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Non-abelian Lubin-Tate theory and Deligne-Lusztig theory

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Let K be any finite extension of the p-adic field $\mathbb{Q}_p$ for a prime number p. The central object studied in the algebraic number theory of K is the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. For example the local class field theory for K gives a canonical homomorphism

$$K^\times \longrightarrow G_K^{ab}$$

where G_K^{ab} is the maximal abelian quotient of G_K. The non-abelian generalization of the local class field theory, aiming at the understanding of the whole of G_K, not only G_K^{ab}, is called the *local Langlands correspondence*. It asserts that for any positive integer n there exists an one-to-one correspondence

$$\{\text{irreducible admissible representations of } GL_n(K)\} \longleftrightarrow \{n\text{-dim. representations of } G_K\}$$

(the case where $n = 1$ is nothing but the local class field theory). This correspondence has been an important conjecture in this field for a long time, but it was eventually proven by Harris-Taylor [HT] in 1999. There are conjectures regarding the representations of $G(K)$ for general reductive algebraic group G over K (not only GL_n), which still remain to be solved.

This solution of the local Langlands conjecture involved a highly sophisticated technique of the arithmetic geometry of Shimura varieties and the automorphic representations over global fields. This is natural because, as in the case of the class field theory, the local Langlands correspondence is the "local version" of the *global Langlands correspondence*, namely the non-abelian generalization of the global class field theory of algebraic number fields. Just as when the proof of the local class field theory was first deduced from the global class field theory, at present the proof of the local Langlands correspondence relies heavily on the partially established theory of the global Langlands correspondence. Here the global correspondence is realized in the etale cohomology groups of unitary Shimura varieties, which is the generalization of the realizations of the global class field theory using complex multiplications. Consequently, they could prove that the local Langlands correspondence is realized in the etale cohomology groups of the Shimura varieties over the local field K.

Therefore in the present situation much more extensive study is wanting for the arithmetic geometry concerned with the local Langlands correspondence, in particular the thorough understanding of the reason why it is realized in the cohomology groups of Shimura varieties over K (see [H] for further questions). One important goal is the purely local proof of the so-called *non-abelian Lubin-Tate theory* formulated by Carayol[Car], which asserts that the local Langlands correspondence for GL_n is realized in the vanishing cycle cohomology of the deformation spaces of the formal O_K-module with Drinfeld level structures, which are the local rings of the integral model of the Shimura variety used in the theory of Harris-Taylor. This actually can be deduced from the work of Harris-Taylor, but it does not give the local understanding. Our result obtained so far in this direction ([Y]) is a purely local proof of the non-abelian Lubin-Tate theory in a special case, in which we can show that it is geometrically equivalent to the theory of Deligne-Lusztig concerning the representations of reductive groups over finite fields.

Now denote the integer ring of K by O_K and residue field by $k \cong \mathbb{F}_q$. Fix a uniformizer π of O_K and a positive integer $n \geq 1$. Let X be the spectrum of the deformation ring of formal O_K-module of height n with level π structure ([Dr]), which is a scheme of relative dimension $n-1$ over the integer ring $W = \widehat{O_K^{\mathrm{ur}}}$ of the completed maximal unramified extension $\widehat{K^{\mathrm{ur}}}$ of K, which is a complete DVR with the residue field $\overline{k}$. We are interested in ℓ-adic etale cohomology groups $H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$ ($\ell \neq \mathrm{char}\, k$) of the geometric generic fiber $X_{\overline{\eta}} = X \times_{\mathrm{Spec}\, W} \mathrm{Spec}\, \overline{\widehat{K^{\mathrm{ur}}}}$, which are finite dimensional $GL_n(k) \times I_K$-modules, where I_K is the inertia group of K.

On the other hand, let DL be the Deligne-Lusztig variety over $\overline{k}$ for $GL_n(k)$, associated to the element $(1, \ldots, n)$ of the Weyl group of GL_n regarded as the symmetric group of n letters, or equivalently to a non-split torus T with $T(k) \cong k_n^\times$ where k_n is the extension of k of degree n ([DL]). As this variety has an action of $GL_n(k)$ and $T(k) \cong k_n^\times$, we can regard $H_c^i(DL, \overline{\mathbb{Q}}_\ell)$ as a $GL_n(k) \times I_K$-module by the canonical surjection $I_K \to k_n^\times$. We denote the alternating sums of these cohomology groups as follows:

$$H^*(X_{\overline{\eta}}) = \sum_i (-1)^i [H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)], \quad H^*(DL) = \sum_i (-1)^i [H_c^i(DL, \overline{\mathbb{Q}}_\ell)]$$

which are regarded as elements of the Grothendieck group of $GL_n(k) \times I_K$-modules. Then our main theorem on the vanishing cycle cohomology groups of X can be stated as follows:

Theorem 1. (i) *We have the equality* $H^*(X_{\overline{\eta}}) = H^*(DL)$.

(ii) *Among the* $H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$, *cuspidal representations* π *of* $GL_n(k)$ *and generic inertia characters* χ *of* I_K *(here* generic *means* χ *does not factor through any* $k_m^\times$ *with* $m \mid n$, $m < n$ *via the norm map* $k_n^\times \to k_m^\times$*) occur only in* $H^{n-1}(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$, *where they are coupled as* $\bigoplus \pi_\chi \otimes \chi$ *by the usual correspondence* $\pi_\chi \otimes \mathrm{St} = \mathrm{Ind}_{T(k)}^{GL_n(k)} \chi$ *where* St *is the Steinberg representation of* $GL_n(k)$.

This theorem gives the local proof of the non-abelian Lubin-Tate theory in this particular case of level π, because of the following. The character χ of I_K that factors through $I_K \to k_n^\times$ can be extended to a character $\tilde{\chi}$ of G_L, where L is the unramified extension of K of degree n (N.B. $I_K = I_L$). If χ is generic, $\mathrm{Ind}_{G_L}^{G_K} \tilde{\chi}$ is an irreducible representation of G_K. On the other hand, we can pull back the cuspidal representation π_χ by the natural surjection $GL_n(O_K) \longrightarrow GL_n(k)$, and then induce (compact modulo center induction) to $GL_n(K)$ to obtain a supercuspidal representation of $GL_n(K)$ which corresponds to $\mathrm{Ind}_{G_L}^{G_K} \tilde{\chi}$ via the local Langlands correspondence.

References

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