Non Abelian Lubin-Tate theory
and beyond

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§1. GL_n の Langlands 対応.

$n \geq 1$.

代数的整数論　　　　　　表現論

（ℚの絶対 Galois群 $G_{\mathbb{Q}}$ を調べる）　　　　（保型形式）

$$\begin{tikzcd}
\text{$G_{\mathbb{Q}}$ の n 次元表現} \arrow[rr, leftrightarrow] \arrow[dr] & & \text{$GL_n/\mathbb{Q}$ の保型表現} \arrow[dl] \\
& \boxed{\text{L 関数}} & \\
& \text{代数幾何 $/\mathbb{Q}$} \arrow[u] \arrow[uul, "\text{e'tale cohomology}"] \arrow[uur, dash] &
\end{tikzcd}$$

rank n.

ℚ 上の代数多様体（のモチーフ）

量子力学 / 場の量子論.

群の表現 ⇒ 作用素

関数	← 作用素
（可換な対象）	（演算子）
$tr(AB) = tr(BA)$	非可換 $AB \neq BA$
有限（次元）	無限次元.
⇑	⇑
観測できる対象	背後に隠れているもの

関数 … 空間の性質を反映
幾何.

整数論　$\boxed{\text{Spec $\mathbb{Z}$ の関数}}$

$\mathrm{Spec}\,\mathbb{Z} = \{(0), 2, 3, 5, \cdots\}$　スキーム.

← 隠れた $\boxed{\text{調和に}}$.
rigid.

Euler積

$$\prod_p (1 - a_p p^{-s})^{-1} \quad \left(= \sum_{n \geq 1} \frac{a_n}{n^s} \right)$$

$\Longrightarrow$ $\boxed{\text{解析接続 + 関数等式}}$ （Re $s > 1$ $\Rightarrow$ $\mathbb{C}$ にのびる）

値

"a_p" $\longleftarrow$ 何かのトレース.

◎ Langlands … よい Euler積はすべて GL_n の保型表現からくる.

◎ 代数的整数論

（$\mathbb{Q}$ の絶対 Galois群 $G_\mathbb{Q}$ を調べる）

$$G_\mathbb{Q} = \varprojlim_{\substack{K/\mathbb{Q} \\ \text{fin. Gal.}}} \mathrm{Gal}(K/\mathbb{Q})$$

$\forall p$. Frob_p … $\mathrm{Gal}(K/\mathbb{Q})$ の共役類

関数 a_p

$\Uparrow$

$G_\mathbb{Q}$ の表現 R

$a_p = \mathrm{tr}(R(\mathrm{Frob}_p))$

◎ 代数幾何

Weil予想

$X/\mathbb{Q}$: 代数多様体

$\Rightarrow X_p := X \bmod \mathbb{F}_p \quad \forall p$.

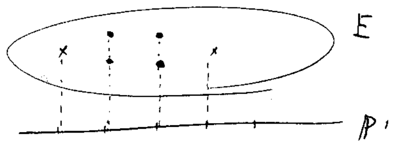
$p \longmapsto$ $\mathbb{F}_p$-有理点の個数 $\# X_p(\mathbb{F}_p)$ $\longleftarrow$ Frob_p の X の étale コホモロジーへの作用のトレースと解釈できる.

(Lefschetz trace formula)

<u>例.</u> $E/\mathbb{Q}$: 楕円曲線.

$y^2 = (x$ の 3次式$)$

$$\begin{tikzcd} E \arrow[d, "2次\ covering"] & \\ \mathbb{P}^1 & \mathbb{P}^1(\mathbb{F}_p) = p+1 \arrow[l] \end{tikzcd}$$



Weil予想
$|a_p| \leq 2\sqrt{p}$.

$\Longrightarrow \# E(\mathbb{F}_p) \fallingdotseq p+1$.

$a_p := \# E(\mathbb{F}_p) - (p+1)$.

$\longrightarrow L(s, E) = \prod_p (1 - a_p p^{-s} + p^{1-2s})^{-1}$.

(ほとんどすべてのp)

$L(s, E)$ を与える

- ◦ Galois表現 (2次元)
- ・ GL_2の保型表現

の対応は確立. (<u>谷山-志村予想 50</u>, 1999 解決).

$R : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}_\ell)$

↖ ℓ進数.

$\mathbb{Z} \ni a_p = tr(R(Frob_p))$
$= \alpha_p + \beta_p$.

(Weil予想)

$\alpha_p, \beta_p \in \overline{\mathbb{Q}}$ $\quad$ <u>$|\alpha_p| = |\beta_p| = \sqrt{p}$</u>

$\alpha_p = \sqrt{p}\, e^{i\theta}, \quad \beta_p = \sqrt{p}\, e^{-i\theta} \Longrightarrow a_p = 2\sqrt{p} \cos \theta_p$.

$p \longmapsto \theta_p$ の分布が…
正確に予想された.
↑
$\boxed{GL_n \text{ の Langlands 対応の系}}$

(<u>佐藤-Tate予想</u> '60s)
j-inv $\notin \mathbb{Z}$
なら解決. 2006.

GL_1 の Langlands 対応 = 類体論
↑　　　　　　　　　　　　{ 大域
局所 Langlands 対応　　　　　局所.

◎ <u>大域対応</u>

「有限」を除いてすべて」 ←

= ほとんどすべての p での
様子で決定してしまう.

Galois表現 … Chebotarev Density Thm.
保型表現 … Multiplicity One Thm

Spec($\mathbb{Z}$) は <u>rigid</u>

。「関数の値」(数) 以上の情報が
悪い素数 (特異点) には隠れている.

。局所 Langlands 対応は証明された (1999) が,
大域理論を用いる証明しか知られていない.
⇐ 局所的証明が欲しい.

◎ ところが, GL_1-case (局所類体論)
は局所的に証明できる.
(Lubin-Tate 理論. ⇒ 非可換できる)

§ 2.

p : prime.

$\mathbb{Z}_p := \varprojlim_m \mathbb{Z}/p^m\mathbb{Z}$. $\qquad \mathbb{Q}_p := \mathbb{Z}_p[\frac{1}{p}]$

$[K : \mathbb{Q}_p] < \infty$.

K : CDVF. $\qquad$ complete disc. val.

$\mathcal{O}_K$: integral closure of $\mathbb{Z}_p$ in K.

$$\mathcal{O}_K \supset \underset{\substack{\| \\ (\varpi_K)}}{\mathfrak{p}_K} : \exists! \text{ maximal ideal.} \qquad \varpi_K : \text{uniformizer.}$$

$k := \mathcal{O}_K/\mathfrak{p}_K \cong \mathbb{F}_q$.

$G_K := \mathrm{Gal}(\overline{K}/K)$.

$K(\mu_N) :=$ decomposition field of $X^N - 1$.

<u>Hensel's lemma</u> $p \nmid N$.

$\mu_N \subset k \Longleftrightarrow \mu_N \subset K$.

このとき.

$$\begin{tikzcd} \mu_N \arrow[r, "\sim"] \arrow[d, phantom, "\cap"] & \mu_N \arrow[d, phantom, "\cap"] \\ \mathcal{O}_K \arrow[r] & k \end{tikzcd}$$

$p \nmid N$.

$$\begin{tikzcd} \mathrm{Gal}(K(\mu_N)/K) \arrow[d, "\wr"] & & \\ \mathrm{Gal}(\underset{\substack{\| \\ \mathbb{F}_{q^f}}}{k(\mu_N)}/\underset{\substack{\| \\ \mathbb{F}_q}}{k}) \arrow[r, phantom, "="] & \langle x \longmapsto x^q \rangle \arrow[r, phantom, "\cong"] & \mathbb{Z}/f\mathbb{Z}. \end{tikzcd}$$

$N = q^f - 1$

$K^{ur} := \bigcup_{p \nmid N} K(\mu_N)$: maximal unramified ext'n of K.

$\mathrm{Gal}(K^{ur}/K) \cong \mathrm{Gal}(\overline{k}/k) \cong \hat{\mathbb{Z}}$.

normalization.

$$\begin{tikzcd} \varphi : G_K \arrow[r, two heads] & \mathrm{Gal}(K^{ur}/K) \arrow[r, phantom, "\cong"] & \mathrm{Gal}(\bar{k}/k) \arrow[r, phantom, "\cong"] & \hat{\mathbb{Z}} . \\ & \mathrm{Frob}_K \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & (x \mapsto x^q)^{-1} \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & 1 \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

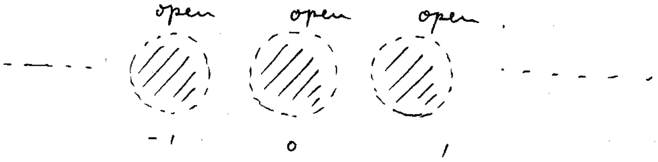
(geometric Frobenius)

$W_K := \varphi^{-1}(\mathbb{Z})$: Weil group of K.

$I_K := \ker \varphi$: inertia group of K.

$$\begin{tikzcd} 1 \arrow[r] & I_K \arrow[r] \arrow[d, equal] & W_K \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & \overset{\text{discrete}}{\boxed{\mathbb{Z}}} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \underset{\text{profinite}}{I_K} \arrow[r] & \underset{\text{profinite}}{G_K} \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1 \end{tikzcd}$$

$W_K = \coprod_{i \in \mathbb{Z}} I_K \cdot \mathrm{Frob}_K^i .$



$\Rightarrow$ topology on W_K

(stronger than the induced topo from G_K)

Local Class Field Theory.

$$1 \longrightarrow \mathcal{O}_K^\times \longrightarrow K^\times \xrightarrow{v_K} \mathbb{Z} \longrightarrow 1$$

$$\left[\begin{array}{l} x \in \mathcal{O}_K \supset \mathfrak{p}_K \supset \mathfrak{p}_K^2 \supset \mathfrak{p}_K^3 \supset \cdots\cdots \\ v_K(x) := m \iff x \in \mathfrak{p}_K^m \setminus \mathfrak{p}_K^{m+1}. \\ v_K(x^{-1}) = -v_K(x), \quad v_K(xy) = v_K(x) + v_K(y) \end{array}\right]$$

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_K^\times \arrow[r] \arrow[d, "\text{ramified}"'] & K^\times \arrow[r] \arrow[d, "\mathrm{Art}_K"', "\wr"] & \mathbb{Z} \arrow[r] \arrow[d, equal, "\text{unramified}"] & 1 \\
1 \arrow[r] & \mathrm{Ker} \arrow[r] & W_K^{ab} \arrow[r] & \mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

$$W_K^{ab} \subset G_K^{ab} \twoheadleftarrow G_K .$$

maximal abelian quotient.

LCFT.

$$\begin{tikzcd}
 & & \mathbb{Z} \arrow[r, phantom, "\subset"] & \mathrm{Gal}(K^{ur}/K) \\
\exists!\ \mathrm{Art}_K : K^\times \arrow[rr] & & W_K^{ab} \arrow[u, two heads] \arrow[r, phantom, "\subset"] & G_K^{ab} \arrow[u, two heads]
\end{tikzcd}$$

s.t.

1) ϖ : uniformizer

$\Longrightarrow \mathrm{Art}_K(\varpi)|_{K^{ur}} = \mathrm{Frob}_K$.

2) $\forall K'/K$: fin. abel. $\forall x \in (K')^\times$

$\mathrm{Art}_K(N_{K'/K}(x))|_{K'} = id$

2) comes from

<u>Base change</u> K'/K : fin.

$$\begin{tikzcd} K'^{\times} \arrow[r, "\sim", "\mathrm{Art}_{K'}"'] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & W_{K'}^{ab} \arrow[d] & \subset & G_K^{ab} \arrow[d] & G_{K'} \arrow[l, two heads] \arrow[d, hook, "\text{fin. index}"] \\ K^{\times} \arrow[r, "\sim", "\mathrm{Art}_K"'] & W_K^{ab} & \subset & G_K^{ab} & G_K \arrow[l, two heads] \end{tikzcd}$$

$$(K^\times : \underbrace{N_{K'/K}(K'^\times)}_{\text{index } = [K':K]}) < \infty$$

<u>Rem</u>

- Art_K : characterized by BC (functoriality).

Local Langlands Corresp should be characterized by functoriality.

<u>Lubin-Tate theory</u>. "LCFT via LT theory by T. Y."

--- construction of Art_K using <u>formal groups</u>.

<u>Philosophy</u>. --- Complex Multiplication.

$\sigma \longleftrightarrow f_\sigma$

CM

$\boxed{\text{Galois gp} \overset{\text{act}}{\circlearrowright} \text{geometric obj}}$.

$\boxed{\text{Geometric 1J action}}$ に書き換える.

(vect.sp.への作用が同じ)

~~Galois~~ repn. $\circlearrowright$ $\Big\Downarrow$ cohomology

vector sp / module (lin. alg. obj)

$$\begin{tikzcd} X \arrow[rr, "f_\sigma"] \arrow[dr] & & X \arrow[dl] \\ & \mathrm{Spec}\,K & \end{tikzcd}$$

$$\boxed{f_\sigma^* : H(X) \to H(X)}$$

$$\begin{tikzcd} G_K \circlearrowright \bar{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ G_K \circlearrowright \mathrm{Spec}\,\bar{K} \arrow[r] & \mathrm{Spec}\,K \end{tikzcd}$$

$G_K \ni \sigma$

$$\boxed{\sigma^* : H(X) \longrightarrow H(X)}$$

trace : Lefschetz formula で分かる.

ex 1) SGA5. $K = \mathbb{F}_p$. $X/\mathbb{F}_p$. 気分：coefficient を p乗.

$\underline{\mathrm{Frob}_p^{-1}} \curvearrowright H^i_{et}(X, \mathbb{Q}_\ell)$

$F : X \longrightarrow X$ relative Frob. (coordinate を p乗)

$\longrightarrow$ Letschetz trace formula

$\underline{F^*} \curvearrowright H^i_{et}(X, \mathbb{Q}_\ell)$.

これらの作用が同じ

(∵ F と Frob_p^{-1} を合成 ‥ abs. Frob ⇒ cohomology には identity.)

ex 2) CM-theory. $K/\mathbb{Q}$. imag. quad. $\mathcal{O}_K \hookrightarrow \mathrm{End}(E)$ ← ell. curve.

E/H : H : Hilbert Class field of K.

$\mathcal{O}_K \curvearrowright H^1_{et}(E, \mathbb{Q}_\ell) \circlearrowleft G_H$.

dim 2. ⟵ Galois repn $\mathcal{O}_K \otimes \mathbb{Q}_\ell$-rk 1.] rk 1 i.e. abel. G_H から G_K にのびる.

H の ほとんどすべての よい 素点で ex 1 の議論.

⟹ $\underline{G_H \text{ の } \mathrm{Frob}_{\mathfrak{p}}}$ の作用は. Galois → geometric.

E mod 𝔭 への F ($\underline{\text{relative Frob}}$)

↑ 𝔭 の下にある K の素イデアルを

$\mathfrak{q} = (\varpi)$ in H $\mathfrak{q} \subset \mathcal{O}_K$

とすると. $N_{H/K}(\varpi) \in \mathcal{O}_K \subset \underline{\mathrm{End}(E)}$ geometric.

の E への action で なければならない.

これが類体論を実現

G_K の char ⟺ $\mathbb{A}_K^\times / K^\times$ の char.

CM theory で.

$\mathfrak{p}$ が E の bad prime. ($H_\mathfrak{p}$: local field)

$\Longrightarrow$ E mod $\mathfrak{p}$ が singular.

$$\begin{tikzcd}
H^1(E,\mathbb{Q}_\ell) \arrow[r, phantom, "\circlearrowleft"] & G_H \\
 & G_{H_\mathfrak{p}} \arrow[u, phantom, "\cup"] \arrow[d] \\
\mathrm{Frob}_\mathfrak{p} \in \hat{\mathbb{Z}} & G_{H_\mathfrak{p}}^{ab} \arrow[l] & \ker \arrow[l] \\
 & & \mathcal{O}_{H_\mathfrak{p}}^\times \arrow[u, phantom, "\cong \text{ LCFT}"]
\end{tikzcd}$$

ramified $\circlearrowleft$

これが non-triv に作用する.

Lubin-Tate theory.

この ramified Galois action を geometric に書く.

$\ell \neq p$: prime.

$$H^1_{et}(E, \mathbb{Q}_\ell) := H^1(E, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell.$$

$$H^1_{et}(E, \mathbb{Z}_\ell) = (T_\ell E)^\vee$$

E/F perfect $\quad T_\ell E := \varprojlim_m E[\ell^m]$ Tate module.

$$G_F \circlearrowright E[n] := \{ x \in E(\bar{F}) \mid [n](x) = 0 \}$$

$\overset{\text{as}}{\text{abel gp}} \cong (\mathbb{Z}/n\mathbb{Z})^2$ 等分点.

· ramified の場合, 等分点は E の formal completion at origin だけみて調べられる.

CM elliptic curve E $\longrightarrow$ formal comp.
Lubin-Tate formal gp
(の最も単純な場合)

$K/\mathbb{Q}_p$ fin.

<u>Def</u>. A : $\mathcal{O}_K$-alg.

<u>formal $\mathcal{O}_K$-module</u> $\mathcal{F} := (F, [\cdot])$ $/A$.

... $F \in A[[X, Y]]$, s.t.

1) $F(X, Y) \equiv X + Y$ mod deg 2)

2) $F(F(X,Y), Z) = F(X, F(Y, Z))$.

3) $F(X, Y) = F(Y, X)$.

$$[\cdot] : \mathcal{O}_K \hookrightarrow \underset{\underset{\left\{ f \in A[[X]] \;\middle|\; \begin{array}{l} f \equiv 0 \mod X \\ F(f(X), f(Y)) = f(F(X,Y)) \end{array} \right\}}{\parallel}}{\mathrm{End}_A(F)}$$

... non commutative ring by.

$$\begin{cases} f \underset{F}{+} g := F(f, g) \\ f \circ g := f(g(X)) \end{cases}$$

$$F(f \underset{F}{+} g(X), f +_F g(Y)) = \cdots$$
$$= f +_F g(F(X,Y))$$

s.t. $\forall a \in \mathcal{O}_K$.

$$\underset{\underset{A[[X]].}{\cap}}{[a](X)} \equiv \underset{\underset{(\text{via } \mathcal{O}_K \to A)}{\uparrow}}{aX} \mod X^2.$$

<u>Lubin-Tate group</u> ——— GL$_1$.

= formal $\mathcal{O}_K$-mod of height ①

i.e. $[\varpi](X) \equiv X^{q^{\textcircled{1}}}$ (mod $\mathfrak{p}_K$).

$q = |\mathcal{O}_K/\mathfrak{p}_K|$.

$\varphi : G_K \longrightarrow \hat{\mathbb{Z}}$ ではなく．しばらく．

φ で Arith. Frob. Frob_K^{-1} を表す．

$$L = \begin{cases} \text{fin. unram. ext. of } K \\ \qquad \text{or} \\ \underline{\text{completion } \boxed{\hat{K}^{ur}} \text{ of }} \ \underline{K^{ur}}. \end{cases} \qquad \begin{aligned} &\leftarrow \text{residue field} \\ &\quad k \text{ が拡大} \end{aligned}$$

$\cup$

$\mathcal{O}_L$

integral closure of $\mathcal{O}_K$ 　　　 residue field : $\bar{k} = \overline{\mathbb{F}_q}$.

uniformizer ϖ of K is also a uniformizer of L.
i.e. $\mathfrak{p}_K \mathcal{O}_L = \mathfrak{p}_L$.

Lubin-Tate group の構成．

($L = \hat{K}^{ur}$ のとき．$\mathcal{O}_L$ 上では unique)

<u>Prop.</u> Fix $\varpi \in \mathcal{O}_L$: unif.

Let $f \in \mathcal{O}_L[[X]]$ satisfy.

$$\circledast \begin{cases} f(X) \equiv \varpi X & (\text{mod deg } 2) \\ f(X) \equiv X^q & (\text{mod } \mathfrak{p}) \end{cases}$$

$\Longrightarrow \exists\ F_f / \mathcal{O}_L$: formal group s.t.

$f \in \mathrm{Hom}(F_f, F_f^{\varphi})$

(F_f^{φ} で F の係数に φ を作用させたべき級数を表す).

<u>Lemma.</u> $\varpi, \varpi' \in \mathcal{O}_L$: uniformizer.

Let f, f' satisfy for ϖ, ϖ' (resp.)

Let $\alpha_1, \cdots, \alpha_n \in \mathcal{O}_L$ satisfy.

$\varpi' \alpha_i = \varpi \alpha_i^{\varphi}$ $(1 \leq i \leq n)$. $\leftarrow$ [☆ mod 2]

$\Longrightarrow \exists !\ F \in \mathcal{O}_L[[X_1, \cdots, X_n]]$. s.t.

$$\begin{cases} F \equiv \alpha_1 X_1 + \cdots + \alpha_n X_n & \text{mod deg } 2. \\ f^{\varphi} \circ F = F^{\varphi} \circ f. & \text{---} ☆. \end{cases}$$

(pf). For each $m \geq 1$.
$\exists !$ F_m : pol of degree $\leq m$.
satisfying ✪ mod deg $m+1$.

$m=1$ OK.

$$G_{m+1} := f' \circ F_m - F_m^{\varphi} \circ f.$$

$$G_{m+1} \equiv F_m^{q} - F_m^{\varphi}(X_1^{q}, \cdots, X_n^{q})$$

$$\equiv 0 \quad \text{mod } \mathfrak{p} \qquad (\because \varphi : \text{mod } \mathfrak{p} \text{ すると q乗写像})$$

$$\Downarrow$$

$$\varpi' \mid G_{m+1}.$$

$H_{m+1} := F_{m+1} - F_m$ を決める.

✪ $$f' \circ F_{m+1} - F_{m+1}^{\varphi} \circ f$$
$$= G_{m+1} + (f' \circ H_{m+1} - H_{m+1}^{\varphi} \circ f)$$
$$\equiv G_{m+1} + (\varpi' \cdot H_{m+1} - \varpi^{m+1} H_{m+1}^{\varphi}) \quad \text{mod deg } m+2$$

For each monomial X of deg $m+1$.
$\varpi' \beta$: coeff of X in G_{m+1}
α : coeff of X in H_{m+1}.

$$\varpi' \beta + (\varpi' \alpha - \varpi^{m+1} \alpha^{\varphi}) = 0$$
$$\alpha = -\beta - \sum_{i=1}^{\infty} \left(\frac{\varpi^{m+1}}{\varpi'} \right)^{1+\varphi+\cdots+\varphi^{i-1}} \beta^{\varphi^{i}} \qquad \boxed{/\!/}$$

<u>Completeness</u>

<u>pf of Prop.</u>

Apply lemma to the case

$\varpi = \varpi'$
$f = f'$
$n = 2$
$\alpha_1 = \alpha_2 = 1$

$$\Longrightarrow \exists ! \ F_f \quad f \circ F_f = F_f^{\varphi} \circ f.$$

- $F(X, F(Y, Z))$, $F(F(X,Y), Z)$
  both satisfy the lemma for $n=3$, $\alpha_1=\alpha_2=\alpha_3=1$
  $\Rightarrow$ coincide.
- $F(X,Y) = F(Y,X)$: same. ▨

Next. $\mathcal{O}_K$-action.

$\varpi, \varpi' \in \mathcal{O}_L$: unif.

$$A^L_{\varpi,\varpi'} := \{\theta \in \mathcal{O}_L \mid \varpi'\theta = \varpi\theta^{\varphi}\} \longleftarrow \boxed{A^L_{\varpi,\varpi} = \mathcal{O}_K}$$

Let f, f' satisfy $(*)$ for ϖ, ϖ' (resp.)
Then for $\forall \theta \in A^L_{\varpi,\varpi'}$

Lemma gives $[\theta]_{f,f'} \in \mathcal{O}_L[[X]]$.

$$\text{s.t.} \begin{cases} [\theta]_{f,f'} \equiv \theta X \mod \deg 2 \\ f' \circ [\theta]_{f,f'} = [\theta]^{\varphi}_{f,f'} \circ f. \end{cases}$$

Prop. 1) $[\theta]_{f,f'} \in \mathrm{Hom}_{\mathcal{O}_L}(F_f, F_{f'})$.

2) $[\cdot]_{f,f'} : A^L_{\varpi,\varpi'} \longrightarrow \mathrm{Hom}_{\mathcal{O}_L}(F_f, F_{f'})$ is injective and

$$[\theta]_{f,f'} +_{F_{f'}} [\theta']_{f,f'} = [\theta + \theta']_{f,f'}$$

$$[\theta']_{f',f''} \circ [\theta]_{f,f'} = [\theta\theta']_{f,f''}$$

In particular, we have

$$[\cdot]_f := [\cdot]_{f,f} : \mathcal{O}_K \longrightarrow \mathrm{End}_{\mathcal{O}_L}(F_f)$$

When $f = f'$ $\quad \left(\overset{\Uparrow}{A^L_{\varpi,\varpi}} = \mathcal{O}_K\right)$

pf 1) Check. $[\theta]\circ F_f = F_{f'}\circ[\theta]$.

both sides satisfy the condition of Lemma for

$$\varpi = \varpi' \quad , \ n = 2 , \quad \alpha_1 = \alpha_2 = \theta .$$

$$f'\circ([\theta]\circ F_f) = [\theta]^{\varphi}\circ f\circ F_f$$
$$= [\theta]^{\varphi}\circ F_f^{\varphi}\circ f = ([\theta]\circ F_f)^{\varphi}\circ f \quad \text{etc.}$$

2) $[\theta']\circ[\theta]$ satisfies the condition of Lemma. which characterizes $[\theta'\theta]$ etc. ▨

[Later we will see that

$\Sigma_f \simeq \Sigma_{f'}$ over $\mathcal{O}_{\hat{K}^{ur}}$ $(\forall f, f')$]

Next Construct abelian extns. (Lubin-Tate extns) of L by adjoining division pts of Σ_f.

Prop. $\varpi \in \mathcal{O}_L$: unif.

$f \in \mathcal{O}_L[X]$: monic pol. satisfying (*).

$$\left(\text{i.e. } f(X) = \underbrace{\varpi X + \cdots\cdots}_{\text{divisible by } \varpi} + X^q \right)$$

For $\forall m \geq 1$. define

$$\boxed{f_m := f^{\varphi^{m-1}}\circ\cdots\cdots\circ f^{\varphi}\circ f} \in \mathcal{O}_L[X].$$

$L' = L(\mu_{f,m})$:= splitting field of f_m.

$\mathcal{O}_{L'} \supset \mu_{f,m} := \{ \alpha \in L(\mu_{f,m}) \mid f_m(\alpha) = 0 \}$

1) $\mu_{f,m}$ is an $\mathcal{O}_K$-module by $+_{F_f}$, $[\cdot]_f$.

2) ${}^\forall \alpha \in \mu_{f,m} \setminus \mu_{f,m-1}$ ("primitive division pt")

$$\mathcal{O}_K/\mathfrak{p}^m \ni a \xmapsto{\sim} [a]_f(\alpha) \in \mu_{f,m}.$$

3) L'/L : tot. ram.

$-\alpha$: unif of L' s.t. $N_{L'/L}(-\alpha) = \varpi^{\varphi^{m-1}}$.

4) $\rho_{f,m}$: $\mathrm{Gal}(L'/L) \xrightarrow{\sim} (\mathcal{O}_K/\mathfrak{p}^m)^\times$

$\underset{\text{indep of } \alpha.}{}$ $(\alpha \mapsto [u]_f(\alpha)) \longmapsto u \bmod \mathfrak{p}^m$.

<u>ex.</u> $L = K = \mathbb{Q}_p$. $f(X) = (1+X)^p - 1$.

$\Rightarrow f_m(X) = (1+X)^{p^m} - 1$. $\Rightarrow \mu_{f,m} = \{\zeta - 1 \mid \zeta \in \mu_{p^m}\}$

$\Rightarrow L(\mu_{f,m}) = \mathbb{Q}_p(\mu_{p^m})$

<u>pf.</u>

1) 2) $\mu_{f,m} \setminus \mu_{f,m-1}$ = set of all roots of $h(X) := f_m(X)/f_{m-1}(X)$

$$\begin{cases} h(X) \equiv \varpi^{\varphi^{m-1}} & (\bmod X) \\ h(X) \equiv X^{(q-1)q^{m-1}} & (\bmod \mathfrak{p}) \end{cases}$$

$\therefore {}^\forall \alpha$. $0 = h(\alpha) \equiv \alpha^{(q-1)q^{m-1}} \quad (\bmod \mathfrak{p})$

$\Rightarrow \alpha \notin \mathcal{O}_{L'}^\times$, $\alpha \in \mathfrak{p}_{L'}$

m was arbitrary

$\Rightarrow \mu_{f,m} \subset \mathfrak{p}_{L'}$

$\Rightarrow$ can substitute elements of $\mu_{f,m}$ into $+_{F_f}$, $[\cdot]_f$.

by def. $\begin{cases} f \circ F_f = F_f^{\varphi} \circ f \\ f \circ [a] = [a]^{\varphi} \circ f \end{cases} \quad (a \in \mathcal{O}_K).$

$\longrightarrow \begin{cases} f_m \circ F_f = F_f^{\varphi^m} \circ f_m \\ f_m \circ [a] = [a]^{\varphi^m} \circ f_m \end{cases} \Rightarrow \mu_{f,m}$ is closed under $+_{F_f}$, $[\cdot]_f$

1) ▨

2) $|\mu_{f,m}| \le \deg f_m = q^m$.

$\Downarrow$

$\mathcal{O}_K$-hom

Ker $(\mathcal{O}_K \ni a \longmapsto [a](\alpha) \in \mu_{f,m})$

高々 q^m 個

$\supset \mathfrak{p}_K^m$.

あとは $\boxed{\mathfrak{p}_K^m \supset \mathrm{Ker}}$ を示せばよい.

[Lemma. $g \in \mathcal{O}_L[[X]]$.
$\forall \alpha \in \mu_{f,m}$, $g(\alpha) = 0 \Rightarrow f_m(X) \mid g(X)$.]

ϖ_0 : unif of K.

$[\varpi_0^m]$ kills $\mu_{f,m}$.

$\underset{\text{Lemma}}{\Longrightarrow}$ $f_m \mid [\varpi_0^m]$.

$\Downarrow$ coeff of X in f_m, $[\varpi_0^m]$ have valuation m in L.

$[\varpi_0^m] = f_m \cdot g_m$ $g_m(X)$ has const term $\in \mathcal{O}_L^\times$ (val 0).

$\Downarrow$

$\forall \alpha \in \mu_{f,m}$ $g_{m-1}(\alpha) \in \mathcal{O}_L^\times$, hence non-zero.

$\Rightarrow [\omega_0^{m-1}](\alpha) = 0$ implies $f_{m-1}(\alpha) = 0$

i.e. $\alpha \in \mu_{f,m-1}$.

$\overset{\text{対偶}}{\Longrightarrow} \left(\alpha \in \mu_{f,m} \setminus \mu_{f,m-1} \Rightarrow [\varpi_0^{m-1}](\alpha) \neq 0 \right)$.

3) 2)より. $\mu_{f,m} = \{ [a](\alpha) \mid a \in \mathcal{O}_K \}$.

$\Rightarrow L(\mu_{f,m}) = L(\alpha)$.

α : root of h

$\Rightarrow [L(\alpha):L] \le \deg h = (q-1)q^{m-1}$.

一方. const. term of h = $\varpi^{\varphi^{m-1}} = \prod_{\alpha \in \mu_{f,m} \setminus \mu_{f,m-1}} (-\alpha)$.

$$\begin{tikzcd}[row sep=small] L' = L(\alpha) \arrow[d, dash, "rm"] \\ L \arrow[d, dash, "ur"] \\ K \end{tikzcd}$$

take $v_{L'}$ of both sides.

$[L':L] \geq e_{L'/L} = \sum v_{L'}(-\alpha) \geq |\mu_{f,m} \setminus \mu_{f,m-1}|$

$= (q-1)q^{m-1}$.

$\therefore$ $L' = L(\alpha)/L$: tot. ram.

$[L':L] = (q-1)q^{m-1}$.

h : irred.

$\mu_{f,m} \setminus \mu_{f,m-1} =$ all conjugates of α

$\varpi f^{m-1} = N_{L'/L}(-\alpha)$.

4) $$\begin{tikzcd} \mathrm{Gal}(L'/L) \arrow[r, hook, "\rho"] & (\mathcal{O}_K/\mathfrak{p}^m)^\times \\ (\alpha \mapsto [u](\alpha)) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & u \bmod \mathfrak{p}^m \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

$\Longrightarrow$ ρ is group hom.

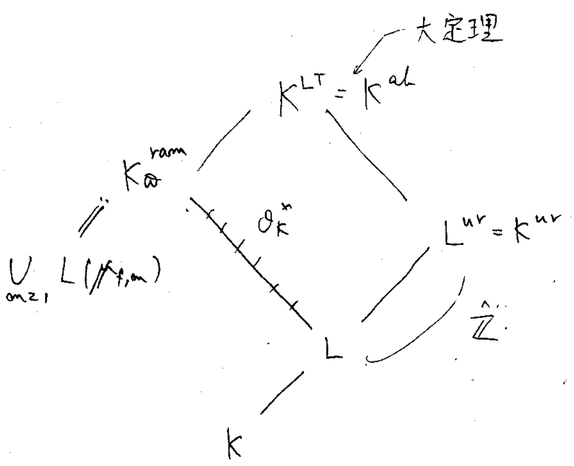
$$\left.\begin{array}{l} \alpha \mapsto [u](\alpha) \\ \alpha \mapsto [u'](\alpha) \end{array}\right\} \longrightarrow \alpha \mapsto \underset{[uu']}{\underset{\shortparallel}{[u][u']}}(\alpha)$$

- injective, same order $\Rightarrow$ isom.

- $\overset{\text{Galois}}{\sigma}(\alpha) = \overset{\text{geom.}}{[u]}(\alpha)$

$$\begin{aligned} \sigma(\underbrace{[u']\alpha}_{\alpha'}) &= [u'][u](\alpha) \\ &= [u'u](\alpha) \\ &= [u]([u'](\alpha)) \\ &= [u](\alpha'). \end{aligned}$$

▨



$\varpi \in \mathcal{O}_L$.

$$\begin{tikzcd}
\mathrm{Gal}(L(\mu_{f,m'})/L) \arrow[r, phantom, "\simeq"] \arrow[d] & (\mathcal{O}_K/\mathfrak{p}^{m'})^\times \arrow[d] \\
\mathrm{Gal}(L(\mu_{f,m})/L) \arrow[r, phantom, "="] \arrow[ur, phantom, "\circlearrowright"] & (\mathcal{O}_K/\mathfrak{p}^{m})^\times
\end{tikzcd}$$

$m' \geq m$.

Prop. (uniqueness over $\hat{K^{ur}}$) $\varpi, \varpi' \in \mathcal{O}_{\hat{K^{ur}}}$, unif.

f, f' satisfy $(*)$ for ϖ, ϖ' resp.

$\Rightarrow$ for $\theta \in A^{\hat{K^{ur}}}_{\varpi,\varpi'} \cap \mathcal{O}^\times_{\hat{K^{ur}}} \neq \phi$

we have $[\theta]_{f,f'} \in \mathrm{Isom}_{\mathcal{O}_{\hat{K^{ur}}}}(F_f, F_{f'})$

Prop (Classification $/L$) L/K : unr. of deg n.

$\varpi, \varpi' \in \mathcal{O}_L$, f, f' as usual.

Take $[\theta]_{f,f'} \in \mathcal{O}_{\hat{K^{ur}}}[\![X]\!]$ as above

$$\Rightarrow [\theta]^{\overset{\text{Gal}}{\boxed{\varphi^n}}} = [\theta] \circ \underbrace{[N_{L/K}(\varpi'/\varpi)]}_{\text{geometric}}{}_f .$$

In particular,

$[\theta] \in \mathcal{O}_L[\![X]\!] \iff N_{L/K}(\varpi') = N_{L/K}(\varpi)$

(1/12)

LCFT の構成.

Thm $K/\mathbb{Q}_p$ fin, $G_K := \mathrm{Gal}(\overline{K}/K)$.

$\exists !$ $\mathrm{Art}_K : K^\times \longrightarrow G_K^{ab}$

1) $\forall \varpi \in \mathcal{O}_K$ unif. $\Rightarrow$ $\mathrm{Art}_K(\varpi)|_{K^{ur}} = \mathrm{Frob}_K$

2) $\forall K'/K$ fin. abel. $\forall x \in K'^\times$ $\mathrm{Art}_K(N_{K'/K}(x))|_{K'} = id.$

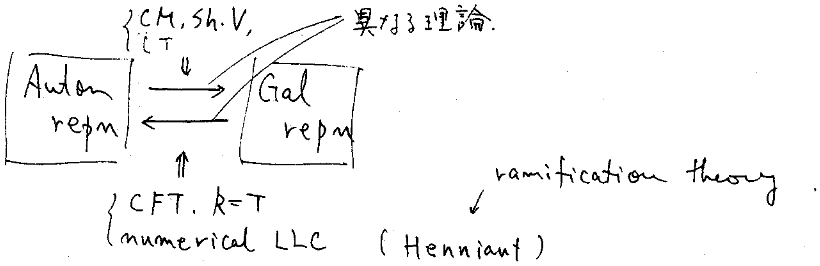
Moreover, $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{ab} \subset G_K^{ab}$

$$\begin{tikzcd}
K^{ab} \arrow[d, dash, "\mathcal{O}_K^\times"] \\
\bigcup_{p\nmid n} K(\mu_n) = K^{ur} \arrow[d, dash, "\hat{\mathbb{Z}}"] \\
K
\end{tikzcd}$$

Lubin-Tate theory $\overset{\text{全く別の理論}}{\dashrightarrow}$ turns out to be $K^{LT} = K^{ab}$

… construct abelian ext K^{LT}/K

Local Kronecker-Weber Thm
(ramification theory)



Lubin-Tate theory.

L/K : $\begin{cases} \text{fin. unram. or} \\ L = \widehat{K^{ur}} \end{cases}$

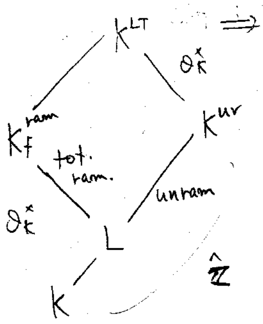
$\Rightarrow$ $\exists$ Lubin-Tate group $\Sigma_f := (F_f, [\cdot]_f)/\mathcal{O}_L$.

for $\forall f \in \mathcal{O}_L[[X]]$. s.t.

$f(x) \equiv \boxed{\varpi} X$ (mod deg 2), $f(x) \equiv X^q$ (mod p)

$\uparrow$ unif. of L.

Σ_f の relative Frob となるべき射 (べき級数)



$\Rightarrow$

$\bigcup_{m\geq1} L(\mathfrak{p}_{f,m})$

$\Uparrow$

$\{\alpha \mid f_m(\alpha)=0\} = \{\alpha \mid [\mathfrak{p}_K^m](\alpha)=0\}$;

$\mathfrak{p}^m$-division pts of Σ_f.

$\mathrm{Gal}(K_f^{ram}/L) \xrightarrow{\sim} \mathcal{O}_K^\times$.

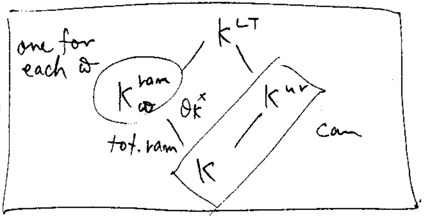
Prop 1 $\Rightarrow$ $K_f^{ram}\cdot\hat{K}^{ur}$: indep of f, ϖ, L $\Rightarrow$ $K^{LT} := K_f^{ram}\cdot K^{ur}$, canonical.
特に. $L=K$ とすれば. $\mathrm{Gal}(K^{LT}/K) = \mathcal{O}_K^\times \times \hat{\mathbb{Z}}$.
Prop 2 $\Rightarrow$ K_f^{ram} depends only on $N_{L/K}(\varpi)$, write $K_{N_{L/K}(\varpi)}^{ram}$

$L = \underset{\underset{\varpi}{\rotatebox{90}{$\in$}}}{K}$ とし.

$$\begin{array}{ccccc} \mathrm{Art}_K^{\varpi} : K^\times & \xrightarrow{\sim} & \mathcal{O}_K^\times \times \mathbb{Z} & \longrightarrow & \mathrm{Gal}(K_\varpi^{ram}/K)\times\mathrm{Gal}(K^{ur}/K) = \mathrm{Gal}(K^{LT}/K) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ u\cdot\varpi^n & \longmapsto & (u,n) & \longmapsto & ((\rho_{f,m}^{-1}(u))_{m\geq1}, \mathrm{Frob}_K^n) \end{array}$$

と定めると. これは <u>indep of ϖ</u> (Prop 2 を用いる).

$$\boxed{\begin{tikzcd} 1 \arrow[r] & \mathcal{O}_K^\times \arrow[r, hook, "\mathrm{can}"'] & K^\times \arrow[r] & \mathbb{Z} \arrow[r] \arrow[l, dashed, bend right, "\varpi"'] & 1 \end{tikzcd}} \Longleftrightarrow$$



ϖ の choice が対応

さらに. K'/K : fin に対し. Base Change を示すには.
Coleman Norm Operator を用いる.
(T.Y. LCFT via LT theory).

⊙ LCFT の構成 $\begin{cases} \text{LT theory} \cdots \text{Gal (abel) ext } K^{LT} \subset K^{ab} \text{ の構成} + \begin{cases} Art_K \\ B.C. \end{cases} \\ \text{local KW} \cdots K^{LT} = K^{ab} \end{cases}$

$\rightsquigarrow$

Local Langlands Corresp に拡張したい.

Non-abelian LT-theory

$\cdots$ [étale cohomology $H^{n-1}(X \otimes \widehat{\overline{K^{ur}}}, \overline{\mathbb{Q}}_\ell)$ of the moduli sp.
$X / \mathcal{O}_{\widehat{K^{ur}}}$ of [formal $\mathcal{O}_K$-module of ht n + Drinfeld level str]
realizes the LLC

-- proven via Sh. V. (global) $\longleftarrow$ [GL_1:
Harris - Taylor
CM-theory $\Rightarrow$ LT theory
$E \supset \mathcal{O}_K$
$K/\mathbb{Q}$: imag. quad. $\Rightarrow \begin{cases} K = \mathbb{Q}_p \\ [K:\mathbb{Q}_p] = 2 \end{cases}$

Goal : Local construction of

$$\left\{ \begin{array}{l} \text{irred. } \sout{\text{smooth}} \text{ (adm.)} \\ \text{repn of } GL_n(K) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} n\text{-dim Weil-Deligne} \\ \text{rep'n of } W_K \end{array} \right\}$$

$n = 1$; $\quad K^\times \qquad W_K^{ab}$.

$n = 1$. ◦ LT-theory :

$$\begin{tikzcd} W_K^{ab} \arrow[r, two heads, "\text{(じつは}\cong)"'] & W_K^{LT} \arrow[r, "\sim", "Art_K^{-1}"'] & K^\times \\ G_K^{ab} \arrow[u, phantom, sloped, "\subset"] \arrow[r, two heads] & G_K^{LT} \arrow[u, phantom, sloped, "\subset"] & \end{tikzcd}$$

$K^\times$ の char $\chi \Rightarrow W_K'$ の char $\chi \circ Art_K^{-1}$.

- Prop 1: $\exists ! \ \Sigma / \mathcal{O}_{\hat{K^{ur}}}$: LT gp

  … moduli sp X_0 of LT-groups $/ \mathcal{O}_{\hat{K^{ur}}} = S$:

  relative 0-dim.

  $X_0 = \operatorname{Spec} \mathcal{O}_{\hat{K^{ur}}}$

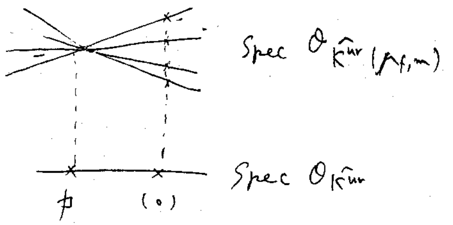
<u>moduli</u>

$$\{ \text{obj} / S' \} \underset{\text{bij}}{\xleftrightarrow{\ f^* \Sigma^{univ}/_{S'} \longleftarrow\!\shortmid f\ }} \left\{ \begin{matrix} S\text{-mor} \\ S' \longrightarrow X \end{matrix} \right\}$$

Σ^{univ} / X moduli sp of $\underset{\Sigma}{\underline{\text{Obj's}}}$.

$\Sigma / \mathcal{O}_{\hat{K^{ur}}}$: LT groups moduli of {LT groups + Drinfeld $\mathfrak{p}^m$-str}

$= \underline{\operatorname{Spec} \mathcal{O}_{\hat{K^{ur}}(\boxed{\mu_{f,m}})}}^{\curvearrowleft \mathcal{O}_K}$.

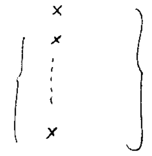


CDVF obtained by adjointing $\mathfrak{p}^m$-division pts $\Sigma[\mathfrak{p}^m] \circlearrowleft \mathcal{O}_K$

$\Sigma \xrightarrow{\sim} \Sigma' / \mathcal{O}_{\hat{K^{ur}}}$

$$X \otimes_{\mathcal{O}_{\hat{K^{ur}}}} \overline{\hat{K^{ur}}} = \coprod_{[\hat{K^{ur}}(\mu_{f,m}) : \hat{K^{ur}}]} \operatorname{Spec} \overline{\hat{K^{ur}}}$$

$\circlearrowleft$ $\operatorname{Gal}(\overline{\hat{K^{ur}}} / \hat{K^{ur}}) = I_K$



$\circlearrowleft I_K$

ich $H^{1-1} = H^0$ $\wr\wr$ $\overline{\mathbb{Q}}_\ell^{[\hat{K^{ur}}(\mu_{f,m}), \hat{K^{ur}}]}$

primitive $\mathfrak{p}^m$-division pts $\mu_{f,m} \setminus \mu_{f,m-1}$

$\circlearrowleft \operatorname{Gal}(\hat{K^{ur}}(\mu_{f,m}) / \hat{K^{ur}}) \twoheadleftarrow I_K$

$\downarrow \wr\wr \ \rho_{f,m}$.

$\circlearrowleft$ $(\mathcal{O}_K / \mathfrak{p}^m)^\times \twoheadleftarrow \mathcal{O}_K^\times$

$\overline{\mathbb{Q}}_\ell^{[\hat{K^{ur}}(\mu_{f,m}) : \hat{K^{ur}}]}$

$\circlearrowright \quad \circlearrowleft$

$I_K \underset{\text{Art}_K}{\hookrightarrow} \mathcal{O}_K^\times$

$\bigoplus_{\chi} (\chi \circ \text{Art}_K^{-1}) \otimes \chi$.

$n \geq 1$.

> Art_K はないが、 $\chi \longmapsto \chi \circ \mathrm{Art}_K^{-1}$
> にあたる対応 $\pi \longmapsto \mathcal{L}(\pi)$ を実現する v.s. $/ \overline{\mathbb{Q}_\ell}$
> $$V = \bigoplus_{\pi} \pi \otimes \mathcal{L}(\pi)$$
> を作る.

◎ X_m の方程式をもとめる.

$$R = \mathcal{O}_{\widehat{K^{ur}}} \qquad R/\mathcal{O}_K \qquad q = \#(\mathcal{O}_K/\mathfrak{p}_K).$$

$n \geq 1$. fix

Def A : R-alg. local $\mathfrak{m} \subset A$. complete Noetherian $\overline{\mathbb{F}_q} \xrightarrow{\sim} A/\mathfrak{m}$
Σ : formal $\mathcal{O}_K$-module $/ A$.

Σ : height n $\Longleftrightarrow$ $[\varpi](X) \equiv X^{q^n}$ $(\mathrm{mod}\ \mathfrak{m} + \deg q^m)$
for all unif $\varpi \in \mathcal{O}_K$.

このとき. Σ, deformation of Σ_n to A.

Rem $A = \overline{\mathbb{F}_q} \Rightarrow {}^{\exists !} \Sigma_n / \overline{\mathbb{F}_q}$: formal $\mathcal{O}_K$-mod of ht n.

Thm (Drinfeld) $R_0 = R[[T_1, \cdots, T_{n-1}]]$ represents the functor:

$$\mathcal{C} = \left(\begin{array}{l} \text{local comp. noeth.} \\ R\text{-alg. w/ res. field } \overline{\mathbb{F}_q} \end{array} \right) \longrightarrow \text{Sets}$$

$x \hookrightarrow$ 

$\mathrm{Spec}\,\overline{\mathbb{F}_q} \hookrightarrow \mathrm{Spec}\,A$.

mor. local R-hom.

$$(A, \mathfrak{m}) \longmapsto \left\{ (\Sigma, i)/_{\cong} \;\middle|\; \begin{array}{l} \Sigma : \text{formal } \mathcal{O}_K\text{-mod of ht } n \\ i : \Sigma_n \xrightarrow{\sim} \Sigma \bmod \mathfrak{m} \end{array} \right\}$$

deformation of Σ_n.

$$\cong \quad \mathrm{Hom}_{\mathcal{C}}(R_0, A)$$

$${}^{\exists} f \quad \Sigma = \Sigma^{\mathrm{univ}} \otimes_{R_0, f} A.$$

Rem $\circ$ $\Sigma_2/\overline{\mathbb{F}_q}$ formal comp. of a supersingular elliptic curve $/\overline{\mathbb{F}_q}$.
$K = \mathbb{Q}_p$

$\circ$ Deformation sp $\overset{\uparrow}{=}$ strict complete local ring of modular curve $/\mathbb{Z}_p$ (2-dim sp) at ssing pt.

Serre-Tate $\left[\text{def. of ell. curve } E = \text{def of } p\text{-div gp } E[p^\infty] \right]$

$R[\![T]\!] = \hat{\mathbb{Z}}_p^{ur}[\![T]\!]$ .. 2-dim local ring

Level str. $(A, m) \in \mathcal{C}$. Σ/A : height n.
Drinfeld ϖ^m-str. of Σ. $\mathcal{O}_K$-module by

$$\varphi : (\mathcal{O}_K/\varpi^m)^n \xrightarrow[\mathcal{O}_K\text{-hom}]{} m \circlearrowleft \quad (+_\Sigma, [\cdot])$$

s.t. $$[\varpi^m](X) = (\text{unit}) \cdot \prod_{x \in (\mathcal{O}_K/\varpi^m)^n} (X - \varphi(x))$$

$\longrightarrow$ higher dim p-div gp ··· ???
$\rightsquigarrow$ $\left[\begin{matrix} \text{Diaudonne} \\ \text{Display} \end{matrix} \right]$ interpretation.

Thm (Drinfeld). Functor:
$\left\{ \begin{matrix} n \geq 1 \\ m \geq 1 \end{matrix} \right.$

$$\mathcal{C} \longrightarrow \text{Sets}.$$
$$(A, m) \longmapsto \{ (\Sigma, i, \varphi)/\cong \}$$
$\nwarrow$ Drinfeld ϖ^m-str.

is represented by a fin. flat R-alg. R_m,
which is regular, regular parameters $X_i := \varphi(e_i)$
$\{ \underset{\overset{\shortparallel}{(1,0,\cdots,0)}}{e_1}, \cdots, \underset{\overset{\shortparallel}{(0,\cdots,0,1)}}{e_n} \}$: canonical basis of $(\mathcal{O}_K/\varpi^m)^n$

φ^{univ}: level ϖ^m str on Σ^{univ}/R_m look at the leading term

$$\boxed{\varpi = (\text{unit}) \cdot \prod_{x \in (\mathcal{O}_K/\mathfrak{p})^n \setminus \{0\}} \varphi(x) = (\text{unit}) \prod_{(a_1, \cdots, a_n) \in (\mathcal{O}_K/\mathfrak{p})^n \setminus \{0\}} \Big([a_1](X_1) + \cdots + [a_n](X_n) \Big) \quad \underbrace{X_1 \cdots X_n}_{\text{reg. par. of } R_1}}$$

$$\begin{tikzcd} \operatorname{Spec} R_m = X_m \arrow[d, "\text{fin. flat}"] \\ \operatorname{Spec} R_0 \end{tikzcd}$$

$\wr$

$R_1 = R[\![X_1, \cdots, X_n]\!] / (\varpi - \prod(\cdots))$

$\boxed{X_m} = \operatorname{Spec} R_m$. $m \geq 1$. $R = \mathcal{O}_{\widehat{K^{ur}}}$

$\boxed{R_m} =$ deform ring rep'ting.

$\mathcal{C} \longrightarrow \mathcal{S}ets.$

$(A, \mathfrak{m}) \longmapsto \{ (\Sigma, i, \varphi)/\cong \}$

Σ / A : formal $\mathcal{O}_K$-mod, $\underline{\text{ht}}$ n.

i : $\Sigma_n \xrightarrow{\sim} \Sigma \bmod \mathfrak{m} \,/\, \overline{\mathbb{F}_q}$

φ : $(\mathcal{O}_K/\mathfrak{p}^m)^n \longrightarrow \mathfrak{m}$ Drinfeld $\mathfrak{p}^m$-str.

$\{e_1, \cdots, e_n\}$ $\operatorname{Im}(\varphi) =$ $\mathfrak{p}^m$-division pts.

$X_i := \varphi^{univ}(e_i)$ $1 \leq i \leq n$.

$\uparrow$ regular param of R_m.

$$R_m = R[\![X_1, \cdots, X_n]\!] \Big/ \Big(\varpi - \prod_{(a_1, \cdots, a_n) \in (\mathcal{O}_K/\mathfrak{p}^m)^n \setminus (\mathfrak{p}\mathcal{O}_K/\mathfrak{p}^m)^n} \big([a_1](X) + \cdots + [a_n](X_n) \big) \Big)$$

$\uparrow$ $\boxed{R_m : \underline{\text{regular}} \Rightarrow 1 \text{ eqn defines } R_m}$

$$\left(\overset{n=1}{} \quad \varpi^{\varphi^m} = \prod_{\alpha \in \mu_{f,m} \setminus \mu_{f,m-1}} (-\alpha) \right)$$

$H^{n-1}_{\text{ét}}(X_m \otimes_R \overline{\hat{K}^{ur}}, \overline{\mathbb{Q}}_\ell)$ $\ell \neq p$.

↑ DVR

geom. generic fibre をとる.

$K/\mathbb{Q}_p$: fin.

$$\boxed{H^{n-1}_{\text{ét}}(X_m \otimes_R \overline{\hat{K}^{ur}}, \overline{\mathbb{Q}}_\ell)}$$

↑ DVR

geom. gen. fibre をとる.

$\mathcal{O}_D^{\times}$ ↻ ↻ ↻

$GL_n(\mathcal{O}_K/p^m)$

$I_K = Gal(\overline{\hat{K}^{ur}}/\hat{K}^{ur})$

$\quad = Gal(\bar{K}/K^{ur})$

D: div. alg $/K$

dim n^2

Hasse inv : $1/n$.

$\mathcal{O}_D = End(\Sigma_n)$

$\ni d \quad (i \mapsto i \circ d)$

$\ni g : (\varphi^{univ} \mapsto \varphi^{univ} \circ g)$

$H^{n-1} = \bigoplus \rho \otimes \pi \otimes \sigma$

Jacquet-Langlands corresp.

↑ trace identity で defines

F の表ω

Local Langlands corresp

→ $D^{\times}$, $GL_n(K)$, $G_K(W_K)$ の repn にのばす.

Local Langlands Corresp

$$\left\{\begin{array}{c}\text{irred. smooth. (admissible)}\\ \text{rep'n of } GL_n(K)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}n\text{-dim Frobenious s.s.}\\ \text{Weil-Deligne repn of } W_K\end{array}\right\}$$

Def $\underline{\mathcal{P}=(\pi,V)}$: repn of $GL_n(K)$, $\quad \left\{\begin{array}{l}\pi : GL_n(K) \longrightarrow GL(V)\\ V : \text{vect. sp} / \overline{\mathbb{Q}}_\ell \ (\text{ふつう}\infty\text{次元}).\end{array}\right.$

$U \overset{\cup}{.}$ open cpt

$$V^U := \{ v \in V \mid \pi(\sigma)\cdot v = v \quad (\forall \sigma \in U) \}$$

(π, V) : smooth $\Leftrightarrow$ ${}^\forall v \in U,\ {}^\exists U,\quad v \in V^U$

$\Leftrightarrow V = \bigcup_{U \triangleleft G} V^U$ ..... $\underbrace{G/U}_{\text{discrete}} \curvearrowright V^U$ $\quad$ smooth $\Leftrightarrow$ (discrete gp の repn のつみかさね)

(π, V) : admissible $\Leftrightarrow$ $\forall U$. $\quad \dim V^U < \infty$

Rem G : reductive p-adic Lie gp

irred. smooth $\Rightarrow$ admissible (Casselman)

- Bernstein-Zelevinsky .

Def. $\underline{R=(r,N)}$: Weil-Deligne rep of $W_K := \varphi^{-1}(\mathbb{Z})$

$q = \# \mathcal{O}_K/\mathfrak{p}_K$. $\quad \left\{\begin{array}{l} r : W_K \longrightarrow \underset{\text{discrete}}{GL(V)} \\ N \in \mathrm{End}(V)\end{array}\right.$ $\quad \dim_{\overline{\mathbb{Q}}_\ell} V < \infty$.

s.t. $r(\sigma)\cdot N = q^{-\varphi(\sigma)}\cdot N \cdot r(\sigma) \quad {}^\forall \sigma \in W_K$ $\quad$ ($\Rightarrow N$: nilp.)

- (r, N) : Frob s.s. $\Leftrightarrow$ r : s.s.
- $r|_{I_K}$: factors thru fin. quot.

$$\begin{tikzcd} \varphi : G_K \arrow[r, two heads] & G_K^{ur} \arrow[r] & \hat{\mathbb{Z}} \\ W_K \arrow[u, phantom, "\cup" rotate=180] & \mathrm{Frob}_K \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & 1 \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

Thm (Grothendieck) $\ell \neq p$.

$$\left\{\begin{array}{l} n\text{-dim} \\ \text{WD-repn}/\bar{\mathbb{Q}}_\ell \text{ of } W_K \end{array}\right\} \xrightarrow{1:1} \left\{\begin{array}{l} n\text{-dim} \\ \text{conti. repn. of } G_K \end{array}\right\}$$

$$G_K \longrightarrow \underbrace{GL_n(\bar{\mathbb{Q}}_\ell)}_{\ell\text{-adic top.}}$$

$\cup$ tame

$I_K \supset P_K$

(wild interia pro-p)

$$I_K/P_K \cong \prod_{\ell \neq p} \mathbb{Z}_\ell. \qquad t_\ell : I_K \longrightarrow \mathbb{Z}_\ell \text{ tame char.}$$

$$(r, N) \longmapsto \rho \quad \text{s.t.}$$

$$\rho(\sigma) = r(\sigma)\cdot \exp(t_\ell(\sigma)\cdot N)$$

$$WD(\rho) \longleftarrow\!\!\mid \rho$$

- ℓ-adic monodromy.
- $\ell = p$. Fontaine p-adic Hodge.

$\rho \longmapsto WD(\rho)$ D_{pst}-functor.

Ⓡ n-dim Frob s.s. WD repn of W_K

$$\boxed{R = \bigoplus_{i=1}^{t} \underbrace{\overbrace{Sp_{s_i}(R_i)}^{\text{indecomposable}}}_{s_i n_i\text{-dim}}}$$

R_i : irred. $(r_i, 0)$ n_i-dim WD-rep.

irrd.

$$n = \sum_{i=1}^{t} s_i n_i \qquad Sp_s(R) = Sp_s \otimes R.$$

non-triv N $\to Sp_s = \langle e_1, \cdots, e_s \rangle$

reducible

indecomposable.

$$\begin{cases} r(\sigma)e_i = |\mathrm{Art}_K^{-1}(\sigma)|_K^{i-1} \cdot e_i \\ N e_i = e_{i+1}. \end{cases}$$

$$\begin{tikzcd}
 & \mathcal{O}_K^\times \arrow[d] & & W_K \arrow[d] \arrow[dddl, bend left, "|\mathrm{Art}_K^{-1}|_K"] \\
\mathrm{Art}_K : & K^\times \arrow[rr, leftrightarrow, "\cong"] \arrow[d] & \ni \varpi \arrow[d, mapsto] & W_K^{ab} \\
 & \mathbb{Z} \arrow[d] & \ni 1 \arrow[d, mapsto] & \\
 & \bar{\mathbb{Q}}_\ell^\times & \ni q^{-1} &
\end{tikzcd}$$

$$\boxed{|\mathrm{Art}_K^{-1}|_K : \mathrm{Frob} \mapsto q^{-1} \text{ .. cyclotomic char } \chi.}$$

(ρ) : irred smooth repn of $GL_n(K)$

$$\boxed{\rho = \boxplus_{i=1}^{t} Sp_{s_i}(\rho_i)}$$

ρ_i : supercuspidal rep of $GL_{n_i}(K)$
(matrix coeff is cpt supported).

↓

$Sp_{s_i}(\rho_i)$; square integrable (discrete series) rep of $GL_{s_i n_i}(K)$.

特別なタイプの
parab. ind.

↓

$$\boxplus_{i=1}^{t} Sp_{s_i}(\rho_i) = n\text{-}\mathrm{Ind}_H^G \Big(\prod_{i=1}^{t} \underbrace{Sp_{s_i}(\rho_i)}_{GL_{s_i n_i}(K)\text{ の rep.}} \Big)$$

Langlands sum.
(parabolic ind)

$$H = \begin{pmatrix} \boxed{s_1 n_1} & & * \\ & \boxed{s_2 n_2} & \\ 0 & & \ddots \end{pmatrix} \subset GL_n(K) = G.$$

⑨ LLC の characterization -- 現状では. L-factor, ε-factor (of pairs) というものを使う.

supercuspidal ⟷ irred. WD-repn $(r, 0)$

$r : W_K \to GL(V)$ を $r|I_K$ にすると I_K の fin. quot. を factor する repn.

Brauer's thm

$$r|I_K = \sum_i \left(\mathrm{Ind}_{I_{L_i}}^{I_K} \chi_i\right)^{n_i} \qquad n_i \in \mathbb{Z}.$$

$$\begin{tikzcd} \chi_i : I_{L_i} \arrow[r] \arrow[d, leftrightarrow, "LCFT"] & \overline{\mathbb{Q}}_\ell^\times \\ \mathcal{O}_{L_i}^\times \arrow[r] & \overline{\mathbb{Q}}_\ell^\times . \end{tikzcd}$$

$$\boxed{\text{Automorphic Induction}}\ ?? \longleftarrow \boxed{\text{Induction}}\ ?$$

LLC is characterized by functoriality

$$\left\{\begin{array}{lcl} n=1 & : & \mathrm{LCFT} \\ \mathrm{AI} & \longleftrightarrow & \mathrm{Ind} \\ \mathrm{PI} & \longleftrightarrow & \oplus \\ \mathrm{Sp} & \longleftrightarrow & \mathrm{Sp} \end{array}\right.$$

さらに、これを Non-abelian LT-theory (geometric) で realize.

$m=1$ … R_1 を blow up.

$$\Rightarrow \boxed{\begin{array}{c}\text{Deligne-Lusztig theory} \\ GL_n(\mathbb{F}_q) \circlearrowright \mathbb{F}_{q^n}^\times\end{array}}$$

$\downarrow$

$GL_n(\mathbb{F}_q)$ の cusp. irred. rep.

$\uparrow$

$GL_n(\mathcal{O}_K)$

$\}$ c-Ind

super cusp. $\boxed{GL_n(K) \text{ の rep}}$

$[L:K]=n$

$\mathbb{F}_{q^n}^\times$ の char

$\Uparrow$

$I_K = I_L$ の char

W_L から W_K に ind $\Rightarrow$ n-dim.

Dream : …
- ∘ $X_m = \mathrm{Spec}\, R_m$ の geometry
  - $\Rightarrow$ [char からの Ind に対応する AI が geometric にできる? (DL の拡張)
- ∘ $Sp_s \longleftrightarrow Sp_s$ の geometric realization
  - … [Iwahori Hecke alg の rep'n の場合
  - $Sp_s(\mathbb{1})$ χ : unram.

Non abelian L-T が local geometric にできたらいいな.

$$\mathrm{Iw}_n \subset GL_n(\mathcal{O}_K) \subset GL_n(K)$$

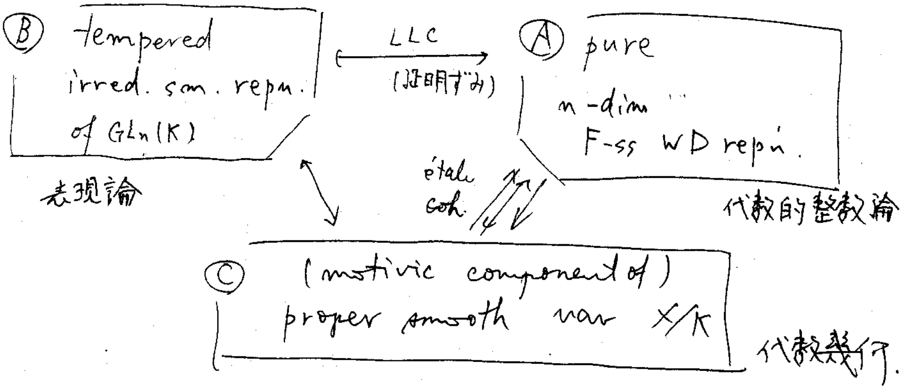
$$\mathrm{Iw}_n = \left\{ g \;\middle|\; g \equiv \begin{pmatrix} \diagdown & * \\ 0 & \diagdown \end{pmatrix} \bmod \mathfrak{p} \right\}$$

V : V^{Iw_n} で generate された Am. rep.

V^{Iw_n} : $\overline{\mathbb{Q}_\ell}[\mathrm{Iw}_n \backslash GL_n(K) / \mathrm{Iw}_n]$ の f.d. rep.

- Dream 2.

$K/\mathbb{Q}_p$: fin



Ⓒ → Ⓐ … Weight-Monodromy conj.
(Weil conj の一般化).

Global な l-adic Gal rep.
$G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$ の local comp … pure (と考えられる)

Autom rep $\Pi = \otimes \Pi_p$ of $GL_n(\mathbb{A})$
の local comp … tempered
(Generalized Ramanujan conj)

$$\begin{tikzcd} Ⓐ \arrow[r] \arrow[dr] & Ⓒ \\ & Ⓑ \arrow[u] \end{tikzcd}$$

できる!

Ⓐ → Ⓑ. $R = (r, N) \overset{LLC}{\rightsquigarrow} p = (\pi, V)$ を local comp にもつような autom. rep. $\Pi = \otimes \Pi_p$ of $GL_n(\mathbb{A}_F)$
($\Pi_p = p$)

Shimura Var の cohomology に Global Langlands が実現
$\left[\begin{array}{l} F : \text{CM-field} \\ \Pi : \text{regular cuspidal self-dual} \\ \vdots \end{array}\right.$

$G_F \times GL_n(\mathbb{A}_F^\infty) \curvearrowright H^{n-1}(Sh_V, \overline{\mathbb{Q}}_l)[\Pi^\infty]$: l-adic Galois rep of G_F.

↓ (次頁)

↓

この Galois rep を $G_K \subset G_F$ に制限すると R に戻る　　$K = F_{\mathfrak{p}}$

（して．WD をとる）

(Global-Local Compatibility　Taylor-Yoshida)

$\mathfrak{X} \otimes K \cong K$　　　X/K : strictly semistable.

$\Longleftarrow {}^{\exists}\mathfrak{X}/\mathcal{O}_K$　　$\mathfrak{X}$: Zariski local に

étale cover $\mathcal{O}_K[T_1, \cdots, T_n]/(\varpi - T_1 \cdots T_n)$

local には NALT でいうと

Iwahori-Hecke case.