Global and Local Langlands Correspondences for GL_n

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1. Cyclotomic Theory

$G_F := \mathrm{Gal}(\overline{F}/F)$: absolute Galois group of a perfect field F,
$G_F^{\mathrm{ab}} = \mathrm{Gal}(F^{\mathrm{ab}}/F)$, where F^{ab} : maximal abelian extension of F in $\overline{F}$.
$F^{\mathrm{ab}} \supset F^{\mathrm{cyc}} := \bigcup_N F(\boldsymbol{\mu}_N)$, $\widehat{\mathbb{Z}} := \varprojlim_N \mathbb{Z}/N$.

Theorem 1 (A) : $\mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q}) \xrightarrow{\sim} \widehat{\mathbb{Z}}^\times$.
(B) : $G_{\mathbb{Q}}^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q})$, i.e. $\mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}^{\mathrm{cyc}}$.

- $\mathbb{A}^\infty := \widehat{\mathbb{Z}} \otimes \mathbb{Q}$, $\mathbb{A} := \mathbb{R} \times \mathbb{A}^\infty$ ... $\mathbb{Q}$-alg.
  $\Longrightarrow$ natural ring hom. $\widehat{\mathbb{Z}} \to \mathbb{A}$, $\mathbb{Q} \to \mathbb{A}$, $\mathbb{R} \to \mathbb{A}$.
  $\Longrightarrow$ $\widehat{\mathbb{Z}}^\times \mathbb{Q}^\times \mathbb{R}_{>0}^\times \xrightarrow{\sim} \mathbb{A}^\times$, $\widehat{\mathbb{Z}}^\times \xrightarrow{\sim} \mathbb{A}^\times/\mathbb{Q}^\times \mathbb{R}_{>0}^\times$.

- Chinese Remainder Th. $\Longrightarrow$ $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$, where $\mathbb{Z}_p := \varprojlim_m \mathbb{Z}/p^m$.

Local version. $\mathbb{Q}_p := \mathbb{Z}_p \otimes \mathbb{Q}$ ... a field (p-adic field).

- $\mathbb{Q}_p^{\mathrm{cyc}} = \mathbb{Q}_p^{\mathrm{ur}} \cdot \mathbb{Q}_p^{\mathrm{ram}}$, where $\begin{cases} \mathbb{Q}_p^{\mathrm{ur}} := \bigcup_{p \nmid N} \mathbb{Q}_p(\boldsymbol{\mu}_N), \\ \mathbb{Q}_p^{\mathrm{ram}} := \bigcup_m \mathbb{Q}_p(\boldsymbol{\mu}_{p^m}). \end{cases}$

- $$\begin{array}{ccccc} \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) & \xrightarrow{\sim} & \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) & \xrightarrow{\sim} & \widehat{\mathbb{Z}} \\ (\zeta \longmapsto \zeta^a) & \longmapsto & (\zeta \longmapsto \zeta^a) & & & (\zeta \in \boldsymbol{\mu}_N,\ p \nmid N) \\ \mathrm{Frob}_p^{-1} & \overset{\text{Def.}}{\longmapsto} & (\zeta \longmapsto \zeta^p) & \longmapsto & -1 \end{array}$$

- $$\begin{array}{ccc} \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ram}}/\mathbb{Q}_p) & \xrightarrow{\sim} & \mathbb{Z}_p^{\times} \\ (\zeta \longmapsto \zeta^{a \bmod p^m}) & \longmapsto & a & & (\zeta \in \boldsymbol{\mu}_{p^m}) \end{array}$$

Rem. $\mathbb{Z}_p^{\times} \times \mathbb{Z} \ni (a, b) \overset{\cong}{\longmapsto} a \cdot p^b \in \mathbb{Q}_p^{\times}$.

Theorem 2 (A) The following is an isom. onto the preimage of $\mathbb{Z}$ under $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{cyc}}/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \cong \widehat{\mathbb{Z}}$:

$$\begin{array}{ccc} \mathbb{Q}_p^\times & \longrightarrow & \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{cyc}}/\mathbb{Q}_p). \\ a \cdot p^{-b} & \longmapsto & \begin{cases} \zeta \mapsto \zeta^{a \bmod p^m} & (\zeta \in \boldsymbol{\mu}_{p^m}) \\ \zeta \mapsto \zeta^{p^b} & (\zeta \in \boldsymbol{\mu}_N,\ p \nmid N) \end{cases} \end{array}$$

(B) $G_{\mathbb{Q}_p}^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{cyc}}/\mathbb{Q}_p)$, i.e. $\mathbb{Q}_p^{\mathrm{ab}} = \mathbb{Q}_p^{\mathrm{cyc}}$.

2. Class Field Theory

<u>Local CFT.</u> $K/\mathbb{Q}_p$: fin., $K^{\mathrm{ur}} := \bigcup_{p \nmid N} K(\mu_N)$.

$\mathcal{O}_K$: integral closure of $\mathbb{Z}_p$ in K ... DVR
$\mathfrak{m} = (\pi)$: its maximal ideal, π : <u>uniformizer</u>,
$k = \mathcal{O}_K/\mathfrak{m}$: residue field, $\cong \mathbb{F}_q$ where q : power of p.

$$\begin{array}{ccccccc}
\bullet\ \mathrm{Gal}(K^{\mathrm{ur}}/K) & \xrightarrow{\sim} & \mathrm{Gal}(\overline{k}/k) & \xrightarrow{\sim} & \widehat{\mathbb{Z}} & & \\
(\zeta \longmapsto \zeta^a) & \longmapsto & (\zeta \longmapsto \zeta^a) & & & & (\zeta \in \boldsymbol{\mu}_N,\ p \not| N) \\
\mathrm{Frob}_K^{-1} & \stackrel{\text{Def.}}{\longmapsto} & (\zeta \longmapsto \zeta^q) & \longmapsto & -1 & &
\end{array}$$

Def. <u>The Weil group</u> $W_K :=$ the preimage of $\mathbb{Z}$ under the surj. $G_K \to \mathrm{Gal}(K^{\mathrm{ur}}/K) \cong \widehat{\mathbb{Z}}$. Its kernel $I_K \ldots$ <u>the inertia group</u>.

Theorem 3 (<u>LCFT</u>) $\exists$ unique isom. $\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}} \subset G_K^{\mathrm{ab}}$ s.t.
(1) $\forall \pi$: unif. $\Longrightarrow \mathrm{Art}_K(\pi)|_{K^{\mathrm{ur}}} = \mathrm{Frob}_K$.
(2) $\forall K'/K$: fin. abel. $\Longrightarrow \forall x \in N_{K'/K}(K'^\times),\ \mathrm{Art}_K(x)|_{K'} = \mathrm{id}$.

Rem. $$\begin{array}{ccc}
\{\text{char. } K^\times \longrightarrow \mathbb{C}^\times\} & \stackrel{1:1}{\longleftrightarrow} & \{\text{char. } W_K \longrightarrow \mathbb{C}^\times\}. \\
\chi & \longmapsto & \mathrm{rec}(\chi) := \chi \circ \mathrm{Art}_K^{-1}
\end{array}$$

Global CFT. $L/\mathbb{Q}$: fin., $\mathbb{A}_L := \mathbb{A} \otimes_{\mathbb{Q}} L$, $L_\infty := \mathbb{R} \otimes_{\mathbb{Q}} L$
$L_\infty^\times \supset L_\infty^{>0}$: connected component of 1 (totally pos. elts).

Theorem 4 (GCFT) The product of local Artin maps

$$\mathrm{Art}_L := \prod_v \mathrm{Art}_{L_v} : \mathbb{A}_L^\times \longrightarrow G_L^{\mathrm{ab}}$$

induces the isom. $\mathrm{Art}_L : \mathbb{A}_L^\times \Big/ \overline{L^\times L_\infty^{>0}} \xrightarrow{\sim} G_L^{\mathrm{ab}}$.

Rem. $$\begin{array}{ccc} \{\text{char. } \mathbb{A}_L^\times / L^\times \longrightarrow \mathbb{C}^\times\} & \overset{1:1}{\longleftrightarrow} & \{\text{char. } G_L \longrightarrow \mathbb{C}^\times\}. \\ \chi & \longmapsto & R(\chi) := \chi \circ \mathrm{Art}_L^{-1} \end{array}$$

- Global Langlands correspondence (roughly):

$$\{\text{automorphic rep. of } GL_n(\mathbb{A}_L)\} \overset{1:1}{\longleftrightarrow} \{n\text{-dim. rep. of } G_L\}.$$

3. Langlands Correspondence

Local LC. $K/\mathbb{Q}_p$: fin.

Def. A Weil-Deligne rep. (r, N) of W_K over $\mathbb{C}$ (or $\overline{\mathbb{Q}}_\ell$) : a pair of $r : W_K \longrightarrow GL(V)$ ($\dim_{\mathbb{C}} V < \infty$) and $N \in \mathrm{End}(V)$ s.t. $\forall \sigma \in W_K$:

$$r(\sigma) N r(\sigma)^{-1} = q^{-b} N \quad \text{if } \sigma|_{K^{\mathrm{ur}}} = \mathrm{Frob}_K^b.$$

(r, N) is called Frob.-s.s. if r is semisimple. $(r, N)^{F\text{-ss}} := (r^{\mathrm{ss}}, N)$.

- For a prime ℓ, $\exists$ a corres. (bij. if $\ell \neq p$):

$$\begin{array}{ccc} \{\ n\text{-dim. } \ell\text{-adic rep.}/\overline{\mathbb{Q}}_\ell \text{ of } W_K\} & \longrightarrow & \{\ \text{WD-rep.}/\overline{\mathbb{Q}}_\ell \text{ of } W_K\}. \\ \rho & \longmapsto & \mathrm{WD}(\rho) \end{array}$$

For $\ell = p$, assume ρ : de Rham, then WD is Fontaine's D_{pst}.

Theorem 5 (<u>LLC</u>) $\exists$ a bij.

$$\{\text{ irred. adm. rep.}/\mathbb{C} \text{ of } GL_n(K)\} \xleftrightarrow{1:1} \{\ n\text{-dim. F-s.s. WD-rep.}/\mathbb{C} \text{ of } W_K\},$$
$$\pi \longmapsto \mathrm{rec}(\pi)$$

characterized by certain properties. (LCFT if $n = 1$.)

<u>Unramified LLC.</u> (r, N) : <u>unramified</u> if $r|_{I_K} = 1$ and $N = 0$.

$$\{\text{ unram. principal series }\} \xleftrightarrow{1:1} \{\text{ unram. F-s.s. WD-rep. }\},$$
$$\pi \longmapsto (r, 0)$$

is given by:

$$\{\text{ Satake parameters of } \pi\ \} = \{\text{ eigenvalues of } r(\mathrm{Frob}_K)\ \}.$$

Global LC. $L/\mathbb{Q}$: fin.

Conjecture 6 (GLC) Fix ℓ : prime and $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. $\exists$ a bij.

$$\{ \text{ alg. autom. rep. of } GL_n(\mathbb{A}_L) \} \stackrel{1:1}{\longleftrightarrow} \{ n\text{-dim. } \ell\text{-adic rep. of } G_L \},$$
$$\Pi \longmapsto R_{\ell,\iota}(\Pi)$$

where ℓ-adic rep. $R : G_L \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ is assumed to be (i) de Rham at ℓ, (ii) $R|_{W_{L_v}}$ is unram. for almost all v.
When $\Pi = \bigotimes_v \Pi_v$, it is characterized by the compatibility

$$\iota \mathrm{WD}(R_{\ell,\iota}(\Pi)|_{W_{L_v}})^{F\text{-ss}} = \mathrm{rec}\Big(\Pi_v^\vee \otimes |\det|^{\frac{1-n}{2}}\Big) \tag{1}$$

for almost all v.

Rem. • For a.a. v, Π_v is unram. principal series.
• GCFT if $n = 1$.

4. Geometric Constructions of (A)

Lubin-Tate theory (LCFT). $K/\mathbb{Q}_p$: fin., π: unif.
Note that $\mathcal{O}_K^\times \times \mathbb{Z} \ni (u,b) \overset{\cong}{\longmapsto} u \cdot \pi^{-b} \in K^\times$.

• $\exists F_\pi$: 1-dim. $\mathcal{O}_K$-formal mod./$\mathcal{O}_K$ of ht 1.
Let $K_\pi^{\mathsf{ram}} := \bigcup_m K\Big(F_\pi[\pi^m]\Big)$, $\quad K_\pi^{\mathsf{LT}} := K^{\mathsf{ur}} \cdot K_\pi^{\mathsf{ram}}$.

Theorem 7 (B) $G_K^{\mathsf{ab}} \xrightarrow{\sim} \mathsf{Gal}(K_\pi^{\mathsf{LT}}/K)$, i.e. $K^{\mathsf{ab}} = K_\pi^{\mathsf{LT}}$.
(A) $\mathsf{Art}_K : K^\times \longrightarrow W_K^{\mathsf{ab}} \subset G_K^{\mathsf{ab}}$ is described by:

$$\begin{array}{ccc} K^\times & \longrightarrow & \mathsf{Gal}(K_\pi^{\mathsf{LT}}/K). \\ u \cdot \pi^{-b} & \longmapsto & \begin{cases} \zeta \mapsto [u \bmod \pi^m] \cdot \zeta & (\zeta \in F_\pi[\pi^m]) \\ \zeta \mapsto \zeta^{q^b} & (\zeta \in \boldsymbol{\mu}_N,\ p \nmid N) \end{cases} \end{array}$$

<u>**Complex Multiplication**</u> (GCFT). $L = \mathbb{Q}(\sqrt{-d})/\mathbb{Q}$: imag. quad., $\mathcal{O}_L$: integral closure of $\mathbb{Z}$ in L. Fix $\tau : L \subset \mathbb{C}$.

• $\exists E$: elliptic curve/$\mathbb{C}$ with $\mathcal{O}_L$-action, s.t. induced action on $\mathrm{Lie}\, E$ is compatible with τ.

Theorem 8 (B) $L(j(E))$ is the HCF of L, and $L^{\mathrm{ab}} = \bigcup_N L(j(E))[E[N]]$.
(A) Art_{L_v} is described by the LT theory of $E[v^\infty]$.

Rem. (A) for L : CM-field ... Taniyama-Shimura theory.

Non-abelian LT theory (Harris-Taylor). $K/\mathbb{Q}_p$: fin. The LLC rec is realized in the ℓ-adic vanishing cycle cohomology $H^{n-1}_{\text{et}}(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$ of the deformation space X of 1-dim. $\mathcal{O}_K$-formal module/$\overline{k}$ of ht n.

Non-abelian CM theory (Kottwitz, Clozel). $L/\mathbb{Q}$: CM-field. For certain cuspidal Π, the GLC $R_{\ell,\iota}(\Pi)$ is realized in the ℓ-adic étale cohomology $H^{n-1}_{\text{et}}(X_{\overline{L}}, \overline{\mathbb{Q}}_\ell)$ of an $(n-1)$-dim. unitary Shimura var./L, i.e. the moduli space of $n[L:\mathbb{Q}]/2$-dim. abelian var. with $\mathcal{O}_L$-action.

Theorem 9 (Harris-Taylor, Taylor-Y) For these Π, the compatibility (1) holds for all $v \not| \ell$.