WEEK 6: ETALE COHOMOLOGY

TERUYOSHI YOSHIDA

ABSTRACT. Etale cohomology.

Lecture 15 (Oct. 27, 2008)

1. Etale sheaves

1.1. **Big and small etale sites.** Let $\mathcal{O}$ be our fixed (often noetherian) base ring, and consider the category $(\mathcal{O}\text{-alg})^\vee$ of functors $\mathcal{F} : (\mathcal{O}\text{-alg}) \to (\text{Sets})$. Let $X \in (\mathcal{O}\text{-alg})^\vee$ be an *algebraic space*, i.e. a functor which is an etale sheaf and etale locally representable. Let $(\mathcal{O}\text{-alg}/X)$ be a category of pairs (R, φ), where $R \in (\mathcal{O}\text{-alg})$ and $\varphi : \operatorname{Spec} R \to X$ (the *structure morphism*). The morphisms $(R, \varphi) \to (R', \varphi')$ are the morphisms $R \to R'$ which commute with the structure morphisms. We think of an object of $(\mathcal{O}\text{-alg}/X)$ as giving a functor in $(\mathcal{O}\text{-alg}/X)^\vee$, and call it an *affine scheme over* X.

More generally, let $(\mathcal{O}\text{-alg})^\vee/X$ be a category of pairs $(\mathcal{F}, \psi)$, where $\mathcal{F} \in (\mathcal{O}\text{-alg})^\vee$ and $\psi : \mathcal{F} \to X$ is a morphism of functors. An object $(\mathcal{F}, \psi) \in (\mathcal{O}\text{-alg})^\vee/X$ gives a functor $F \in (\mathcal{O}\text{-alg}/X)^\vee$ (see the lemma below). In fact, we have:

Lemma 1.1. *The following is an equivalence of categories:*

$$(\mathcal{O}\text{-alg})^\vee/X \xrightarrow{\cong} (\mathcal{O}\text{-alg}/X)^\vee,$$
$$(\mathcal{F}, \psi) \longmapsto F$$

where F corresponding to $(\mathcal{F}, \psi)$ is defined as

$$F : (R, \varphi) \longmapsto \{\ f \in \operatorname{Hom}(\operatorname{Spec} R, \mathcal{F}) \mid \psi = \varphi \circ f\ \},$$

and $(\mathcal{F}, \psi)$ corresponding to F is defined as:

$$\mathcal{F}(R) := \coprod_{\varphi \in X(R)} F(R, \varphi) \xrightarrow{\psi} \coprod_{\varphi \in X(R)} \{\cdot\} = X(R).$$

Proof. It is straightforward to check that these functors give quasi-inverse to each other. □

We will often identify $(\mathcal{O}\text{-alg})^\vee/X$ and $(\mathcal{O}\text{-alg}/X)^\vee$ via the equivalence in the lemma, and we will often omit the structure morphism ψ and write $\mathcal{F} \in (\mathcal{O}\text{-alg})^\vee/X$. A collection of morphisms $\{\ \mathcal{F}'_i \to \mathcal{F}\ \}_{i\in I}$ in $(\mathcal{O}\text{-alg})^\vee/X$ is a *Zariski* (resp. *etale*) *covering* if it is a Zariski (resp. etale) covering in $(\mathcal{O}\text{-alg})^\vee$. In particular, a collection of morphisms:

$$\left\{\ f_i : (\operatorname{Spec} R'_i,\ \varphi\circ f_i) \to (\operatorname{Spec} R, \varphi)\ \right\}_{i\in I}$$

in $(\mathcal{O}\text{-alg}/X)$ is a Zariski/etale covering if $\{\ f_i : \operatorname{Spec} R'_i \to \operatorname{Spec} R\ \}_{i\in I}$ is a Zariski (resp. etale) covering in $(\mathcal{O}\text{-alg})$.

In general, if a category $\mathscr{C}$ is equipped with a *Grothendieck topology*, i.e. the notion of a collection of morphisms in $\mathscr{C}$ being a *covering* or not, then we can talk of a functor in $\mathscr{C}^\vee$ being a *sheaf* or not, and the category $\mathscr{C}$ is called a *site*. In our case, we call $(\mathcal{O}\text{-alg}/X)$ a *big Zariski site* X_{Zar} (resp. *big etale site* X_{Et}) of X, when equipped with the notion of Zariski (resp. etale) covering of $(\operatorname{Spec} R, \varphi)$.

To have a good analogy with the usual topology (of open subsets) and make functorial constructions, usually we think of a smaller category than $(\mathcal{O}\text{-alg}/X)$. This is the full subcategory $(\mathcal{O}\text{-alg. et}/X)$ of $(\mathcal{O}\text{-alg}/X)$, consisting of pairs (R,φ), where $R \in (\mathcal{O}\text{-alg})$ and $\varphi : \operatorname{Spec} R \to X$ is an *etale* morphism. As we know what it means for a collection of morphisms $\{\ (\operatorname{Spec} R'_i, \varphi_i) \to (\operatorname{Spec} R, \varphi)\ \}_{i\in I}$ in $(\mathcal{O}\text{-alg. et}/X)$ to be an etale covering, we obtain a site X_{et}, which we call the *small etale site* of X. In the big etale site X_{Et}, the notion of etale coverings are defined for any collection of morphisms in $X_{\text{Et}}^\vee = (\mathcal{O}\text{-alg}/X)^\vee$, i.e. we knew what it means to cover a functor by other functors. In the small etale site of X, it is only definied for morphisms in $X_{\text{et}} = (\mathcal{O}\text{-alg. et}/X)$ itself, i.e. the morphisms between representable functors,

1.2. **Etale sheaves.** By an *etale sheaf* on X, we usually mean a sheaf on the small etale site X_{et}. If $\{\ f_i : (\operatorname{Spec} R'_i, \varphi'_i) \to (\operatorname{Spec} R, \varphi)\ \}_{i\in I}$ is an etale covering in X_{et}, then $\varphi'_i = \varphi\circ f_i$, and its self fiber product $f_{ij} : \operatorname{Spec}(R'_i \otimes_R R'_j) \to \operatorname{Spec} R$ is *etale*, hence setting $\varphi'_{ij} := \varphi\circ f_{ij}$, we have $(R'_i\otimes_R R'_j, \varphi'_{ij}) \in X_{\text{et}}$ for all i, j.

Definition 1.2. An *etale sheaf* on X_{et} is a functor $F \in X_{\text{et}}^\vee$, i.e. a functor

$$F : (\mathcal{O}\text{-alg. et}/X) \longrightarrow (\text{Sets}),$$

which is a *sheaf* with respect to the etale topology, i.e. for any etale covering $\{\ (\operatorname{Spec} R'_i, \varphi'_i) \to (\operatorname{Spec} R, \varphi)\ \}_{i\in I}$ in $X_{\text{et}} = (\mathcal{O}\text{-alg. et}/X)$, the following sequence of maps is exact:

$$F(R,\varphi) \longrightarrow \textstyle\prod_i F(R'_i, \varphi'_i) \rightrightarrows \prod_{i,j,} F(R'_i\otimes_R R'_j, \varphi'_{ij})\ .$$

Etale sheaves $\mathcal{F}$ on X_{Et} is defined in the exactly similar way.

Remark 1.3. We could have defined X_{et} as the category of all algebraic spaces that are etale over X. On the other hand, here we are inclined to think of the etale sheaves as objects of $X_{\text{Et}}^\vee$ (see the next subsection).

Example 1.4. Let $X = \operatorname{Spec} R$, where R is a complete noetherian ring with a separably closed residue field. Then $(\text{Sets}) \ni I \overset{\cong}{\longmapsto} (R^I, \amalg_I \text{id}) \in X_{\text{et}}$, and a sheaf on X_{et} is uniquely determined by the set $F(R, \text{id})$, because $F(R^I, \amalg_I\text{id}) = F(R,\text{id})^I$.

There is an obvious restriction functor:

$$u_* : X_{\mathrm{Et}}^{\vee} = (\mathcal{O}\text{-alg}/X)^{\vee} \longrightarrow X_{\mathrm{et}}^{\vee} = (\mathcal{O}\text{-alg. et}/X)^{\vee},$$

and a sheaf on X_{Et} gives a sheaf on X_{et} by restriction. (Usually we think of this as taking a "direct image" with respect to a "morphism of sites" $u : X_{\mathrm{Et}} \to X_{\mathrm{et}}$, which is denoted by an inverse arrow of the functor $X_{\mathrm{et}} \to X_{\mathrm{Et}}$.) We will define a functor u^* which gives the left adjoint of this functor, i.e. $F \cong u_* u^* F$ for every etale sheaf F.

1.3. **Etale spaces.** To an etale sheaf $F \in X_{\mathrm{et}}^{\vee}$, we associate an *algebraic space* $\mathcal{F} \in X_{\mathrm{Et}}^{\vee} = (\mathcal{O}\text{-alg})^{\vee}/X$, which is etale and locally separated over X (for the notion of etale space in the classical sheaf theory, see Exercise II.1.13. of Hartshorne).

Definition 1.5. An *etale space* over X is an algebraic space $\mathcal{F} \in (\mathcal{O}\text{-alg})^{\vee}/X$, which is etale and locally separated over X.

Let $F \in X_{\mathrm{et}}^{\vee}$ be an etale sheaf. The starting point is to think of each "section", i.e. an element $f \in F(R, \varphi)$ as giving a morphism $f : \operatorname{Spec} R \to \mathcal{F}$, and all the sections as giving an etale covering of $\mathcal{F}$ by affine schemes. Therefore define an algebraic space $U \in (\mathcal{O}\text{-alg})^{\vee}/X$ by:

$$U := \coprod_{(R,\varphi,f),\ f \in F(R,\varphi)} \operatorname{Spec} R \xrightarrow{\amalg\varphi} X.$$

As each $f \in F(R, \varphi)$ gives a morphism $(\operatorname{Spec} R, \varphi) \to F$ in $X_{\mathrm{et}}^{\vee}$, by thinking of $U \in X_{\mathrm{Et}}^{\vee}$ as an etale sheaf on X, we have a morphism:

$$u_* U \xrightarrow{\amalg f} F \qquad \text{in } X_{\mathrm{et}}^{\vee},$$

and hence we can think of F as "patched together" from U:

$$u_* U \times_F u_* U \rightrightarrows u_* U \longrightarrow F.$$

We would like to give an algebraic space V associated to $u_* U \times_F u_* U$, so that we can define the etale space $\mathcal{F}$ by patching:

$$V \rightrightarrows U \dashrightarrow \mathcal{F}.$$

Note that U is a disjoint union of affine schemes, hence a scheme, and $U \times_X U$ is etale over U, hence also a scheme. We construct V as an open subscheme of $U \times_X U$.

Lemma 1.6. *There is a unique open subspace $j : V \to U \times_X U$ such that $u_*(j)$ gives the canonical morphism $u_* U \times_F u_* U \to u_* U \times_X u_* U$ in $X_{\mathrm{et}}^{\vee}$, and V gives an etale equivalence relation on U.*

Proof. For each $(R, \varphi) \in X_{\mathrm{et}}$, morphisms $(\operatorname{Spec} R, \varphi) \to u_* U \times_F u_* U$ are in bijection with pairs of morphisms $f_i : (\operatorname{Spec} R, \varphi) \to u_* U$ $(i = 1, 2)$ such that $(\amalg f) \circ f_1 = (\amalg f) \circ f_2$. Let V be the union of the images of such $(f_1, f_2) : (\operatorname{Spec} R, \varphi) \to U \times_X U$ (these are etale morphisms, and the image is open). [**Fill in**] □

Definition 1.7. The algebraic space $\mathcal{F} \in X_{\mathrm{Et}}^{\vee} = (\mathcal{O}\text{-alg})^{\vee}/X$ obtained by taking the quotient of $V \rightrightarrows U$ is etale and locally separated over X, i.e. an etale space over X. We call $\mathcal{F}$ the *etale space* of F. We also write $\mathcal{F} = u^*F$, and we have a functor:

$$u^* : (\text{Sheaves on } X_{\mathrm{et}}) \longrightarrow (\text{Sheaves on } X_{\mathrm{Et}}),$$

such that $F \cong u_* u^* F$ for an etale sheaf $F \in X_{\mathrm{et}}^{\vee}$.

Conversely, let $\mathcal{F} \in (\mathcal{O}\text{-alg})^{\vee}/X$ be an etale space over X, and let $F := u_*\mathcal{F}$. Each element $f \in F(R,\varphi) = \mathcal{F}(R,\varphi)$ for $(R,\varphi) \in X_{\mathrm{et}}$ gives a morphism $f : (\operatorname{Spec} R, \varphi) \to \mathcal{F}$ in $(\mathcal{O}\text{-alg})^{\vee}/X$, which is etale because φ is etale. Doing the same construction $V \rightrightarrows U$ from F, we see that $\amalg f : U \to \mathcal{F}$ is an etale covering of $\mathcal{F}$ because $\mathcal{F}$ has an etale covering by affine schemes, and $V \cong U \times_{\mathcal{F}} U$. Therefore we recover $\mathcal{F}$ as the etale space of F, i.e. $\mathcal{F} \cong u^* u_* \mathcal{F}$. Thus:

Proposition 1.8. *We have an equivalence of categories:*

$$(\text{Sheaves on } X_{\mathrm{et}}) \underset{u_*}{\overset{u^*}{\rightleftarrows}} (\text{Etale spaces over } X) \subset (\text{Sheaves on } X_{\mathrm{Et}}).$$

1.4. **Pull-back of etale sheaves.** Let $X, S \in (\mathcal{O}\text{-alg})^{\vee}$ be algebraic spaces and $f : X \to S$ be a morphism. There is a natural functor

$$(\mathcal{O}\text{-alg})/X \ni (R,\varphi) \longmapsto (R, f\circ\varphi) \in (\mathcal{O}\text{-alg})/S,$$

which gives the functor

$$f^* : (\mathcal{O}\text{-alg})^{\vee}/S \ni \mathcal{F} \longmapsto f^*\mathcal{F} := X \times_S \mathcal{F} \in (\mathcal{O}\text{-alg})^{\vee}/X,$$

and this f^* sends sheaves to sheaves, i.e.

$$f^* : (\text{Sheaves on } S_{\mathrm{Et}}) \longrightarrow (\text{Sheaves on } X_{\mathrm{Et}}),$$

If $F \in (\mathrm{Sh}/S_{\mathrm{et}})$ and $\mathcal{F} = u^*F$, then $f^*\mathcal{F} = X \times_S \mathcal{F}$ is an etale space over X, hence $f^*\mathcal{F} = u^*u_*(f^*\mathcal{F})$. Therefore we can define $f^*F := u_* f^*\mathcal{F}$, which is a sheaf on X_{et}, and we have $f^*\mathcal{F} = u^* f^* F$.

Lecture 16 (Oct. 29, 2008)

2. Etale Cohomology and Higher Direct Image Sheaves

2.1. **Etale cohomology.** From now on, we think of *abelian sheaves* on X_{et}, i.e. the functors $F : X_{\mathrm{et}} \longrightarrow (\mathrm{Ab})$, where (Ab) denotes the category of abelian groups, that are etale sheaves when considered as $F \in X_{\mathrm{et}}^{\vee}$ (similarly for X_{Et}). We denote the category of abelian sheaves on X_{et} (resp. X_{Et}) by $(\mathrm{Sh}/X_{\mathrm{et}})$ (resp. $(\mathrm{Sh}/X_{\mathrm{Et}})$). Both $(\mathrm{Sh}/X_{\mathrm{et}})$ and $(\mathrm{Sh}/X_{\mathrm{Et}})$ are abelian categories (note that we need to use the *sheafification* [**Explain**] when we take cokernels), and we have the additive functors

$$u^* : (\mathrm{Sh}/X_{\mathrm{et}}) \ni F \longmapsto \mathcal{F} = u^*F \in (\mathrm{Sh}/X_{\mathrm{Et}}),$$
$$u_* : (\mathrm{Sh}/X_{\mathrm{et}}) \ni \mathcal{F} \longmapsto F = u_*\mathcal{F} \in (\mathrm{Sh}/X_{\mathrm{et}}),$$

which are adjoint to each other, i.e.

$$\mathrm{Hom}_{X_{\mathrm{Et}}}(u^*F,\mathcal{G}) \xrightarrow{\cong} \mathrm{Hom}_{X_{\mathrm{et}}}(F,u_*\mathcal{G}),$$

and u^* is *exact* (**Proof!**).

We can consider the global section functor on $(\mathrm{Sh}/X_{\mathrm{Et}})$. We will think of sections of sheaves as homomorphisms between algebraic spaces, i.e. Hom in $X_{\mathrm{Et}}^{\vee} = (\mathcal{O}\text{-alg})^{\vee}/X$. We will write $\mathrm{Hom}_{X_{\mathrm{Et}}^{\vee}}$ as Hom_X for simplicity. For any algebraic space $Y \in (\mathcal{O}\text{-alg})^{\vee}/X$, if $f : U \to Y$ is an affine etale covering, and V' is an affine Zariski covering of $U \times_Y U$, then we have

$$\mathrm{Hom}_X(Y,\mathcal{F}) = \mathrm{Ker}\big(\ \mathrm{Hom}_X(U,\mathcal{F}) \rightrightarrows \mathrm{Hom}_X(V',\mathcal{F})\ \big),$$

thus $\mathrm{Hom}_X(Y,\mathcal{F})$ is an abelian group. We define the *global section* functor Γ as:

$$\Gamma : (\mathrm{Sh}/X_{\mathrm{Et}}) \ni \mathcal{F} \longmapsto \mathrm{Hom}_X(X,\mathcal{F}) \in (\mathrm{Ab}),$$

which is a left exact functor, and has a right derived functor:

$$R^q\Gamma = H^q(X,-) : (\mathrm{Sh}/X_{\mathrm{Et}}) \ni \mathcal{F} \longmapsto H^q(X,\mathcal{F}) \in (\mathrm{Ab}),$$

for all $q \geq 0$, where $\Gamma = H^0(X,-)$.

We define the global section functor on $(\mathrm{Sh}/X_{\mathrm{et}})$ as

$$\Gamma := \Gamma \circ u^* : (\mathrm{Sh}/X_{\mathrm{et}}) \ni F \longmapsto \mathrm{Hom}_X(X,u^*F) \in (\mathrm{Ab}).$$

As $u^* : (\mathrm{Sh}/X_{\mathrm{et}}) \to (\mathrm{Sh}/X_{\mathrm{Et}})$ is exact, the right derived functor of Γ on $(\mathrm{Sh}/X_{\mathrm{et}})$ (the *etale cohomology*) is the same as

$$R^q\Gamma = H^q(X,-) : (\mathrm{Sh}/X_{\mathrm{Et}}) \ni F \longmapsto H^q(X,F) := H^q(X,u^*F) \in (\mathrm{Ab}).$$

2.2. **Higher direct image sheaves.** Let $X, S \in (\mathcal{O}\text{-alg})^{\vee}$ be algebraic spaces and $f : X \to S$ be a morphism. The pull-back functor:

$$f^* : (\mathrm{Sh}/S_{\mathrm{Et}}) \longrightarrow (\mathrm{Sh}/X_{\mathrm{Et}})$$
$$(\text{resp. } f^* : (\mathrm{Sh}/S_{\mathrm{et}}) \longrightarrow (\mathrm{Sh}/X_{\mathrm{et}}))$$

is exact. We will define the *push-forward* functor

$$f_* : (\mathrm{Sh}/X_{\mathrm{Et}}) \ni \mathcal{F} \longmapsto f_*\mathcal{F} \in (\mathrm{Sh}/S_{\mathrm{Et}}).$$

For $\mathcal{F} \in (\mathrm{Sh}/X_{\mathrm{Et}})$, the functor $f_*\mathcal{F}$ is defined as:

$$f_*\mathcal{F} : S_{\mathrm{Et}} \ni (R,\varphi) \longmapsto f_*\mathcal{F}(R,\varphi) := \mathrm{Hom}_X(X \otimes_S R,\mathcal{F}) \in (\mathrm{Ab}),$$

where we denote by $X \otimes_S R$ the fiber product:

$$\begin{array}{ccc} X \otimes_S R & \longrightarrow & \mathrm{Spec}\, R \\ \downarrow & & \downarrow \varphi \\ X & \xrightarrow{f} & S. \end{array}$$

This $f_*\mathcal{F}$ is an sheaf on S_{Et}, because for every etale covering $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$, the morphism $X \otimes_S R' \to X \otimes_S R$ is an etale covering, therefore the sequence

$$\mathrm{Hom}_X(X \otimes_S R,\mathcal{F}) \longrightarrow \mathrm{Hom}_X(X \otimes_S R',\mathcal{F}) \rightrightarrows \mathrm{Hom}_X(X \otimes_S (R' \otimes_R R'),\mathcal{F})$$

is exact by the sheaf property of $\mathcal{F}$. The functor f_* is left exact, because $\mathrm{Hom}_X(X \otimes_S R, -)$ is left exact. This functor has the right derived functors

$$R^q f_* : (\mathrm{Sh}/X_{\mathrm{Et}}) \ni \mathcal{F} \longmapsto R^q f_* \mathcal{F} \in (\mathrm{Sh}/S_{\mathrm{Et}}),$$

for all $q \geq 0$, where $f_* = R^0 f_*$.

Similarly, we have a left exact functor

$$f_* : (\mathrm{Sh}/X_{\mathrm{et}}) \ni F \longmapsto f_* F \in (\mathrm{Sh}/S_{\mathrm{et}}),$$

where the functor $f_* F$ is defined as:

$$f_* F : S_{\mathrm{et}} \ni (R, \varphi) \longmapsto f_* F(R, \varphi) := \mathrm{Hom}_X(X \otimes_S R, u^* F) \in (\mathrm{Ab}),$$

which is a sheaf by exactly the same reasoning. Moreover, it is clear from the definition that if $F = u_* \mathcal{F}$, then $f_* F = u_*(f_* \mathcal{F})$, i.e. $f_* u_* \mathcal{F} = u_* f_* \mathcal{F}$. But in general it is not clear whether $f_* \mathcal{F} = u^*(f_* F)$ when $\mathcal{F} = u^* F$ (see the next subsection). This left exact functor f_* has the right derived functor (the *higher direct image sheaves*) $R^q f_* : (\mathrm{Sh}/X_{\mathrm{et}}) \to (\mathrm{Sh}/S_{\mathrm{et}})$.

Example 2.1. Let $f : X \to S = \mathrm{Spec}\, R$, where R is a complete noetherian ring with a separably closed residue field. Then we have an equivalence of categories (see Example 1.4):

$$(\mathrm{Sh}/S_{\mathrm{et}}) \ni F \overset{\cong}{\longmapsto} F(R, \mathrm{id}) \in (\mathrm{Ab}),$$

and under this identification we have $f_* \cong \Gamma$ and $R^q f_* F \cong H^q(X, F)$.

Lemma 2.2. *The sheaf $R^q f_* \mathcal{F} \in (\mathrm{Sh}/S_{\mathrm{Et}})$ (resp. $R^q f_* F \in (\mathrm{Sh}/S_{\mathrm{et}})$) is obtained as the sheafification of the functor:*

$$R^q f_*^0 \mathcal{F} : S_{\mathrm{Et}} \ni (R, \varphi) \longmapsto H^q(X \otimes_S R, \mathcal{F}) \in (\mathrm{Ab})$$
$$(\text{resp. } R^q f_*^0 F : S_{\mathrm{et}} \ni (R, \varphi) \longmapsto H^q(X \otimes_S R, F) \in (\mathrm{Ab})),$$

where we denoted the pull back of $\mathcal{F}$ (resp. F) to $X \otimes_S R$ by the same symbol.

Proof. Denote the category of all functors $X_{\mathrm{Et}} \to (\mathrm{Ab})$ by X^*_{Et}. Then the direct image functor is obtained as the composite:

$$f_* : (\mathrm{Sh}/X_{\mathrm{Et}}) \xrightarrow{f_*^0} S^*_{\mathrm{Et}} \xrightarrow{s} (\mathrm{Sh}/S_{\mathrm{Et}}),$$

where s is the sheafification, the functor f_*^0 is given by

$$f_*^0 \mathcal{F} : S_{\mathrm{Et}} \ni (R, \varphi) \longmapsto \mathrm{Hom}_X(X \otimes_S R, \mathcal{F}) \in (\mathrm{Ab}).$$

As we have $\mathrm{Hom}_X(X \otimes_S R, \mathcal{F}) = \mathrm{Hom}_{X \otimes_S R}(X \otimes_S R, \mathcal{F} \otimes_S R) = \Gamma(\mathcal{F} \otimes_S R)$, we see that $R^q f_*^0 : (\mathrm{Sh}/X_{\mathrm{Et}}) \to S^*_{\mathrm{Et}}$ defined in the lemma are the right derived functors of f_*^0. As s is exact, we have the claim. □

3. Proper Base Change Theorem

3.1. **Constructible sheaves.**

Definition 3.1. Let X be a locally noetherian algebraic space. An etale sheaf $F \in (\mathrm{Sh}/X_{\mathrm{et}})$ is called a *constructible sheaf* if its etale space $\mathcal{F} = u^*F \in (\mathcal{O}\text{-alg})^\vee/X$ is of finite presentation over X.

Example 3.2. Let $X = \operatorname{Spec} R$, where R is a complete noetherian ring with a separably closed residue field. Then the identification of Example 1.4 restricts to:

$$(\mathrm{CSh}/S_{\mathrm{et}}) \ni F \overset{\cong}{\longmapsto} F(R, \mathrm{id}) \in (\mathrm{FAb}),$$

where (FAb) is the category of finite abelian groups.

Proposition 3.3. *The full subcategory* $(\mathrm{CSh}/X_{\mathrm{et}})$ *of* $(\mathrm{Sh}/X_{\mathrm{et}})$ *consisting of constructible etale sheaves on* X_{et} *is an abelian category.*

3.2. **Proper base change theorem.**

Let $\mathcal{O}$ be a noetherian ring, and $S \in (\mathcal{O}\text{-alg})^\vee$ be an algebraic space, separated and locally of finite presentation (hence locally noetherian). Let X be an algebraic space and $f : X \to S$ be a separated morphism locally of finite presentation (hence X is also separated and of locally of finite presentation).

Theorem 3.4. *Let* $f : X \to S$ *be proper. If* $F \in (\mathrm{CSh}/X_{\mathrm{et}})$ *and* $\mathcal{F} := u^*F$*, then:*

(i) *For all* $q \geq 0$*, the etale sheaf* $R^q f_* \mathcal{F} \in (\mathrm{Sh}/S_{\mathrm{Et}})$ *is an etale space over* X*.*
(ii) *For all* $q \geq 0$*, the etale sheaf* $R^q f_* F \in (\mathrm{Sh}/S_{\mathrm{et}})$ *is constructible.*

Corollary 3.5. *Let* $f : X \to S$ *be a proper morphism and* $F \in (\mathrm{CSh}/X_{\mathrm{et}})$*.*

(i) *We have an isomorphism between two functors* $(\mathrm{CSh}/X_{\mathrm{et}}) \to (\mathrm{Sh}/S_{\mathrm{Et}})$*:*

$$u^* R^q f_* \overset{\cong}{\longrightarrow} R^q f_* u^*, \quad \text{i.e.} \quad u^*(R^q f_* F) \cong R^q f_* \mathcal{F} \quad \text{for} \quad \mathcal{F} := u^* F.$$

We also have an isomorphism between two functors $(\mathrm{CSh}/X_{\mathrm{et}}) \to (\mathrm{Sh}/S_{\mathrm{et}})$*:*

$$R^q f_* \overset{\cong}{\longrightarrow} u_* R^q f_* u^*, \quad \text{i.e.} \quad R^q f_* F \cong u_*(R^q f_* \mathcal{F}) \quad \text{for} \quad \mathcal{F} := u^* F.$$

(ii) (Proper base change theorem) *Let* $g : S' \to S$ *be any morphism of algebraic spaces, and let* $X' := X \times_S S'$*:*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

we have an isomorphism between two functors $(\mathrm{CSh}/X_{\mathrm{et}}) \to (\mathrm{CSh}/S'_{\mathrm{et}})$*:*

$$g^* R^q f_* \overset{\cong}{\longrightarrow} R^q f'_* g'^*.$$

(iii) (Finiteness) *Let* $S = \operatorname{Spec} R$*, where* R *is a complete noetherian ring with a separably closed residue field. Then* $R^q f_* F = H^q(X, F) \in (\mathrm{Ab})$ *is a finite abelian group.*

3.3. Sketch of Proof 1. Rephrasing (i) using representability.

Proposition 3.6. *To prove Theorem 3.4(i) for $q \geq 0$, it is enough to show that, for every geometric point $\xi \in S(k)$ and $(R, \iota) \in \mathscr{C}_k/S$, the following is an isomorphism:*

$$H^q(X \otimes_S R, \mathcal{F}) \xrightarrow{\cong} H^q(X \otimes_S k, \mathcal{F}).$$

3.4. Sketch of Proof 2. Proof of (i) for Relative dimension ≤ 1.

3.5. Sketch of Proof 3. Proof of (ii) for Relative dimension ≤ 1.

3.6. Sketch of Proof 4. Devissage.

4. Comparison with Betti Cohomology for Varieties over $\mathbb{C}$

5. Notes on the Literature

References

[SGA4] Artin, M., Grothendieck, A., Verider, J.-L., et. al., *Théorie de Topos et Cohomologie Etale des Schémas (SGA 4)*, Springer LNM 269, 270, 305, 1972/73.

[SGA4$\frac{1}{2}$] Deligne, P., et. al., *Cohomologie Etale (SGA 4$\frac{1}{2}$)*, Springer LNM 569, 1977.

[Art] Artin, M., *Théorèmes de Représentabilité pour les Espaces Algébriques*, Les Presses de l'Univ. Montréal, 1973.

Harvard University, Department of Mathematics, 1 Oxford Street, Cambridge, MA 02138, USA

E-mail address: yoshida@math.harvard.edu