

Compatibility of local and global Langlands correspondences

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(joint work with Richard Taylor)

Let L be a number field (finite over \mathbb{Q}), n a positive integer, ℓ a fixed prime and $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ a fixed field isomorphism. We denote the absolute Galois group of L by $G_L = \text{Gal}(\overline{L}/L)$. Conjectural global Langlands correspondence predicts a correspondence between (A) algebraic automorphic representation Π of $GL_n(\mathbb{A}_L)$ and (B) n -dimensional ℓ -adic Galois representation $R : G_L \rightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ which is (1) almost everywhere unramified and (2) de Rham at ℓ . The cuspidals in (A) should correspond to irreducibles in (B). We are interested in cuspidal Π , and we denote the (conjectural) Galois representation attached to Π by $R = R_{\ell, \iota}(\Pi)$. Up to semisimplification, it is characterized by the property that the eigenvalues of Frobenius Frob_v equal the Satake parameters of Π_v for almost all places v . One of the most general results in the direction $\Pi \mapsto R$ is the one obtained by Kottwitz [K], Clozel [C] and Harris-Taylor [HT], which constructs the semisimple representation $R_{\ell, \iota}(\Pi)$ when L is an imaginary CM-field and Π is cuspidal, satisfying (1) Π is conjugate self-dual, (2) Π is regular algebraic and (3) there is a finite place x of L where Π_x is (essentially) square-integrable. We show the compatibility of this correspondence with the local Langlands correspondence at all places outside ℓ ; we need to fix notations for the local Langlands correspondence.

Let K be a finite extension of \mathbb{Q}_p and n a positive integer. We denote the maximal unramified extension of K by K^{ur} , the (geometric) Frobenius by Frob , and the Weil group by $W_K = \{\sigma \in G_K \mid \sigma|_{K^{ur}} \in \text{Frob}^{\mathbb{Z}}\}$. The local Langlands correspondence rec (proved by Harris-Taylor [HT] and Henniart [He]) gives the correspondence from (A') irreducible admissible representations (over \mathbb{C}) of $GL_n(K)$ to (B') n -dimensional F -semisimple Weil-Deligne representations (over \mathbb{C}) of W_K . Recall (see Tate [Ta]) that a Weil-Deligne representation is a pair (r, N) of a finite dimensional representation $r : W_K \rightarrow GL(V)$ and an $N \in \text{End}(V)$ such that $r(\sigma)N = \chi(\sigma)Nr(\sigma)$ for all $\sigma \in W_K$, where $\chi : W_K \rightarrow \mathbb{Q}^\times$ is the composite of the local reciprocity map $W_K \rightarrow W_K^{ab} \cong K^\times$ (sending lifts of Frob to uniformizers) and the normalized absolute value $||_K : K^\times \rightarrow \mathbb{Q}^\times$. We can define the F -semisimplification $r^{F\text{-ss}}$ of r , and write $(r, N)^{F\text{-ss}} = (r^{F\text{-ss}}, N)$ and $(r, N)^{\text{ss}} = (r^{F\text{-ss}}, 0)$. The cuspidal (resp. square integrable, tempered) representations in (A') correspond to irreducible (resp. indecomposable, pure) representations in (B'). For the definition of pure Weil-Deligne representations, see [TY]. For a prime ℓ and an ℓ -adic Galois representation ρ of G_K (assume de Rham when $\ell = p$), denote the associated Weil-Deligne representation (over $\overline{\mathbb{Q}}_\ell$) of W_K by $\text{WD}(\rho)$ (for $\ell \neq p$ use quasi-unipotence of Grothendieck; for $\ell = p$ use Berger's theorem and Fontaine's functor D_{pst}).

Theorem A. (Harris-Taylor [HT], Taylor-Yoshida [TY]) Let L be a CM-field, ℓ a prime and fix $\iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$. Let Π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ satisfying the three conditions above, and $R_{\ell, \iota}(\Pi)$ be the associated

ℓ -adic representation of G_L . Then, for all finite place v of L not dividing ℓ :

$$\iota\mathrm{WD}(R_{\ell,\iota}(\Pi)|_{G_{L_v}})^{F\text{-ss}} \cong \mathrm{rec}(\Pi_v^\vee \cdot |\det|_{\mathbb{K}^{\frac{1-n}{2}}})$$

as Weil-Deligne representations over \mathbb{C} of W_{L_v} .

Some remarks on the theorem:

- (1) As we assumed the existence of x where Π_x is square-integrable, we obtain the irreducibility of $R_{\ell,\iota}(\Pi)$ if x does not divide ℓ . This is because $R_{\ell,\iota}(\Pi)$ is semisimple by definition, and the theorem tells us that its restriction at x is indecomposable.
- (2) The equality of $(\)^{\mathrm{ss}}$ of both sides in the theorem was one of the main results of Harris-Taylor ([HT], Introduction, Theorem C). The new result in [TY] is the determination of the monodromy operator N .
- (3) In [TY], the theorem for v dividing ℓ is shown to follow from the functoriality of the p -adic weight spectral sequence of Mokrane [M].

We sketch the idea of proof of the theorem. First we note that the temperedness of Π_v is shown in [HT] (Introduction, Theorem C). Hence, by (Theorem A)^{ss}, it suffices to prove that the left hand side is pure (in particular, this follows from the Weight-Monodromy Conjecture). Using the global base change, we reduce to the case when Π_v has a fixed vector by the Iwahori subgroup $\mathrm{Iw}_n = \{g \in GL_n(\mathcal{O}_{L,v}) \mid g \bmod v \text{ is upper triangular}\}$. We descend Π to an automorphic representation π of a unitary group G which locally at v looks like GL_n and at infinity looks like $U(1, n-1) \times U(0, n)^{[L:\mathbb{Q}]/2-1}$. Then we realise $R_{\ell,\iota}(\Pi)$ in the cohomology of a Shimura variety X associated to G with Iwahori level structure at v . More precisely (assume the infinitesimal character to be trivial for simplicity), the representation $R_{\ell,\iota}(\Pi)$ appears inside the semisimplification of the π^p -isotypic component of $H^{n-1}(X, \overline{\mathbb{Q}}_\ell)$. We show that X has strictly semistable reduction at v with a nice moduli-theoretic definition of the strata of the special fiber (the reduction of X at v), and use the results of [HT] to compute the cohomology of these (smooth, projective) strata as a virtual $G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_v^{\mathbb{Z}}$ -module. This description and the temperedness of Π_v shows that the π^p -isotypic component of the cohomology of any strata is concentrated in the middle degree. This implies that the π^p -isotypic component of the Rapoport-Zink weight spectral sequence ([RZ], [S]) degenerates at E_1 , which shows that $\mathrm{WD}(H^{n-1}(X, \overline{\mathbb{Q}}_\ell)[\pi^p]|_{G_{L_v}})$ is pure.

In the special case that Π_v is a twist of a Steinberg representation and Π_∞ has trivial infinitesimal character, the above theorem presumably follows from the results of Ito [I]. After we had posted the first version of this paper, Boyer [B] has announced an alternative proof with presumably stronger results.

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