STATEMENT OF PURPOSE

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Abstract. My research interest lies in the area of arithmetic geometry related to Langlands correspondence, where the theory of automorphic forms and algebraic geometry intersect. It is one of the most interesting and actively studied subject of algebraic number theory, whose origin goes back to the theory of complex multiplication. My research is currently focused on the local aspect of this theory, namely the realization of the local Langlands correspondence in the cohomology groups of algebraic varieties over local fields.

1. Introduction

First we briefly give the overview of the theory of local Langlands correspondence for GL_n over local fields, in order to clarify the backgrounds of my research. Let K be any finite extension of the p-adic field $\mathbb{Q}_p$ for a prime number p. The central object studied in the algebraic number theory of K is the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. For example the local class field theory for K asserts that there is a canonical homomorphism from $K^\times$, the multiplicative group of K, into the maximal abelian quotient G_K^{ab} of G_K. The non-abelian generalization of the local class field theory, aiming at the understanding of the whole of G_K, not only G_K^{ab}, is called the local Langlands correspondence. It asserts that for any positive integer n the n-dimensional representations of G_K are in one-to-one correspondence with the irreducible admissible representations of $GL_n(K)$ (the case where $n = 1$ is nothing but the local class field theory). This correspondence has been an important conjecture in this field for a long time, but it was eventually proven by Harris-Taylor [HT] in 1999. There are conjectures regarding the representations of $G(K)$ for general reductive algebraic groups over K (not only GL_n), which still remain to be solved.

This solution of the local Langlands conjecture involved a highly sophisticated technique of the arithmetic geometry of Shimura varieties and the automorphic representations over global fields. This is natural because, as in the case of the class field theory, the local Langlands correspondence is the "local version" of the global Langlands correspondence, namely the non-abelian generalization of the global class field theory of algebraic number fields. This theory tries to understand the representations of the absolute Galois group over global fields by establishing a correspondence with an analytic object called automorphic representations, which are generalizations of the

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characters of ideal class groups. Just as when the proof of the local class field theory was first deduced from the global class field theory, at present the proof of the local Langlands correspondence relies heavily on the partially established theory of the global Langlands correspondence. Here the global correspondence is realized in the etale cohomology groups of unitary Shimura varieties, which is the generalization of the realizations of the global class field theory by complex multiplications. Consequently, they could prove that the local Langlands correspondence is realized in the etale cohomology groups of the Shimura varieties over the local field K.

Therefore in the present situation much more extensive study is wanting for the arithmetic geometry concerned with the local Langlands correspondence, in particular the thorough understanding of the reason why it is realized in the cohomology groups of Shimura varieties over K. One important goal is the purely local proof of the so-called "non-abelian Lubin-Tate theory" formulated by Carayol[Car], which asserts that the local Langlands correspondence for GL_n is realized in the vanishing cycle cohomology of the deformation spaces of the formal O_K-module with Drinfeld level structures, which are the local rings of the integral model of the Shimura variety used in the theory of Harris-Taylor. This actually can be deduced from the work of Harris-Taylor, but it does not give the local understanding. I have so far obtained a result in this direction, proving in a special case that we can locally understand why this non-abelian Lubin-Tate theory holds via the theory of Deligne-Lusztig concerning the representations of reductive groups over finite fields.

2. Results

Here I explain the result obtained so far, in some detail. Let K be a finite extension of $\mathbb{Q}_p$, with integer ring O_K and residue field $k \cong \mathbb{F}_q$. Fix a uniformizer π of O_K and a positive integer $n \geq 1$. Let X be the spectrum of the deformation ring of formal O_K-module of height n with level π structure ([Dr]), which is a scheme of relative dimension $n-1$ over the integer ring $W = \widehat{O_K^{\mathrm{ur}}}$ of the completed maximal unramified extension $\widehat{K^{\mathrm{ur}}}$ of K, which is a complete DVR with the residue field $\overline{k}$. We are interested in ℓ-adic etale cohomology groups $H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$ ($\ell \neq \operatorname{char} k$) of the geometric generic fiber $X_{\overline{\eta}} = X \times_{\operatorname{Spec} W} \operatorname{Spec} \overline{\widehat{K^{\mathrm{ur}}}}$, which are finite dimensional $GL_n(k) \times I_K$-modules, where I_K is the inertia group of K.

On the other hand, let DL be the Deligne-Lusztig variety over $\overline{k}$ for $GL_n(k)$, associated to the element $(1, \ldots, n)$ of the Weyl group of GL_n regarded as the symmetric group of n letters, or equivalently to a non-split torus T with $T(k) \cong k_n^\times$ where k_n is the extension of k of degree n ([DL]). As this variety has an action of $GL_n(k)$ and $T(k) \cong k_n^\times$, we can regard $H_c^i(DL, \overline{\mathbb{Q}}_\ell)$ as a $GL_n(k) \times I_K$-module by the canonical surjection $I_K \to k_n^\times$. We denote the alternating sums of these cohomology groups as follows:

$$H^*(X_{\overline{\eta}}) = \sum_i (-1)^i [H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)], \quad H^*(DL) = \sum_i (-1)^i [H_c^i(DL, \overline{\mathbb{Q}}_\ell)]$$

which are regarded as elements of the Grothendieck group of $GL_n(k) \times I_K$-modules. Then our main theorem on the vanishing cycle cohomology groups of X can be stated as follows:

Theorem 2.1. (i) *We have the equality* $H^*(X_{\overline{\eta}}) = H^*(DL)$.

(ii) *Among the* $H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$, *cuspidal representations* π *of* $GL_n(k)$ *and generic inertia characters* χ *of* I_K *(here* generic *means* χ *does not factor through any* $k_m^\times$ *with* $m \mid n,\ m < n$ *via the norm map* $k_n^\times \to k_m^\times$*) occur only in* $H^{n-1}(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$, *where they are coupled as* $\bigoplus \pi_\chi \otimes \chi$ *by the usual correspondence* $\pi_\chi \otimes \mathrm{St} = \mathrm{Ind}_{T(k)}^{GL_n(k)} \chi$ *where* St *is the Steinberg representation of* $GL_n(k)$.

These results are proved by purely local arguments, in which we construct a suitable model of X and compute the cohomology of the geometric generic fiber in terms of that of the special fiber. In its course we obtain important informations concerning the local deformation space X as the following:

Theorem 2.2. (i) *The* W*-scheme* X *is isomorphic to*

$$\mathrm{Spec}\, W[[X_1, \dots, X_n]]/(P(X_1, \dots, X_n) - \pi),$$

where $P \in W[[X_1, \dots, X_n]]$ *is of the form:*

$$(\mathrm{unit}) \cdot \prod_{(a_i \bmod \pi)_i \in k^n \setminus \{\underline{0}\}} \big([a_1](X_1) +_{\widetilde{\Sigma}} \cdots +_{\widetilde{\Sigma}} [a_n](X_n)\big)$$

where $[a_i]$ *and* $+_{\widetilde{\Sigma}}$ *denote the formal* O_K*-multiplication and addition of a formal* O_K*-module over* $W[[X_1, \dots, X_n]]$ *obtained by lifting the universal formal* O_K*-module over* X.

(ii) *There exists a generalized semistable model* Z_{st} *of* X *over* W, *i.e. a proper* W*-morphism* $Z_{st} \to X$ *which is an isomorphism on the generic fibers and* Z_{st} *being generalized semistable. Here generalized semistable means that its complete local rings at all the closed points are isomorphic over* W *to*

$$W[[T_1, \dots, T_n]]/(T_1^{e_1} \cdots T_d^{e_d} - \pi) \quad (d \le n)$$

with integers e_i *all prime to* $\mathrm{char}\, k$.

(iii) *Over the tamely ramified extension* $W_n = W(\pi^{1/(q^n-1)})$ *of* W, X *has a model whose special fiber contains a smooth affine variety over* $\overline{k}$ *which is isomorphic to* DL *as schemes with right* $GL_n(k) \times I_K$*-action.*

3. Further research

The above result is only the beginning of the vast area of research that is to be explored in this field. Here I will point out a few of the concrete directions of the further research that I am planning to study in the coming years.

(1) On the above result: the intrinsic reason why we see the Deligne-Lusztig variety inside a certain model of the deformation space of formal modules is to be

clarified by giving a suitable moduli-theoretic interpretation of these models. These are expected to be obtained by illuminating the interplay between the Drinfeld level structures of formal modules and the theory of Dieudonne crystals associated to formal modules.

(2) The generalization to higher level: the above result should be naturally extended to the case of the deformation space with level π^k-structures for $k > 1$, eventually covering the whole of the local Langlands correspondence for GL_n. The case $k > 1$ is known to be representation-theoretically entirely different from the case $k = 1$, and it is expected to have a simple geometric interpretation. For this purpose, the moduli theoretic interpretation alluded to in (1) should become important.

(3) The generalization to other groups: as the Deligne-Lusztig theory exists for the reductive groups other than GL_n, the corresponding representations of the groups over local fields should be realized in the cohomology groups of a certain moduli spaces of formal groups, e.g. some variations of so-called Rapoport-Zink spaces. This would be an important breakthrough towards establishing the local Langlands correspondences for the general reductive groups.

(4) The ϵ-factor: the local Langlands correspondence is formulated in terms of L-factors and ϵ-factors defined on the both representation-theoretic and Galois-theoretic sides. Not being content with only realizing the correspondence via arithmetic geometry, we should clarify the coincidence of ϵ-factors in the local Langlands correspondence via arithmetic geometry.

These are only a few of the possibilities of study that are open in this area, and my interest covers a wider topic of understanding the arithmetic geometry of Shimura varieties. I believe that this is one of the most fundamental area of research in algebraic number theory, where its firm understanding will be essential for the future developments of number theory.

References

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