FINITENESS THEOREMS IN THE CLASS FIELD THEORY OF VARIETIES OVER LOCAL FIELDS

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Abstract. We show that the geometric part of the abelian étale fundamental group of a proper smooth variety over a local field is finitely generated over $\widehat{\mathbb{Z}}$ with finite torsion, and describe its rank by the special fiber of the Néron model of the Albanese variety. As an application, we complete the class field theory of curves over local fields developed by S. Bloch and S. Saito, in which the theorem concerning the p-primary part in positive characteristic case has remained unproven.

Contents

1. Introduction

For a proper smooth geometrically irreducible variety X over a field K, $\pi_1^{ab}(X)$ is the maximal abelian quotient of the étale fundamental group $\pi_1(X)$ classifying finite étale coverings of X ([SGA1]). There is a natural surjection $\pi_1^{ab}(X) \to G_K^{ab}$ where $G_K^{ab} = \mathrm{Gal}(K_{ab}/K)$ is the Galois group of the maximal abelian extension of K, and denote the kernel by $\pi_1^{ab}(X)^{geo}$:

$$0 \longrightarrow \pi_1^{ab}(X)^{geo} \longrightarrow \pi_1^{ab}(X) \longrightarrow G_K^{ab} \longrightarrow 0 \tag{1.1}$$

When X has a K-rational point x, $\pi_1^{ab}(X)^{geo}$ has a geometric interpretation as the group classifying the abelian finite étale coverings of X in which x splits completely.

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For example, Th. 1 of [5] shows that $\pi_1^{ab}(X)^{geo}$ is a finite group when K is an absolutely finitely generated field of characteristic zero.

In this paper, we are interested in the finiteness of the abelian group $\pi_1^{ab}(X)^{geo}$ in the case where K is a *local field*, i.e. a complete discrete valuation field with finite residue field. Our main result is follows :

Theorem 1.1. *Let K be a complete discrete valuation field with finite residue field k, and X a proper smooth geometrically irreducible variety over K. Then $\pi_1^{ab}(X)^{geo}$ has the following structure :*

$$0 \longrightarrow \pi_1^{ab}(X)^{geo}_{tor} \longrightarrow \pi_1^{ab}(X)^{geo} \longrightarrow \widehat{\mathbb{Z}}^r \longrightarrow 0$$

where $\pi_1^{ab}(X)^{geo}_{tor}$ is a finite group, and r is the k-rank of the special fiber of the Néron model of the Albanese variety of X.

Here the *k-rank* of a linear algebraic group over k is the dimension of the maximal k-split subtorus. In particular, we have :

Corollary 1.2. *When X has potentially good reduction, i.e. has a proper smooth model over the integer ring of some finite extension of K, $\pi_1^{ab}(X)^{geo}$ is finite.*

The case $\dim X = 1$ has been considered in the literature. Bloch treated the case of a curve X with good reduction when $\operatorname{char} K = 0$ in [1] Prop. 2.4 (whose proof is attributed to N. Katz), and S. Saito shows our theorem for a general curve X except for the p-primary part in the $\operatorname{char} K = p > 0$ case in [7], Section II-4. The result concerning the remaining p-primary part had been conjectured by Saito ([7], Remark 4.2 of Section II), and our theorem in the case of curves answers this question affirmatively. This enables us to complete the class field theory of curves over local fields developed in [7] in the positive characteristic case (see below).

Also, the author learned after writing the preliminary version of this paper that the higher dimensional case when $\operatorname{char} K = 0$ has been partially treated in Chapter 4 of Raskind[6], and there are related results in Salberger[8], §2.

The method of Bloch [1] employs in particular Tate's theorem on p-divisible groups, and the method of Saito [7] depends on the two-dimensional class field theory. Our approach is a direct generalization of Bloch's method, and we investigate $\pi_1^{ab}(X)^{geo}$ directly by the Tate module of Albanese variety, independently of class field theory. The main technical tool is the theory of the monodromy-weight filtration of degenerating abelian varieties on local fields ([SGA7]), and the recent result of de Jong [3] which removes the condition on $\operatorname{char} K$ in the Tate's theorem on p-divisible groups.

In the final section, we complete the proof of the main theorem of class field theory of curves over local fields of Saito [7], which is stated as follows (For the definition of $V(X)$, see Section 5) :

Theorem 1.3. *Let X be a proper smooth geometrically irreducible curve over a local field K, and denote the maximal divisible subgroup of $V(X)$ by D. Then the reciprocity* *map τ induces an isomorphism of finite groups :*

$$V(X)/D \xrightarrow{\cong} \pi_1^{ab}(X)^{geo}_{tor}$$

Here only the p-primary part of $\pi_1^{ab}(X)^{geo}$ in the char $K = p > 0$ case was remaining, where our finiteness result was the only missing ingredient in the proof.

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Notations. Throughout this paper, K denotes a complete discrete valuation field with residue field k, with $\operatorname{char} k = p > 0$. O_K is the integer ring of K. For any field K, $\overline{K}$ is a separable closure of K, and $G_K = \mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of K. For a variety X over K, $X_{\overline{K}} = X \times_K \mathrm{Spec}(\overline{K})$. $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ is the profinite completion of $\mathbb{Z}$. The term *p-divisible group* refers to a Barsotti-Tate group, and is usually denoted using the corresponding *pro-p* group scheme.

2. Review of the monodromy-weight filtration

Here we review Grothendieck's theory of the monodromy-weight filtration on Tate module of abelian varieties, following [SGA7], Exposé IX, and fix the notations. In this section the residue field k can be an arbitrary perfect field.

2.1. **Raynaud groups.** Let A be an abelian variety over K, and consider its Néron model $\mathscr{A}$, which is a smooth group scheme over O_K of finite type with A as its generic fiber. The special fiber A_k of $\mathscr{A}$ is an extension of the component group Φ by the connected component A_k^0:

$$0 \longrightarrow A_k^0 \longrightarrow A_k \longrightarrow \Phi \longrightarrow 0$$

The connected component $\mathscr{A}^0$ is a group scheme with A as generic fiber and A_k^0 as special fiber.

We assume throughout this section that A has *semistable reduction*, i.e. A_k^0 is an extension of an abelian variety B_k by a torus T_k :

$$0 \longrightarrow T_k \longrightarrow A_k^0 \longrightarrow B_k \longrightarrow 0$$

The *Raynaud group* $A^\natural$ of A is a smooth group scheme over O_K of finite type whose connected component $A^{\natural 0}$ is an extension of an abelian scheme B by an isotrivial torus (i.e. a multiplicative type group scheme of finite type which splits after a finite étale base change) T, which is characterized by the property $\widehat{A^\natural} \cong \widehat{\mathscr{A}}$. Here the $\widehat{\cdot}$ denotes the formal completion along the special fiber. The quotient $A^\natural / A^{\natural 0}$ is a finite étale group scheme over O_K which we also denote by Φ :

$$0 \longrightarrow T \longrightarrow A^{\natural 0} \longrightarrow B \longrightarrow 0$$

$$0 \longrightarrow A^{\natural 0} \longrightarrow A^\natural \longrightarrow \Phi \longrightarrow 0$$

Note that B_k, T_k are the special fibers of B, T, respectively. We denote the character group of T by M, which is a group scheme over O_K étale locally isomorphic to $\mathbb{Z}^r$, where r is the dimension of T.

In fact, all we need are the ℓ-divisible groups coming from $A^\natural, T, B$, so introducing Raynaud groups is not absolutely necessary for our purposes, but it makes the exposition simpler.

2.2. **Tate realizations and the monodromy-weight filtration.** For any smooth group scheme X over O_K of finite type and any prime number ℓ, define the profinite (resp. pro-ℓ) group scheme $T(X)$ (resp. $T_\ell(X)$) by :

$$T(X) = \varprojlim_n X[n], \quad T_\ell(X) = \varprojlim_n X[\ell^n]$$

where $X[m]$ denotes the kernel of the multiplication-by-m map, which is a group scheme over O_K. We use the similar notations for group schemes over K or k. For $X = \mathbb{G}_m$, the multiplicative group scheme, $T(\mathbb{G}_m), T_\ell(\mathbb{G}_m)$ are written as $\widehat{\mathbb{Z}}(1), \mathbb{Z}_\ell(1)$ respectively.

Coming back to our case, the generic fiber of $T_\ell(\mathscr{A}^0)$ is the ℓ-adic Tate module $T_\ell(A)$ of A (a pro-ℓ group scheme over K). As $\mathscr{A}^0[m]$ is quasi-finite in our situation ([SGA7] Exposé IX, Lemme 2.2.1), there is a canonical and functorial decomposition $\mathscr{A}^0[m] = \mathscr{A}^0[m]^f \coprod \mathscr{A}^0[m]'$ where $\mathscr{A}^0[m]^f$ is finite flat over O_K and $\mathscr{A}^0[m]'$ has empty special fiber, we define *the fixed part* $T_\ell(\mathscr{A}^0)^f$ by:

$$T_\ell(\mathscr{A}^0)^f = \varprojlim_n \mathscr{A}^0[\ell^n]^f \subset T_\ell(\mathscr{A}^0)$$

Identifying finite groups over O_K with its formal completions, we have canonical isomorphisms :

$$T_\ell(\mathscr{A}^0)^f \cong T_\ell(\widehat{\mathscr{A}^0}) \cong T_\ell(\widehat{A^{\natural 0}}) \cong T_\ell(A^{\natural 0})$$

and it follows that $T_\ell(\mathscr{A}^0)^f$ is an ℓ-divisible group (Barsotti-Tate group) over O_K, in particular a smooth ℓ-adic sheaf if $\ell \neq p$.

Then we define *the toric part* $T_\ell(\mathscr{A}^0)^t \subset T_\ell(\mathscr{A}^0)^f$ by the subgroup scheme corresponding to $T_\ell(T) \subset T_\ell(A^{\natural 0})$ by the above isomorphism, and we have a filtration on $T_\ell(\mathscr{A}^0)$:

$$W_0 = T_\ell(\mathscr{A}^0) \supset W_{-1} = T_\ell(\mathscr{A}^0)^f \supset W_{-2} = T_\ell(\mathscr{A}^0)^t \supset W_{-3} = 0$$

with $\mathrm{Gr}^W_{-1} \cong T_\ell(B)$. Moreover, if we denote the dual $\mathrm{Hom}(M, \mathbb{Z})$ of the character group of M by $M^\vee$, $\mathrm{Gr}^W_{-2} = T_\ell(T) \cong M^\vee \otimes \mathbb{Z}_\ell(1)$.

To describe the remaining Gr_0^W-part, we introduce the dual abelian variety $A^* = \mathrm{Pic}^0(A)$ of A, and let $T^*, B^*, M^*, \mathscr{A}^*, \dots$ be the corresponding objects for A^*. The orthogonality theorem ([SGA7] Exposé IX, Th. 2.4 and (5.5.9)) states that the toric part $T_\ell(A)^t$ is perpendicular to the fixed part $T_\ell(A^*)^f$ of the dual A^* under the canonical pairing $T_\ell(A) \times T_\ell(A^*) \to \mathbb{Z}_\ell(1)$:

$$T_\ell(A)/T_\ell(A)^t \cong D(T_\ell(A^*)^f), \quad T_\ell(A)/T_\ell(A)^f \cong D(T_\ell(A^*)^t)$$

where D denotes the Cartier dual, i.e. $T_\ell(A)/T_\ell(A)^t$ is the generic fiber of the dual ℓ-divisible group $D(T_\ell(\mathscr{A}^{*0})^f)$, and $\mathrm{Gr}_0^W T_\ell(A) \cong M_K^* \otimes \mathbb{Z}_\ell$.

The corresponding filtration on the generic fiber $T(A)$ of $T(\mathscr{A}^0)$, i.e. the profinite group $T(A)$, is *the monodromy-weight filtration* (see [2], §10 for the treatment in the context of 1-motives) :

$$\begin{array}{lll} & W_0T(A) = T(A) & \mathrm{Gr}_0^W T(A) \cong M_K^* \otimes \widehat{\mathbb{Z}} \\ (2.1) & W_{-1}T(A) = T(A)^f & \mathrm{Gr}_{-1}^W T(A) \cong T(B_K) \\ & W_{-2}T(A) = T(T_K) & \mathrm{Gr}_{-2}^W T(A) \cong M_K^\vee \otimes \widehat{\mathbb{Z}}(1) \end{array}$$

where B_K, T_K are the generic fibers of B, T respectively. Note that the pro-ℓ part of each graded part Gr_i^W is realized as the generic fiber of the ℓ-divisible group over O_K, and if k is a finite field with q elements and $\ell \neq p$, the special fiber of Gr_i^W has weight i as an ℓ-adic G_k-representation (i.e. all the complex conjugates of the eigenvalues of the q-th power arithmetic Frobenius F have the complex absolute value $q^{-i/2}$).

3. Galois coinvariants of the Tate module

From this section, the residue field k is always a *finite field* with q elements.

Now, for an arbitrary abelian variety A over K, we want to analyze the Galois coinvariants of the Galois module obtained by taking the maximal étale quotient $T^{et}(A)$ of $T(A)$, i.e. $T^{et}(A)_{G_K}$. We write $T(X)_G = T^{et}(X)_{G_K}$ for any X/K, for simplicity. The goal of this section is the following :

Proposition 3.1. *For an abelian variety A over K, $T(A)_G$ has the following strucuture :*

$$0 \longrightarrow (T(A)_G)_{tor} \longrightarrow T(A)_G \longrightarrow \widehat{\mathbb{Z}}^r \longrightarrow 0$$

where $(T(A)_G)_{tor}$ is a finite group, and r is the k-rank of the special fiber A_k of the Néron model of A.

First we treat the case where A has semistable reduction (3.1,3.2), where we have the monodromy-weight filtration 2.1. As the functor $(-)^{et}$ taking the maximal étale quotient is exact, and the functor $(-)_{G_K}$ taking coinvariants of the étale part is right exact, we have an exact sequence of abelian groups :

$$(W_{-1}T(A))_G \longrightarrow T(A)_G \longrightarrow (\mathrm{Gr}_0^W T(A))_G \longrightarrow 0$$

We will treat $(W_{-1}T(A))_G$ and $(\mathrm{Gr}_0^W T(A))_G$ separately.

3.1. **The part W_{-1}.** Our proof of the finiteness of $(W_{-1}T(A))_G$, in which the most subtle part is the proof for pro-p part, relies on the celebrated theorem on p-divisible groups :

Theorem 3.2 (Tate[10], de Jong[3])**.** *Let Γ, Γ' be p-divisible groups over O_K, and Γ_K, Γ'_K their generic fibers. Then the natural restriction map* $\mathrm{Hom}(\Gamma, \Gamma') \longrightarrow \mathrm{Hom}(\Gamma_K, \Gamma'_K)$ *is bijective.*

Proposition 3.3. *$(W_{-1}T(A))_G$ is finite.*

Proof. We decompose $W_{-1}T(A)_G$ into :

$$T(T_K)_G \longrightarrow (W_{-1}T(A))_G \longrightarrow T(B_K)_G \longrightarrow 0$$

and show the finiteness of $T(T_K)_G$ and $T(B_K)_G$. Note that both $T(T_K)$ and $T(B_K)$ are generic fibers of the profinite group scheme on O_K, and the finiteness of both parts follows in exactly the same manner, following the argument of Bloch [1] Prop. 2.4. Decompose them into pro-p part and prime-to-p part :

$$T(X) = T'(X) \times T_p(X), \quad T'(X) = \prod_{\ell \neq p} T_\ell(X)$$

where X is any one of T, B, T_K, B_K, T_k, B_k.

First look at $T'(X)$, which is a product of smooth ℓ-adic sheaves on O_K, and G acts through $G_k = \mathrm{Gal}(\bar{k}/k)$ which is topologically generated by q-th power Frobenius F. So looking at the special fiber, we have a commutative diagram with exact rows for $X = T_k, B_k$:

$$\begin{array}{ccccccccc}
 & & & & & & X(k)'_{tor} & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & T'(X) & \longrightarrow & T'(X) \otimes \mathbb{Q} & \longrightarrow & \varinjlim_{(p,n)=1} X[n] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1-F} & & \cong\downarrow{\scriptstyle 1-F} & & \downarrow{\scriptstyle 1-F} & & \\
0 & \longrightarrow & T'(X) & \longrightarrow & T'(X) \otimes \mathbb{Q} & \longrightarrow & \varinjlim_{(p,n)=1} X[n] & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & T'(X)_G & & & & & &
\end{array}$$

where $X(k)'_{tor}$ is the prime-to-p part of the torsion subgroup of the group of k-rational points $X(k)$ of X. The bijectivity of the middle vertical arrow follows from the fact that the eigenvalues of F acting on $T' \otimes \mathbb{Q}$ are not equal to 1, because $T'(T_k), T'(B_k)$ has respectively the weight $-2, -1$. Therefore the snake lemma gives us the finiteness of $T'(X)_G$.

Secondly look at $T_p(X_K)$ for $X = T, B$, and suppose $T_p(X_K)_G$ is not finite. Because $T_p(X_K)$ is a $\mathbb{Z}_p$-module, we must have non-trivial homomorphism $T_p(X_K) \to \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the trivial G_K-module. By Theorem 3.2, we must have a non-trivial homomorphism of p-divisible groups $T_p(X) \to \mathbb{Z}_p$ which necessarily factors through the maximal étale quotient $T_p^{et}(X)$. This must give a non-trivial homomorphism $T_p^{et}(X_k) \otimes \mathbb{Q} \to \mathbb{Q}_p$ at the special fiber, but this is a contradiction because here too the eigenvalue of Frobenius acting on $T_p^{et}(X_k)$ cannot be 1 (For $T_p(B_k)$, see for example [9]; $T_p(T_k)$ is connected and has no étale quotient!). Alternatively, the quotient of X_k corresponding to $T_p^{et}(X_k) \to \mathbb{Z}_p$ gives a variety of finite-type over the finite field k with infinitely many k-rational torsion points, which is impossible. □

3.2. **The part Gr_0^W.** The following proposition completes the proof of Prop. 3.1 in the semistable reduction case (Note that as A and A^* are isogenous, the k-rank of A_k^* is equal to that of A_k) :

Proposition 3.4. $(\mathrm{Gr}_0^W T(A))_G$ *has the following strucuture :*

$$0 \longrightarrow ((\mathrm{Gr}_0^W T(A))_G)_{tor} \longrightarrow (\mathrm{Gr}_0^W T(A))_G \longrightarrow \widehat{\mathbb{Z}}^r \longrightarrow 0$$

where $((\mathrm{Gr}_0^W T(A))_G)_{tor}$ *is a finite group, and* r *is the* k*-rank of the special fiber* A_k^* *of the Néron model of* A^**.*

Proof. By the canonical isomorphism $\mathrm{Gr}_0^W T(A) \cong M_K^* \otimes \widehat{\mathbb{Z}}$ (2.1), we have :

$$(\mathrm{Gr}_0^W T(A))_G \cong (M_K^*)_G \otimes \widehat{\mathbb{Z}}$$

But for M_K^* is the generic fiber of the étale group scheme M^* over O_K which is the character group of the isotrivial torus T^*, the action of G_K factors through G_k, and $(M_K^*)_{G_K} = (M_k^*)_{G_k}$ is a finitely generated $\mathbb{Z}$-module with rank equal to k-rank of T_k^*. □

3.3. **Non-semistable case.** Now we proceed to the non-semistable case and finish the proof of Prop. 3.1.

Proof of Proposition 3.1. By the semistable reduction theorem for abelian varieties ([SGA7], Exposé IX, Th. 3.6), there exists a finite Galois extension K' of K over which A acquires a semistable reduction. Put $A' = A \times_K \mathrm{Spec}(K')$, and let $G', k', T', B', M', \mathscr{A}', \ldots$ be the corresponding objects for A'/K'. Then $T(A')_{G'} = T^{et}(A')_{G_{K'}}$ has the following structure :

$$0 \longrightarrow C' \longrightarrow T(A')_{G'} \longrightarrow (M_K'^*)_{G'} \otimes \widehat{\mathbb{Z}} \longrightarrow 0$$

where C' is finite and M'^* is the character group of T'^*. If we put $\Gamma = \mathrm{Gal}(K'/K)$, the above is an exact sequence of Γ-modules, and by taking Γ-coinvariants we have :

$$C'_\Gamma \longrightarrow T(A)_G \longrightarrow ((M_K'^*)_{G'})_\Gamma \otimes \widehat{\mathbb{Z}} \longrightarrow 0 \qquad (3.1)$$

For C'_Γ is finite and $(M_K'^*)_{G'}$ is a finitely generated $\mathbb{Z}$-module, it suffices to show that the rank of $T(A)_G$ is equal to the k-rank of A_k^*.

This was essentially proven in [7], II-Th. 6.2(1), in the context of treating the jacobian variety of a curve. We reproduce the argument in a slightly different way. By looking at the pro-ℓ part of 3.1 for $\ell \neq p$, it suffices to show that the rank of $T_\ell(A)_G$ is equal to the k-rank of A_k^*. Fix a prime $\ell \neq p$, and consider the perfect duality :

$$T_\ell(A) \times T_\ell(A^*) \longrightarrow \mathbb{Z}_\ell(1)$$

Let I be the inertia subgroup of G, and we have :

$$T_\ell(A)_I \times T_\ell(A^*)^I \longrightarrow \mathbb{Z}_\ell(1)$$

where by definition, $T_\ell(A^*)^I = W_{-1}T_\ell(A^*) \cong T_\ell(A_k^*)$. Moreover, by $G_k \cong G/I$, we have a perfect pairing modulo torsion :

$$T_\ell(A)_G \times T_\ell(A_k^*)^{F=q} \longrightarrow \mathbb{Z}_\ell(1)$$

where F is the Frobenius automorphism and $T_\ell(A_k^*)^{F=q}$ is the kernel of $F - q \cdot \mathrm{id}$ in $T_\ell(A_k^*)$. Hence the rank of $T_\ell(A)_G$ is equal to that of $T_\ell(A_k^*)^{F=q}$.

Now the connected component of A_k^* is the extension of an abelian variety B_k^* by a linear algebraic group L_k^*, which is itself an extension of a unipotent group U_k^* by a torus T_k^*. Hence we have an exact sequence :

$$0 \longrightarrow T_\ell(T_k^*) \longrightarrow T_\ell(A_k^*) \longrightarrow T_\ell(B_k^*) \longrightarrow 0$$

and denoting the character group of T_k^* by M_k^*, $T_\ell(T_k^*) \cong M_k^{*\vee} \otimes \mathbb{Z}_\ell(1)$. Taking the kernel of $F - q \cdot \mathrm{id}$ yields the exact sequence:

$$0 \longrightarrow (M_k^{*\vee})^{G_k} \otimes \mathbb{Z}_\ell(1) \longrightarrow T_\ell(A_k^*)^{F=q} \longrightarrow T_\ell(B_k^*)^{F=q}$$

We see that the last term is a finite group for $T_\ell(B_k^*)$ has weight -1, and the rank of $(M_k^{*\vee})^{G_k}$ is nothing but the k-rank of T_k^*, i.e. k-rank of A_k^*. □

4. Proof of the main theorem

Now we will apply the result of the preceding section to the Albanese variety $\mathrm{Alb}(X)$ of a variety X over K. To deduce the Theorem 1.1, we need the following description of $\pi_1^{ab}(X_{\overline{K}})$:

Lemma 4.1. *Let X be a proper smooth geometrically irreducible variety over any field K which has a K-rational point. Then there is a canonical exact sequence of G_K-modules :*

$$0 \longrightarrow C \longrightarrow \pi_1^{ab}(X_{\overline{K}}) \longrightarrow T^{et}(\mathrm{Alb}(X)) \longrightarrow 0$$

where C is a finite group, and $\mathrm{Alb}(X)$ is the Albanese variety of X over K.

Proof. See [5], III, Lemma 5. □

Now we begin the proof of the main theorem :

Theorem 4.2. *Let K be a complete discrete valuation field with finite residue field k, and X a proper smooth geometrically irreducible variety over K. Then $\pi_1^{ab}(X)^{geo}$ has the following strucuture :*

$$0 \longrightarrow \pi_1^{ab}(X)^{geo}_{tor} \longrightarrow \pi_1^{ab}(X)^{geo} \longrightarrow \widehat{\mathbb{Z}}^r \longrightarrow 0$$

where $\pi_1^{ab}(X)^{geo}_{tor}$ is a finite group, and r is the k-rank of the special fiber of the Néron model of the Albanese variety of X.

First we repeat the argument in [4], Section 3 to prove :

Lemma 4.3. *In the situation of above theorem, $\pi_1^{ab}(X)^{geo} \cong \pi_1^{ab}(X_{\overline{K}})_{G_K}$.*

Proof. The Hochschild-Serre spectral sequence gives an exact sequence :

$$0 \longrightarrow H^1(G_K, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1_{et}(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1_{et}(X_{\overline{K}}, \mathbb{Q}/\mathbb{Z})^{G_K} \longrightarrow H^2(G_K, \mathbb{Q}/\mathbb{Z})$$

But $H^2(G_K, \mathbb{Q}/\mathbb{Z}) = 0$ by Tate duality for local fields, and taking the Pontrjagin dual $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ of abelian groups gives the exact sequence :

$$0 \longrightarrow \pi_1^{ab}(X_{\overline{K}})_{G_K} \longrightarrow \pi_1^{ab}(X) \longrightarrow G_K^{ab} \longrightarrow 0$$

which shows the isomorphism of the lemma. □

Proof of Theorem 4.2. Take a finite Galois extension K' of K such that X has a K'-rational point. By Lemma 4.1, we have the following exact sequence :

$$C_{G_{K'}} \longrightarrow \pi_1^{ab}(X_{\overline{K}})_{G_{K'}} \longrightarrow T^{et}(\mathrm{Alb}(X))_{G_{K'}} \longrightarrow 0$$

Now take the coinvariants by $\mathrm{Gal}(K'/K)$ and apply Lemma 4.3 to get the exact sequence :

$$C_{G_K} \longrightarrow \pi_1^{ab}(X)^{geo} \longrightarrow T^{et}(\mathrm{Alb}(X))_{G_K} \longrightarrow 0$$

As we know that C_{G_K} is finite, the theorem follows by the application of Prop. 3.1 to $\mathrm{Alb}(X)$. □

5. Application to the class field theory of curves over local fields

In this section, we give an application to the class field theory of curves over local fields developed by S. Saito in [7], where the main theorem has been proven except for the p-primary part in the $\mathrm{char}\, K = p > 0$ case. By our finiteness result, we can prove the main theorem also in the remaining case. Raskind[6] contains an exposition of the subject of this section.

For a proper smooth geometrically irreducible curve X over a local field K, and define the group $SK_1(X)$ by :

$$SK_1(X) = \operatorname{Coker}\left(\bigoplus_{x\in P}\partial_x : K_2(K(X)) \longrightarrow \bigoplus_{x\in P}\kappa(x)^\times\right)$$

where P denotes the set of all closed points of X, $K(X)$ is the function field of X, $\kappa(x)$ is the residue field at x, and ∂_x is the boundary map in algebraic K-theory. Let $V(X)$ be the kernel of the norm map $N : SK_1(X) \to K^\times$, induced by the norm map $N_{\kappa(x)/K} : \kappa(x)^\times \to K^\times$ for each x.

In [7], the reciprocity map :

$$\sigma : SK_1(X) \to \pi_1^{ab}(X), \qquad \tau : V(X) \to \pi_1^{ab}(X)^{geo}$$

is defined, which makes following diagrams commute :

$$\begin{array}{ccc} \kappa(x)^\times & \longrightarrow & SK_1(X) \\ \downarrow{\scriptstyle \sigma_x} & & \downarrow{\scriptstyle \sigma} \\ \mathrm{Gal}(\kappa(x)_{ab}/\kappa(x)) & \longrightarrow & \pi_1^{ab}(X) \end{array} \qquad \begin{array}{ccccc} V(X) & \longrightarrow & SK_1(X) & \longrightarrow & K^\times \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma_K} \\ \pi_1^{ab}(X)^{geo} & \longrightarrow & \pi_1^{ab}(X) & \longrightarrow & \mathrm{Gal}(K_{ab}/K) \end{array}$$

(the left diagram exists for all closed points x of X) where σ_x, σ_K denote the reciprocity maps of local class field theory. Then the main theorem of the class field theory of X is stated as follows (cf. [7], Introduction) :

Theorem 5.1. (i) *Let D be the maximal divisible subgroup of $V(X)$. Then the reciprocity map τ induces an isomorphism of finite groups :*

$$V(X)/D \xrightarrow{\cong} \pi_1^{ab}(X)^{geo}_{tor}$$

(ii) *Let E be the maximal divisible subgroup of $SK_1(X)$. Then the reciprocity map σ induces an injection :*

$$SK_1(X)/E \longrightarrow \pi_1^{ab}(X)$$

and the quotient of $\pi_1^{ab}(X)$ by the closure of the image of σ is isomorphic to $\widehat{\mathbb{Z}}^r$, where r is the rank of X defined in [7], *Section II-2.*

The theorem is proven in [7] except for the p-primary part in the $\operatorname{char} K = p > 0$ case.

Proof. The proof of the theorem follows the same argument as in [7], which we reproduce here for completeness. First, note that the rank of X defined in [7], Section II-2 is equal to the k-rank of the special fiber of the Néron model of the jacobian variety of X, by [7], II-Th. 6.2(1).

For the cokernels of the reciprocity maps, we have :

Lemma 5.2 (II-Th. 2.6, II-Prop. 3.5 of [7]). *The quotient of* $\pi_1^{ab}(X)$ *by the closure of the image of* σ*, which is isomorphic to the quotient of* $\pi_1^{ab}(X)^{geo}$ *by the closure of the image of* τ*, is isomorphic to* $\widehat{\mathbb{Z}}^r$*, where* r *is the rank of* X*.*

Combining this with our Theorem 4.2, we know that τ is a surjection onto $\pi_1^{ab}(X)^{geo}_{tor}$. This removes the condition on $\mathrm{char}\, K$ in II-Th. 4.1 (the finiteness of the image of τ), Cor. 4.3, Cor. 4.4 of [7] (and solves affirmatively the conjecture II-Remark 4.2).

Now the determination of the kernels (i.e. proving that the kernels are divisible) in II-Th. 5.1 of [7] is based on the following :

Lemma 5.3 (II-Lemma 5.3 of [7], Prop. 3 of [4] for $\mathrm{char}\, K$-primary part). *For any integer* $n > 0$*, the map* $SK_1(X)/nSK_1(X) \to \pi_1^{ab}(X)/n\pi_1^{ab}(X)$ *induced by the reciprocity map* σ *is an injection.*

Starting from this lemma, the proof of II-Th. 5.1 of [7] can be carried out without any change, as we now have II-Th. 4.1 unconditionally. Also, this removes the condition on $\mathrm{char}\, K$ in II-Cor. 5.2 of [7]. □

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