# WEEK 7: BACK TO SHIMURA VARIETIES

TERUYOSHI YOSHIDA

Abstract. Unitary Shimura varieties. Drinfeld level structures.

**Lecture 18 (Nov. 3, 2008)**

## 1. Unitary Shimura Varieties of Drinfeld Type

Back to Shimura varieties. We are interested in a class of Shimura varieties that are moduli spaces of abelian varieties with Polarization, Endomorphism and Level structure, hence called the *PEL-type* Shimura varieties. We will think of these moduli functors as functors on the category ($\mathcal{O}$-alg), where $\mathcal{O}$ is a number field or a ring contained in it. We will first define the data needed to define such Shimura varieties, i.e. the *PEL-type Shimura data* $(B, *, V, \langle\,,\rangle)$ and the associated connected reductive group $G$ over $\mathbb{Q}$. For each open compact subgroup $U$ of $G(\mathbb{A}^\infty)$, we have a moduli functor of the form:

$$X_U : (\mathcal{O}\text{-alg}) \longrightarrow (\text{Sets}),$$
$$R \longmapsto \big\{(A, \lambda, i, \eta U)/R\big\}/\sim$$

where:
- $A$ is an abelian variety (abelian scheme) over $R$,
- $\lambda : A \to A^\vee$ is an isogeny (*polarization* on $A$),
- $i : B \to \mathrm{End}_R(A) \otimes \mathbb{Q}$ is a $\mathbb{Q}$-algebra homomorphism (*endomorphisms* on $A$),
- $\eta U$ is an (etale) *level $U$ structure* on $A$,

and $\sim$ is given by the isogenies over $R$.

We start with giving precise definitions of this functor, restricting ourselves to the case we are interested in (*unitary Shimura varieties of Drinfeld type*, or *generalized Harris-Taylor type*). Then we will discuss its *representability* as algebraic spaces over $\mathcal{O}$.

### 1.1. Unitary PEL-type Shimura data.

The theory of unitary Shimura varieties is a generalization of the theory of *complex multiplication*, which constructs the abelian extensions (class fields) of a particluar type of number fields called *CM-field*, where CM stands for complex multiplication. The abelian varieties we will classify are generalizations of the *abelian varieties with complex multiplication* (CMAV).

**Definition 1.1.** Let $F^+$ be a totally real field with $[F^+ : \mathbb{Q}] = d$, i.e. $F^+ \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d$, and let $F$ be a totally complex quadratic extension of $F^+$, i.e. $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^d$. We say that $F$ is a

*CM-field*, and the non-trivial element $c \in \mathrm{Gal}(F/F^+)$ is the *complex conjugate* on $F$. In the rest of this course, we *assume* that $F$ is of the form $F = F^+E$ for an imaginary quadratic field $E/\mathbb{Q}$.

The last assumption simplifies the theory of Shimura varieties, and the construction of Galois representations over general CM-fields will be reduced to the case where $F$ satisfies our assumption.

**Definition 1.2.** A *unitary PEL-type Shimura data* is a quadruple $(B, *, V, \langle\,,\rangle)$ as follows:
– $B$ is a central simple algebra (CSA) over a CM-field $F$, i.e. a finite dimensional simple $F$-algebra whose center is $F$,
– $*$ is a *positive involution of the 2nd kind* on $B$, i.e. an $F$-algebra isomorphism $* : B \xrightarrow{\cong} B^{\mathrm{op}}$, where $B^{\mathrm{op}}$ is the opposite algebra of $B$, such that $\mathrm{tr}_{B/\mathbb{Q}}(bb^*) > 0$ for all $b \in B$ and $*|_F : F \cong F$ is the complex conjugation $c$ of $F$,
– $V$ is a finite dimensional $B$-module (modules are always *left* modules),
– $\langle\,,\rangle : V \times V \to \mathbb{Q}$ is a $\mathbb{Q}$-bilinear pairing such that:

$$\langle x, y\rangle = -\langle y, x\rangle \qquad \text{(alternating)},$$
$$\langle bx, y\rangle = \langle x, b^*y\rangle \qquad \text{($*$-Hermitian)}.$$

For the motivation of this definition, see [Mu] and David Geraghty's notes on the correspondence between $\mathbb{Q}$-PHS and $\mathbb{Q}$-PAV over $\mathbb{C}$. Every CSA $B$ over $F$ has dimension equal to a square, i.e. $\dim_F B = n^2$ for $n \in \mathbb{Z}_{>0}$. The existence of the positive involution of the 2nd kind $*$ implies that:

$$B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_n(F \otimes_{\mathbb{Q}} R) \cong \prod_{i=1}^{d} M_n(\mathbb{C}),$$

where the index set of the product corresponds to the $d$ real places of $F^+$, and we chose the complex embedding $F \to \mathbb{C}$ lying above each of these places for the last isomorphism.

Here we make a further remark on the choices of $B$ and $V$. If $B' := M_m(B)$, it is another CSA over $F$, and we have an equivalence of categories (the *Morita equivalence*):

$$(\text{fin. dim. } B\text{-mod}) \ni V \longmapsto V^m \in (\text{fin. dim. } B'\text{-mod}).$$

Thus if $\dim_B V = m$, then $V' := V^m$ is a $B'$-module such that $B' \cong V'$ because $\dim_B V' = m^2 = \dim_B B'$. We can restrict ourselves to the case $B = V$ via this operation. We will *assume* that $B = V$ from now on. Note that $\dim_{\mathbb{Q}} V = 2dn^2$.

Moreover, every CSA $B$ over $F$ with $\dim_F B = n^2$ is written as $B = M_m(D)$ for $m \mid n$ and a *division algebra* $D$ over $F$ (the *Wedderburn theorem*). Therefore, requiring $B = V$, we have two extremal cases:
– (*quasi-split case*) $B = M_n(F)$,
– (*division algebra case*) $B = D$ is a division algebra,
and various intermediate cases for each $m \mid n$. Note that the quasi-split case is equivalent to the case where $B = F$ and $V = F^n$ by the Morita equivalence above.

Let $\mathbb{G}$ be an algebraic group over $F$ defined by

$$\mathbb{G}(R) := \mathrm{Aut}_{B\otimes_F R}(V \otimes_F R) = (B^{\mathrm{op}} \otimes_F R)^\times \quad (\forall R \in (F\text{-alg})).$$

Then we have $\mathbb{G} \cong GL_n$ in the quasi-split case, and in general $\mathbb{G}$ is an *inner form* of $GL_n$, i.e. $\mathbb{G}(\overline{F}) \cong GL_n(\overline{F})$. Therefore automorphic representations of $\mathbb{G}$ will correspond to each other by the *Jacquet-Langlands correspondence*, and the quasi-split case is the most general case. But the analysis of quasi-split case involves the treatment of *endoscopy*, which does not arise in the division algebra case. We will later specialize to the division algebra case.

1.2. **Unitary similitude groups.** Let $(B, *, V, \langle\, ,\rangle)$ be a unitary PEL-type Shimura data, with our additional assumption of $F = F^+E$ and $B = V$. Note that we have:

$$\mathrm{Aut}_{B\otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) = (B^{\mathrm{op}} \otimes_{\mathbb{Q}} R)^\times$$

for every $\mathbb{Q}$-algebra $R$. We associate to this an algebraic group $G$ over $\mathbb{Q}$. We write the pairing obtained from $\langle\, ,\rangle$ by base change to $R$ as:

$$\langle\, ,\rangle_R : (V \otimes_{\mathbb{Q}} R) \times (V \otimes_{\mathbb{Q}} R) \longrightarrow R.$$

**Definition 1.3.** The *unitary similitude group* $G$ associated to a unitary PEL-type Shimura data $(B, *, V, \langle\, ,\rangle)$ is a (connected reductive) algebraic group over $\mathbb{Q}$, defined as:

$$G(R) := \left\{ \; g \in \mathrm{Aut}_{B\otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \mid \exists \lambda \in R^\times, \; \langle gx, gy\rangle_R = \lambda\langle x, y\rangle_R \; (\forall x, y \in V) \; \right\}$$

for every $\mathbb{Q}$-algebra $R$. The element $\lambda \in R^\times$ is uniquely determined by $g \in G(R)$, and we denote the group homomorphism $g \mapsto \lambda$ by:

$$\det : G \longrightarrow GL_1, \qquad \text{given by} \qquad G(R) \ni g \longmapsto \lambda \in R^\times = GL_1(R) \quad (\forall R \in (\mathbb{Q}\text{-alg})).$$

We denote the kernel by $G_0 := \mathrm{Ker}(\det) \subset G$.

**Example 1.4.** If $n = 0$, i.e. $V = 0$, then $G = GL_1$ over $\mathbb{Q}$ (the *cyclotomic case*).

**Lemma 1.5.** *Let $G$ be a unitary similitude group associated to a unitary PEL-type Shimura data, and $G_0 = \mathrm{Ker}(\det : G \to GL_1)$. Then $G_0$ is a restriction of scalars from an algebraic group $G_1$ over $F^+$, i.e. there is an algebraic group $G_1$ over $F^+$ such that:*

$$G_0(R) = G_1(F^+ \otimes_{\mathbb{Q}} R) \quad (\forall R \in (\mathbb{Q}\text{-alg})).$$

*This $G_1$ is a unitary group over $F^+$, or more precisely an outer form of $\mathbb{G}$, associated to the quadratic extension $F/F^+$, i.e.*

$$G_1(R) = \mathbb{G}(R) \quad (\forall R \in (F\text{-alg})).$$

As $\mathbb{G}(F \otimes_{\mathbb{Q}} \mathbb{R}) = (B^{\mathrm{op}} \otimes_{\mathbb{Q}} \mathbb{R})^\times \cong \prod_{i=1}^{d} GL_n(\mathbb{C})$, we have:

$$G_0(\mathbb{R}) = G_1(F^+ \otimes_{\mathbb{Q}} \mathbb{R}) = U(p_1, q_1) \times \cdots \times U(p_d, q_d),$$

where $0 \le p_i \le n$ and $p_i + q_i = n$ for $1 \le i \le d$. These are *signs* of the Hermitian pairing

$$(\, ,)_{\mathbb{R}} : (V \otimes_{\mathbb{Q}} \mathbb{R}) \times (V \otimes_{\mathbb{Q}} \mathbb{R}) \longrightarrow \mathbb{C}.$$

defined by $(x, y)_{\mathbb{R}} := \langle x, iy\rangle_{\mathbb{R}} + i\langle x, y\rangle_{\mathbb{R}}$ using the alternating pairing

$$\langle\, ,\rangle_{\mathbb{R}} : (V \otimes_{\mathbb{Q}} \mathbb{R}) \times (V \otimes_{\mathbb{Q}} \mathbb{R}) \longrightarrow \mathbb{R}.$$

The $2d$ invariants $p_1, q_1, \ldots, p_d, q_d$ correspond to the $2d$ complex embeddings of $F$ into $\mathbb{C}$.

**Definition 1.6.** We say that the unitary PEL-type Shimura data (or the associated unitary similitude group $G$) is of *Drinfeld type* if

$$G_0(\mathbb{R}) = U(1, n-1) \times U(0, n) \times \cdots \times U(0, n).$$

If in addition we are in the division algebra case, i.e. $B$ is a division algebra (note that we always assume $B = V$), then we call it the *Harris-Taylor type.* We denote by $\tau : F \to \mathbb{C}$ the complex embedding giving the invariant $p_1 = 1$.

1.3. **Definition of the moduli functors over $F$.** Let $(B, *, V, \langle\, ,\rangle)$ be a unitary PEL-type Shimura data, with the same assumptions $F = F^+E$ and $B = V$, and $G$ be the associated unitary similitude group over $\mathbb{Q}$.

For each open compact subgroup $U$ of $G(\mathbb{A}^\infty)$, we will define a moduli functor:

$$X_U : (\mathcal{O}\text{-alg}) \longrightarrow (\text{Sets}),$$
$$R \longmapsto \big\{(A, \lambda, i, \eta U)/R\big\}/\sim$$

where we will define each objects one by one.

Note that over $\mathbb{C}$, we compared the $\mathbb{Q}$-vector space $V_{\mathrm{B}}A$ with $V$, the $\mathbb{C}$-vector space $V_{\mathrm{dR}}A = \operatorname{Lie} A$ with $V \otimes \mathbb{R}$ and the $\mathbb{A}^\infty$-module $V_{\mathrm{et}}A$ with $V \otimes \mathbb{A}^\infty$. As we have $\dim_{\mathbb{Q}} V = 2dn^2$, the abelian variety $A$ must be of $dn^2$-dimensional:

$A$ is an abelian variety (abelian scheme) over $R$ of dimension $dn^2$,

$\lambda : A \to A^\vee$ is an isogeny over $R$ (a *polarization* on $A$).

This *polarization* is the "motivic" origin of the following realizations over $\mathbb{C}$:
– Betti: the positive involution $*_\lambda$ on $\operatorname{End}_R(A) \otimes \mathbb{Q}$ (the *Rosati involution*),
– de Rham: the Hermitian pairing $(\, ,)_{\mathbb{R}}$ on $\operatorname{Lie} A$ (the *Riemann form*), and
– etale: the *Weil pairing* $\langle\, ,\rangle_{\mathbb{A}^\infty}$ on $V_{\mathrm{et}}A$.
The latter two should correspond to the alternating pairing $\langle\, ,\rangle_{\mathbb{A}}$ on $V \otimes \mathbb{A}$.

- $i : B \to \operatorname{End}_R(A) \otimes \mathbb{Q}$ is a $\mathbb{Q}$-algebra homomorphism (*endomorphisms* on $A$),
- $\eta U$ is an (etale) *level $U$ structure* on $A$,
and $\sim$ is given by the isogenies over $R$.

$\operatorname{Ker}^1(G, \mathbb{Q})$ issue.

**Lecture 19 (Nov. 5, 2008)**

## 2. Cyclotomic Fields and Drinfeld Level Structures

We will prove representability by extending the moduli functor $X_U$ to a functor on ($\mathcal{O}$-alg) (the *integral model* of $X_U$), where $\mathcal{O}$ is a ring contained in $F$ with infinitely many prime ideals, and analyzing its deformation ring at the points over finite fields.

2.1. **Cyclotomic fields as Shimura varieties.**

### 2.2. Integral models: Drinfeld level structures on $\mathbb{G}_m$.

**Lecture 20 (Nov. 7, 2008)**

## 3. Representability as Algebraic Spaces

### 3.1. Integral models of unitary Shimura varieties of Drinfeld type.

## 4. Notes on the Literature

## References

[Art] Artin, M., *Théorèmes de Représentabilité pour les Espaces Algébriques*, Les Presses de l'Univ. Montréal, 1973.
[Mu] Mumford, D, *Abelian Varieties*,.

Harvard University, Department of Mathematics, 1 Oxford Street, Cambridge, MA 02138, USA

*E-mail address*: yoshida@math.harvard.edu