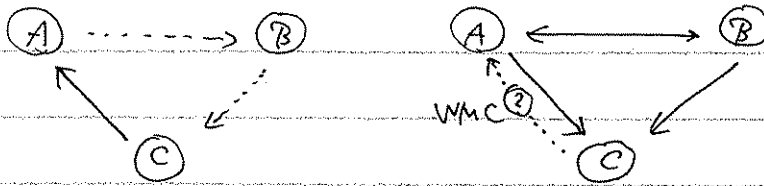


Converse of Weight-Monodromy Conj.

§1.

- L^c →
- (A) Algebraic Number Theory (Galois reps)
 - (B) Representation Theory (Autom. reps of GL_n)
 - (C) Algebraic Geometry (Pure Motives of alg. var.s)

Base field $[L:\mathbb{Q}] < \infty$ $[K:\mathbb{Q}_p] < \infty$



→ known
 partially known

- §1. Intro.
- §2. L^c ↔ L^c corresp.
 - i) local
 - ii) global
- §3. L^c + Motives
 - i) global obj's are motivic
 - ii) motivic obj's \mathbb{F}_q
 - iii) motivic local obj.

§2. i) local. $[K:\mathbb{Q}_p] < \infty$. $\mathcal{O}_K/\mathfrak{p}_K = k \simeq \mathbb{F}_q$. $K^{ur} := \bigcup_{p \nmid N} K(\mu_N)$

(A) $\varphi: \text{Gal}(\bar{K}/K) \rightarrow \text{Gal}(K^{ur}/K) \simeq \text{Gal}(\bar{\mathbb{F}}_q/k) \simeq \hat{\mathbb{Z}}$
 $\text{Frob}_k \mapsto (x \mapsto x^q)^{-1} \mapsto 1$

$W_k := \varphi^{-1}(\mathbb{Z})$. $I_k := \text{Ker } \varphi$. $\Omega := \mathbb{C}$ or $\bar{\mathbb{Q}}_\ell$ (ℓ : prime)

Def. Weil-Deligne rep $(r, N, V) / \Omega$ of W_k : $\left\{ \begin{array}{l} r: W_k \rightarrow GL(V) \text{ (dim}_{\Omega} V < \infty) \\ \text{s.t. } \left\{ \begin{array}{l} \text{Ker}(r|_{I_k}) \triangleleft I_k : \text{fin. index} \\ r(\sigma)N = \chi(\sigma)Nr(\sigma) \text{ (}\forall \sigma \in W_k\text{)} \end{array} \right. \end{array} \right.$ $N \in \text{End}(V)$

$\chi: W_k \rightarrow \Omega^\times$
 $\sigma \mapsto \varphi^{-1}(\sigma)$

Thm. $\left\{ \begin{array}{l} n\text{-dim WD reps} \\ \bar{\mathbb{Q}}_\ell \end{array} \right\} \xleftrightarrow{\text{bij.}} \left\{ \begin{array}{l} \text{cont. rep } W_k \rightarrow GL_n(\bar{\mathbb{Q}}_\ell) \\ \text{cont. rep } W_k \rightarrow GL(V) \end{array} \right.$
 $(r, N, V) \mapsto (\sigma \mapsto r(\sigma) \cdot \exp(t(\sigma)N) \in GL(V))$

Rem. $\left\{ \text{WD rep} / \bar{\mathbb{Q}}_\ell \right\} \simeq \left\{ \text{WD rep} / \mathbb{C} \right\}$. $t: I_k \rightarrow \mathbb{Z}_\ell$

ex. $Sp_n := (r, N, \langle e_1, \dots, e_n \rangle)$. $r(\sigma)e_i := \chi(\sigma)^{i-1}e_i$. $Ne_i := e_{i+1}$

Prop. If r : s.s. (we say Frob-s.s.), then $\exists n_i, r_i$.
 $(r, N) \cong \bigoplus_i Sp_{n_i} \otimes (r_i, 0, V_i)$ ($r_i: W_k \rightarrow GL(V_i)$: irred.)

ⓑ $\pi : GL_n(K) \rightarrow GL(V)$, $V: \Omega$ -vect. sp.

Def. $(\pi, V):$ smooth $\Leftrightarrow \forall v \in V. \exists U \subset GL_n(K):$ open cgt. s.t.
 $v \in V^U := \{v \in V \mid \pi(\sigma)v = v \ \forall \sigma \in U\}$.

Thm (LLC). ⓐ $\left\{ \text{Frob-s.s. } n\text{-dim. WD-rep of } W_K \right\} \xleftrightarrow{\text{bij.}} \left\{ \text{irred. sm. rep of } GL_n(K) \right\}$
 $r = (r.N.V) \leftrightarrow \pi$. ⓑ

ex. $n=1: \chi: W_K \rightarrow \mathbb{C}^\times \leftrightarrow \chi = \text{Art}_K. \text{Art}_K: K^\times \xrightarrow{\sim} W_K^{ab}$
 $r \otimes \chi \leftrightarrow \pi \otimes (\chi \circ \text{Art}_K \circ \det)$
 $r_1 \oplus r_2, r \mapsto Sp_n \otimes r \leftrightarrow \pi_1 \boxplus \pi_2. \pi \mapsto St_n(\pi)$
 $r: \text{irred} \leftrightarrow \pi: \text{cuspidal}$
 $\left\{ r: \text{unram. } (r|_{I_K} = \text{id. } N=0) \right\} \leftrightarrow \left\{ \pi: \text{spherical gen. by } \pi^U. U = GL_n(\mathcal{O}_K) \right\}$
 e.v. of $\text{Frob}_K \downarrow^2 \quad \left(\mathbb{C}^\times \right)^n / \mathfrak{S}_n \leftarrow \sim \left(\mathbb{C}[X_1^\pm, \dots, X_n^\pm] \right)^{\mathfrak{S}_n}$
 e.v. of $X_1, \dots, X_n \hookrightarrow \pi^U$ (Satake param.)

ii) global. $[L:\mathbb{Q}] < \infty. G_L := \text{Gal}(\bar{L}/L). v|\ell \Rightarrow [L_v:\mathbb{Q}_p] < \infty$

ⓐ Galois rep'n: $R: G_L \rightarrow GL_n(\bar{\mathbb{Q}}_l) \dots$ irred. cont. s.t.
 $R|_{G_{L_v}}: i) \text{ unram. for a.a. } v, ii) \text{ de Rham if } v|\ell.$

ⓑ Cuspidal Autom. Rep. $\Pi = \bigotimes_v \Pi_v$ of $GL_n(\mathbb{A}_L)$ in $L^2(GL_n(\mathbb{A}_L) \backslash GL_n(\mathbb{A}_L))_{\text{cusp}}$
 $\left(\Pi_v: \text{irred. sm. rep of } GL_n(L_v) \text{ if } v \nmid \ell \right) \xrightarrow{\text{st.}} \text{algebraic if } v|\ell$

Conj. (GLC). For $\forall v: \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}. \exists \text{bij. } \{R\} \leftrightarrow \{\Pi\}$
 s.t. $rR|_{W_{L_v}} \xleftrightarrow{\text{LLC}} \Pi_v. (\forall v \nmid \ell)$.

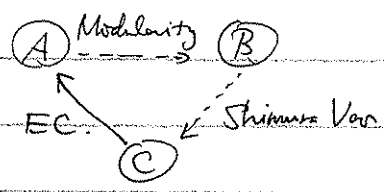
Rem: $R: \text{determined by } R|_{W_{L_v}} \text{ for a.a. } v$ (Chebotarev density).
 $\Pi: \longrightarrow \Pi_v \longrightarrow$ (Strong Multiplicity One).
 $\text{GLC} \Rightarrow \{ \ell\text{-adic Gal. rep. } \}: \text{indep. of } \ell.$

§3. i) global obj's are motivic

① Thm X : proper smooth var./ $L \rightsquigarrow R$: irred. subgt of $H^i(X \otimes \bar{L}, \bar{Q}_\ell)$
 (Étale Cohomology) $\Rightarrow R$: Gal. rep in the above sense $\hookrightarrow GL$

\therefore for a.a. $v. \exists \mathcal{X}/\mathcal{O}_{L_v}$: prop. sm. s.t. $X \otimes L_v \cong \mathcal{X} \otimes L_v$
 $H^i(X \otimes \bar{L}, \bar{Q}_\ell) \rightsquigarrow H^i(X \otimes \bar{L}_v, \bar{Q}_\ell) \rightsquigarrow H^i(\mathcal{X} \otimes \bar{K}_v, \bar{Q}_\ell)$
 $\hookrightarrow GL \leftarrow GL_v \longrightarrow GL_v \hookrightarrow \Rightarrow$ unram at $v.$
 p -adic Hodge theory \Rightarrow de Rham at $v \nmid \ell$

① \rightarrow ② (is motivic)
Conj. (Fontaine-Mazur) All R arises in this way up to Tate twist.
 [\Rightarrow Hodge symmetry for all R]



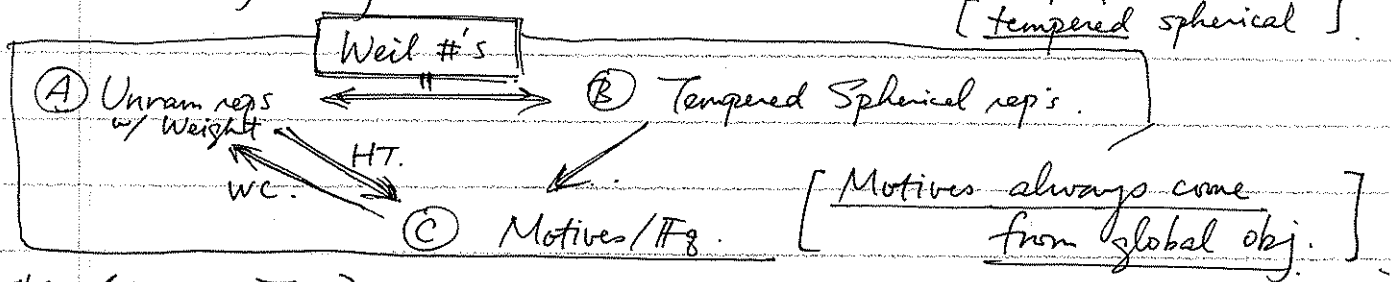
ii) ~~motivic~~ motivic obj's / \bar{F}_q

① Thm (Weil Conj.) $\ell \neq p. X/\bar{F}_q$: proper smooth
 $\Rightarrow \forall$ e.v. of $Frob_p \cap H^i(X \otimes \bar{F}_q, \bar{Q}_\ell)$ are Weil q^i -numbers.
 [i.e. $c \in \bar{Q}$ s.t. $\forall z: \bar{Q} \hookrightarrow \mathbb{C}, z(c), \bar{z}(c) = q^i$]

Rem. Conj... $Frob_p$: always s.s.

① Cor. (\exists Weight) R : motivic $\Rightarrow \exists i \in \mathbb{Z}$ for a.a. $v.$ e.v. of $Frob_v$ are R/W_{L_v}
 Weil q^i -#.

② Cor. (Ramanujan Conj.) Π : motivic $R \Rightarrow$ Satake param of Π_v
 [tempered spherical]

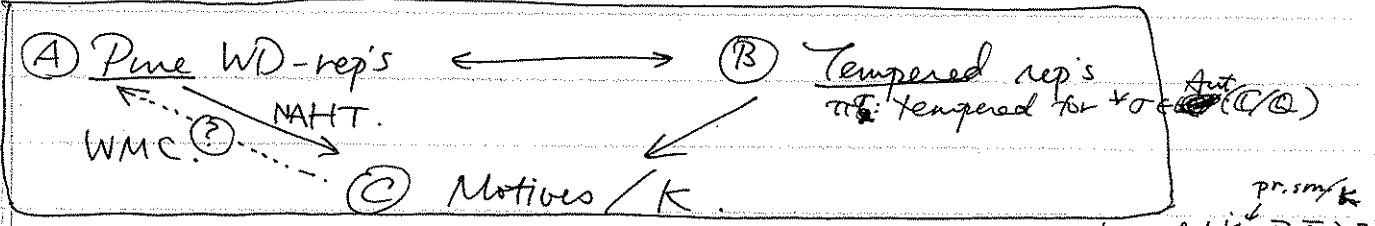


Thm. (Honda-Tate). All Weil #'s are motivic

\therefore realize ② in Π_v of Hecke ch Π /CM-field
 use ② \rightarrow ③ for GL_1 /CM-field. (CM abel var. = 0-dim Sh.V.)

iii) motivic local objects. $[K: \mathbb{Q}_p] < \infty$. $k \cong \mathbb{F}_q$.

- (A) Def. $(r, N) = \bigoplus \text{Spr}_q(r_i)$: pure of wt $i \in \mathbb{Z}$ if:
 - \forall e.v. of one (\Rightarrow any) lift of Frobk in r_i are Weil $q^{i+(n_j-1)/f.s.} \neq \#$.
- (C) Conj. (WMC) $l \neq p$. X/k : pr. sm. $\Rightarrow W_k \hookrightarrow H^i(X \otimes \bar{k}, \bar{\mathbb{Q}}_l)$: pure of wt i .
- (A) Conj. + (A) \rightarrow (C) $\forall R$. $\forall v$. $R/W_{l,v}$: pure.
- (B) Conj. + (B) \rightarrow (C) $\forall \pi$. $\forall v$. π_v : tempered. (Generalized Ramanujan Conj.)
 (\Leftrightarrow assoe. to pure (r, N)).



Thm (Non-abelian Honda-Tate). All pure WD reps are motivic. (up to F.s.s. fixation)

pf. $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$. can twist by (Weil #'s - Hecke char's by HT).
 $H^*(X \amalg Y) = H^*(X) \oplus H^*(Y)$. can reduce to $\text{Sp}_n \otimes r$.

(1) \Rightarrow go to (B). $\text{Sp}_n \otimes r \leftrightarrow \text{St}_n(\pi)$. realize this as π_v .
 π_v CM field. use (B) \rightarrow (C). Harris-Taylor. Taylor-Y.

(2) \Rightarrow $\text{Sp}_n \dots$ use p -adically uniformized var. (Ito)
 (some family w/ big monodromy)
 $r \dots$ go to (B). $r \leftrightarrow \pi$. use (B) \rightarrow (C). Harris-Taylor.

(3) \Rightarrow $r \dots$ direct summand of $\text{Ind}_{K'}^{W_K} \mathbb{1}$. up to twisting by char.
 $[K': K] < \infty$. $X = \text{Spec } K'$. (Honda-Tate).

(Taylor) Q: characterize geometric p -adic Gal. reps? descent. (alg. fields) \rightarrow (tensor cat.)
 (Mazur) l -indep notion of Gal. reps - funct. (top. alg. fields) \rightarrow (tensor cat.)