GL_n(F_v)/\mathrm{Iw}_n\big]$. It is generated by the generators $w_1, \dots, w_{n-1}$ and $T_1^{\pm}, \dots, T_n^{\pm}$ of the extended affine Weyl group, subject to certain relations (Bernstein-Zelevinsky presentation). It naturally contains the Iwahori Hecke algebra of Levi subgroups, say $\mathcal{H}_m \otimes \mathcal{H}_{n-m}$ of $GL_m \times GL_{n-m}$, which is generated by the above set of generators minus w_m, and makes $\mathcal{H}_n$ into a finite $\mathcal{H}_m \otimes \mathcal{H}_{n-m}$-algebra of dimension $\binom{n}{m}$. Now we refine the computation done in [TY]: we compute $H^*(Y^{(m)})$ as $\mathcal{H}_n$-module. By dividing it into open strata, we see that $H^*(Y^{(m)})$ (the alternating sum in the Grothendieck group) is the sum of $\mathcal{H}_n \otimes H_c^*(Y_{I_s}^0)$ for $m \le s \le n$, where $I_s := \{1, \dots, s\}$ and $Y_{I_s}^0 := Y_{I_s} - \bigcup_{I_s \subset I \ne I_s} Y_I$ is the open stratum of Y_{I_s}, and the tensor product is over $\mathcal{H}_m \otimes \mathcal{H}_{s-m} \otimes \mathcal{H}_{n-s}$. Now, the action of $\mathcal{H}_m \otimes \mathcal{H}_{s-m} \otimes \mathcal{H}_{n-s}$ on $H_c^*(Y_{I_s}^0)$ is given as follows: the action of $\mathcal{H}_m \otimes \mathcal{H}_{s-m}$ is computed locally by Theorem A, and the action of $\mathcal{H}_{n-s}$ is computed by counting of points on Igusa varieties via trace formula ([HT], this is where we need global assumptions). The action of $\mathcal{H}_m \otimes \mathcal{H}_{s-m}$ is roughly given by $\mathrm{St}_m \otimes \mathrm{Tr}_{s-m}$, with some unramified twists corresponding to the Frobenius action, where St_n is a 1-dimensional $\mathcal{H}_n$-module given by $w_i \mapsto -1$ and $T_i \mapsto 1$, similarly Tr_n is a 1-dimensional $\mathcal{H}_n$-module given by $w_i \mapsto q := |k(v)|$ and $T_i \mapsto q^{i(n-i)}$, and $\otimes$ denotes the product corresponding to non-normalized induction. When we look at the E_1-term $H^*(Y^{(m)})$ of the weight spectral sequence after taking the limit with respect to U^v, making $H^*(Y^{(m)})$ into a $G(\mathbb{A}^{\infty,v}) \times \mathcal{H}_n \times \mathrm{Frob}^{\mathbb{Z}}$-module, its $\pi^{\infty,v}$-isotypic component, where $\pi = \pi^{\infty,v} \times \pi_v$ is a cuspidal automorphic respresentation of $G(\mathbb{A}^\infty)$, recovers $\pi_v^{\mathrm{Iw}_v}$ as $\mathcal{H}_n$-module and is pure of weight $n-m$ as $\mathrm{Frob}^{\mathbb{Z}}$-module. Two things are used: (1) the global result on the cohomology of $Y_{I_s}^0$ shows that as $\mathcal{H}_{n-s}$-module $H^*(Y_{I_s}^0)[\pi^{\infty,v}]$ is essentially the (Iwahori invariants of) the Jacquet module of π_v to $GL_s \times GL_{n-s}$, and (2) the cancellation $\sum_{m=0}^{s}(-1)^m \mathrm{St}_m \otimes \mathrm{Tr}_{s-m} = 0$ in the Grothendieck group of $\mathcal{H}_s$-modules.

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