WEEK 4-5: SCHEMES, ALGEBRAIC SPACES AND REPRESENTABILITY

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Abstract. Representability. Etale cohomology.

Lecture 10 (Oct. 15, 2008)

1. Introduction

Now we move on to the consideration of the "moduli space" X_U over $\mathbb{Q}$, or even $\mathbb{Z}$. This is possible because the notion of elliptic curves and abelian varieties makes sense over arbitrary fields, or even arbitrary ring — abelian varieties are none other than "proper smooth group varieties", so its definition does not involve any notion that depends on the base field (in particular, things over $\mathbb{C}$).

In this course, we do assume some familiarity with the theory of schemes, but not too much – we start with asking ourselves what schemes are, and why they are needed, in our context. The first answer is that *all schemes are moduli spaces*, or even that *all mathematical object are moduli spaces*, in the sense of *Yoneda's lemma*. Let us recall what it says: for any category $\mathscr{C}$, every object $X \in \mathscr{C}$ gives a (covariant) functor to the category of sets:

$$\mathcal{F}_X : \mathscr{C} \ni Y \longmapsto \mathrm{Hom}_{\mathscr{C}}(X, Y) \in (\mathrm{Sets}),$$

and this functor knows everything about X as an object of $\mathscr{C}$. More precisely, the contravariant functor $X \longmapsto \mathcal{F}_X$ from $\mathscr{C}$ into its dual, i.e. the category $\mathscr{C}^\vee := \mathrm{Funct}(\mathscr{C}, \mathrm{Sets})$ of functors from $\mathscr{C}$ to (Sets), is fully faithful. This is a fancy way of saying the following: whenever we have a morphism $f : X \to X'$ in $\mathscr{C}$, we have a morphism $f^* : \mathcal{F}_{X'} \to \mathcal{F}_X$ of functors defined as:

$$f^*(Y) : \mathcal{F}_{X'}(Y) = \mathrm{Hom}_{\mathscr{C}}(X', Y) \ni \varphi \longmapsto \varphi \circ f \in \mathrm{Hom}_{\mathscr{C}}(X, Y) = \mathcal{F}_X(Y),$$

for all $Y \in \mathscr{C}$. Conversely, any morphism $\mathcal{F}_{X'} \to \mathcal{F}_X$ is of the form f^* for some $f \in \mathrm{Hom}_{\mathscr{C}}(X, X')$. This embedding of $\mathscr{C}$ into $\mathscr{C}^\vee$ means that the functor $\mathcal{F}_X$ determines X up to isomorphism. We say that a functor $\mathcal{F} \in \mathscr{C}^\vee$ is *representable* if $\mathcal{F} \cong \mathcal{F}_X$ for some $X \in \mathscr{C}$, in which case we say $\mathcal{F}$ is *represented by* X, and X is unique up to isomorphism.

Therefore we have a functor (a *moduli problem*):

$$\mathcal{F} : (\mathcal{O}\text{-algebras}) \ni R \longmapsto \left\{ \; (A, \eta_{\mathrm{et}} U) \text{ over } R \; \right\} / \mathrm{isog} \in (\mathrm{Sets}),$$

where $\mathcal{O}$ is some fixed base ring and A are elliptic curves (over a ring R — we need to define what it means), and ask if this functor is representable. If it is represented by an $\mathcal{O}$-algebra A_U and $\mathbb{C}$ is an $\mathcal{O}$-algebra, then we recover X_U as $\mathcal{F}(\mathbb{C}) = \mathrm{Hom}(A_U, \mathbb{C})$. Actually this is the case when U is small enough and $\mathcal{O}$ is a ring over $\mathbb{Z}[1/N]$ where $N \in \mathbb{Z}_{>0}$ such that $U(N) \subset U$; the ring A_U would be the coordinate ring of a certain (affine) algebraic curve over $\mathcal{O}$ (see [KM], p.70-73). This amounts to the question of finding *automorphic functions*.

But this is not what we will aim for. First of all, the Shimura varieties we want to treat are the *compact* ones, hence they cannot be affine varieties, i.e. the above functor cannot be representable. In fact, they will be represented by *projective schemes*. Secondly, we do not really need the fact that they are schemes for the goal of this lecture, namely to compute the etale cohomology of these spaces. We only need these moduli spaces to be something on which we have etale cohomology groups, i.e. *algebraic spaces*. The schemes and algebraic spaces are both certain class of functors from (Rings) to (Sets), which are not necessarily representable but *locally* representable in a suitable sense.

Of course there are more geometric questions we can ask about Shimura varieties, but for the cohomology theory we only need the corresponding local structure (for example in order to compute the cohomology of holomorphic vector bundles we need them to be complex manifolds). And we will eventually see that only thing we need for computing the etale cohomology of Shimura varieties are the complete local rings (the *deformation rings*) and the counting of points (over $\mathbb{F}_p$). So the fact that Shimura variety is actually a *scheme* is somewhat an overkill. (Well, it is also convenient to know that they are schemes for the purpose of reference, because most literature, even on etale cohomology, are written in terms of schemes and not algebraic spaces.)

2. Schemes and Algebraic Spaces

Let us review the definition of schemes in a functorial language. We fix a base ring $\mathcal{O}$, which is an arbitrary ring (a commutative associative ring with a unit) at the moment, but can be thought of as $\mathbb{Z}$, $\mathbb{Z}[1/N]$, $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{Q}_p$ or their extensions according to the context. Let ($\mathcal{O}$-alg) be the category of $\mathcal{O}$-algebras. Just to make sure: an *$\mathcal{O}$-algebra* is a ring R together with a ring homomorphism $\mathcal{O} \to R$ (the *structure morphism*), and *$\mathcal{O}$-homomorphisms* are the ring homorphisms commuting with the structure morphisms. Consider the functors

$$\mathcal{F} : (\mathcal{O}\text{-alg}) \ni R \longmapsto \mathcal{F}(R) \in (\text{Sets}).$$

The representable functors are called *affine schemes*, and we denote by $\operatorname{Spec} A$ the functor represented by $A \in (\mathcal{O}\text{-alg})$:

$$\operatorname{Spec} A : (\mathcal{O}\text{-alg}) \ni R \longmapsto \mathrm{Hom}_{\mathcal{O}}(A, R) \in (\text{Sets}).$$

The *schemes* and *algebraic spaces* are the functors which are not necessarily representable but in a sense “close” to being representable.

2.1. **Ringed spaces versus functors.** Before going into the (functorial) definition of schemes, let us review the two aspects of algebraic geometry, namely geometric and algebraic. Let us start with the *geometric* one, where we think of schemes (or related spaces) as *ringed spaces*, i.e. a topological space with a structure sheaf of rings (usually the stalks are required to be local rings). For this, we introduce the *Zariski topology* on the set $\operatorname{Spec} A$ of prime ideals of A, and *prove* that the structure presheaf naturally defined by localization is actually a sheaf, hence we get a ringed space (an *affine scheme*). A general scheme is defined as a ringed space which locally is isomorphic to affine schemes; we are defining general schemes by patching affine schemes together along Zariski topology. Note that the morphism of schemes are morphisms as ringed spaces, i.e. the category of schemes is a full subcategory of the category of ringed spaces.

Now the other way of looking at schemes is the *algebraic*, or *functorial* one. We think of affine schemes as the representable functors from ($\mathcal{O}$-alg) to (Sets). If A is an $\mathcal{O}$-algebra *of finite presentation*, i.e.

$$A = \mathcal{O}[X_1, \ldots, X_n]/(f_1, \ldots, f_m),$$

then the set $\operatorname{Hom}_{\mathcal{O}}(A, R)$ is the set of solutions of $(f_1, \ldots, f_m)$ in the ring R. We can say: *all ring homomorphisms are solutions of some system of algebraic equations*, in a way.

When we are thinking of algebraic varieties as subsets of affine or projective n-spaces over a fixed algebraically closed field $\overline{k}$, we are only thinking of the set of solutions (the zero set, the *points* on a variety) in $\overline{k}$. But thinking of the functor $\operatorname{Spec} A$ (or $\mathcal{F}$) amounts to considering the set of solutions in an arbitrary $\mathcal{O}$-algebra R; solving the equations (with coefficients in $\mathcal{O}$) in *all* $\mathcal{O}$-algebras at once. A solution in R is an element of $\mathcal{F}(R)$, which is $\operatorname{Hom}_{\mathcal{O}}(A, R)$ if $\mathcal{F} = \operatorname{Spec} A$, and we call an element of $\mathcal{F}(R)$ an *R-valued point* of $\mathcal{F}$. In this sense, all schemes are *moduli spaces*, solving a *moduli problem*, i.e. a functor sending a ring R to the set of solutions in R of a fixed set of equations. The moduli problem is represented by an $\mathcal{O}$-algebra (resp. schemes) *of finite presentation* if and only if the things that are classified are defined (resp. Zariski locally defined) by a finite number of algebraic equations, i.e. they correspond to the solution of finite set of equations. This is the case for most of the interesting moduli problems (Grassmanians or flag varieties, Jacobians, Picard schemes, Hilbert schemes, Shimura varieties, etc).

Exercise 2.1. The set of prime ideals of A is recovered from the functor $\operatorname{Spec} A$ as the set of equivalence classes of field-valued points of $\operatorname{Spec} A$.

2.2. **Schemes.** Now let us describe the *Zariski topology* in the functorial language. A functor $\mathcal{F}'$ is a *subfunctor* of $\mathcal{F}$ if $\mathcal{F}'(R) \subset \mathcal{F}(R)$ for all R. Precisely speaking, we should consider a morphism of functors $f : \mathcal{F}' \to \mathcal{F}$ and say that f is a *monomorphism* if $f(R)$ is injective for all R. Now for a subset E of $A \in$ ($\mathcal{O}$-alg), we define two subfunctors $\mathcal{U}_E, \mathcal{V}_E$ of $\operatorname{Spec} A$:

$$\mathcal{U}_E(R) := \{\varphi \in \operatorname{Hom}_{\mathcal{O}}(A, R) \mid (\varphi(E)) \ni 1\} \subset \operatorname{Hom}_{\mathcal{O}}(A, R),$$
$$\mathcal{V}_E(R) := \{\varphi \in \operatorname{Hom}_{\mathcal{O}}(A, R) \mid \varphi(E) = \{0\}\} \subset \operatorname{Hom}_{\mathcal{O}}(A, R),$$

where $(\varphi(E))$ denotes the ideal of R generated by $\varphi(E)$. A subfunctor $\mathcal{F}$ of $\operatorname{Spec} A$ is called an *open* (resp. *closed*) subfunctor if it is of the form $\mathcal{U}_E$ (resp. $\mathcal{V}_E$) for a subset E of A. Note that for a fixed E, the subsets $\mathcal{U}_E(R)$ and $\mathcal{V}_E(R)$ are disjoint unless R is the zero ring, and we have:

$$\operatorname{Hom}_{\mathcal{O}}(A, R) = \mathcal{U}_E(R) \amalg \mathcal{V}_E(R) \quad \text{if } R \text{ is a } \textit{field}.$$

Exercise 2.2. When $E = \{t\}$, then $\mathcal{U}_E(R) = \mathrm{Hom}_{\mathcal{O}}(A[1/t], R)$, i.e. $\mathcal{U}_E = \mathrm{Spec}\, A[1/t]$ is an affine scheme. But in general $\mathcal{U}_E$ is not an affine scheme. On the other hand, a closed subfunctor $\mathcal{V}_E$ is always an affine scheme $\mathcal{V}_E = \mathrm{Spec}\big(A/(E)\big)$, where (E) is the ideal of A generated by E.

Now we introduce the notion of *Zariski coverings* of a functor $\mathcal{F}$ by a collection of morphisms $\mathcal{F}'_i \to \mathcal{F}$ from functors $\mathcal{F}'_i$, in particular affine schemes. A morphism $f : \mathcal{F}' \to \mathcal{F}$ is called an *open* (resp. *closed*) *immersion* if for any morphism $\mathrm{Spec}\, A \to \mathcal{F}$, the morphism $\mathrm{Spec}\, A \times_{\mathcal{F}} \mathcal{F}' \to \mathrm{Spec}\, A$ obtained by base change from f is isomorphic to an open (resp. closed) subfunctor of $\mathrm{Spec}\, A$.

We call a field-valued point $\xi \in \mathcal{F}(k)$ for a field k a *point* of $\mathcal{F}$. By Yoneda's lemma, we can think of a point as a morphism of functors:

$$\xi : \mathrm{Spec}\, k \to \mathcal{F},$$

and we denote it also with ξ by abuse of notation. A collection of open immersions $\{ f_i : \mathcal{F}'_i \to \mathcal{F} \}_{i \in I}$ is called a *Zariski covering* of $\mathcal{F}$ if for every point $\xi \in \mathcal{F}(k)$, there exists $i \in I$ and a point $\xi' \in \mathcal{F}'_i(k)$ such that $f_i(\xi') = \xi$. We say that the point $\xi \in \mathcal{F}(k)$ *lifts* to a point $\xi' \in \mathcal{F}'_i(k)$ for some i:

$$\begin{tikzcd} \mathrm{Spec}\, k \arrow[r, dotted, "\xi'"] \arrow[dr, "\xi"'] & \mathcal{F}'_i \arrow[d, "f_i \,:\, \text{open immersion}"] \\ & \mathcal{F} \end{tikzcd}$$

We also say that $(\mathcal{F}'_i, \xi', f_i)$ is a *Zariski neighborhood* of ξ.

Definition 2.3. A functor $\mathcal{F} : (\mathcal{O}\text{-alg}) \to (\mathrm{Sets})$ is called a *scheme* if the following two conditions are satisfied:

(i) It is a *Zariski sheaf* on ($\mathcal{O}$-alg), i.e. for any affine scheme $\mathrm{Spec}\, R$ and its Zariski covering $\{ \mathrm{Spec}\, R'_i \to \mathrm{Spec}\, R \}_{i \in I}$ by affine schemes, the following sequence of maps is exact:
$$\mathcal{F}(R) \longrightarrow \textstyle\prod_i \mathcal{F}(R'_i) \rightrightarrows \textstyle\prod_{i,j} \mathcal{F}(R'_i \otimes_R R'_j) .$$

(ii) It is *Zariski locally representable*, i.e. there is a Zariski covering $\{ \mathrm{Spec}\, A'_i \to \mathcal{F} \}_{i \in I}$ of $\mathcal{F}$ by affine schemes.

The condition (ii) is equivalent to say that *every point* $\xi \in \mathcal{F}(k)$ *has an affine Zariski neighborhood*, because then we can take I as the set of all points of $\mathcal{F}$. We need to show that affine schemes are schemes by checking the first property, but we will do it after we introduce etale morphisms and algebraic spaces.

Lecture 11 (Oct. 17, 2008)

2.3. **Etale coverings.** We first extract some finiteness properties that open immersions had. A morphism $\mathcal{F}' \to \mathrm{Spec}\, A$ of functors is called *locally of finite presentation* if there is a Zariski covering $\{ \operatorname{Spec} A'_i \to \mathcal{F}' \}_{i \in I}$ of $\mathcal{F}'$ by affine schemes such that the composite morphism $\operatorname{Spec} A'_i \to \operatorname{Spec} A$ makes A' into an A-algebra of finite presentation. A general morphism $f : \mathcal{F}' \to \mathcal{F}$ is called *locally of finite presentation* if for any morphism $\operatorname{Spec} A \to \mathcal{F}$, the morphism $\operatorname{Spec} A \times_{\mathcal{F}} \mathcal{F}' \to \operatorname{Spec} A$ obtained by base change from f is locally of finite presentation.

Example 2.4. Open immersion $\mathcal{F}' \to \mathcal{F}$ is locally of finite presentation. It is enough to check this when $\mathcal{F} = \operatorname{Spec} A$, and every open subfunctor $\mathcal{U}_E$ of $\operatorname{Spec} A$ is covered by the affine subfunctors $\{ \mathcal{U}_{\{t\}} = \operatorname{Spec} A[1/t] \mid t \in E \}$.

Remark 2.5. If $\mathcal{F}', \mathcal{F}$ are Zariski sheaves, then for any morphism $\operatorname{Spec} A \to \mathcal{F}$, the functor $\operatorname{Spec} A \times_{\mathcal{F}} \mathcal{F}'$ is also a Zariski sheaf. Therefore, for a morphism $f : \mathcal{F}' \to \mathcal{F}$ locally of finite presentation between Zariski sheaves, the functor $\operatorname{Spec} A \times_{\mathcal{F}} \mathcal{F}'$ is a scheme.

Definition 2.6. Let $f : \mathcal{F}' \to \mathcal{F}$ be a morphism, and $\xi' \in \mathcal{F}'(k)$ be a point of $\mathcal{F}'$. We say f is *formally etale* at ξ' if the following holds: for every Artinian local ring $R \in (\mathcal{O}\text{-alg})$ with maximal ideal $\mathfrak{m}$ and $\iota : R/\mathfrak{m} \xrightarrow{\cong} k$, and a commutative diagram of solid arrows as below:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[d, "\iota"'] & \mathcal{F}' \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "\eta"'] \arrow[ur, dotted, "\eta'"] & \mathcal{F} \end{tikzcd}$$

there is a unique morphism η' making the diagram commute. A morphism $\mathcal{F}' \to \mathcal{F}$ is called *etale* if the following two conditions are satisfied:

(i) It is locally of finite presentation.
(ii) It is *formally etale* at all points $\xi' \in \mathcal{F}'(k)$ of $\mathcal{F}'$.

Exercise 2.7. A morphism f is formally etale at every point of $\mathcal{F}'$ if, for every Artinian local ring $R \in (\mathcal{O}\text{-alg})$ and an ideal $I \subset R$ with $I^2 = 0$, the diagram:

$$\begin{tikzcd} \mathcal{F}'(R) \arrow[r] \arrow[d] & \mathcal{F}'(R/I) \arrow[d] \\ \mathcal{F}(R) \arrow[r] & \mathcal{F}(R/I) \end{tikzcd}$$

is cartesian, i.e. induces a bijection from $\operatorname{Spec} A'(R) = \operatorname{Hom}_{\mathcal{O}}(A', R)$ to the fiber product $\mathcal{F}(R) \times_{\mathcal{F}(R/I)} \operatorname{Spec} A'(R/I)$.

Now the coverings. A point $\xi \in \mathcal{F}(k)$ of $\mathcal{F}$ is called a *geometric point* if k is *separably closed*. A collection of etale morphisms $\{ f_i : \mathcal{F}'_i \to \mathcal{F} \}_{i \in I}$ is called an *etale covering* of $\mathcal{F}$ if every *geometric* point $\xi \in \mathcal{F}(k)$ lifts to a (geometric) point $\xi' \in \mathcal{F}'_i(k)$ for some i:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, dotted, "\xi'"] \arrow[dr, "\xi"'] & \mathcal{F}'_i \arrow[d, "f_i \,:\, \text{etale}"] \\ & \mathcal{F} \end{tikzcd}$$

We also say that $(\mathcal{F}'_i, \xi', f_i)$ is an *etale neighborhood* of ξ.

Definition 2.8. A functor $\mathcal{F} : (\mathcal{O}\text{-alg}) \to (\text{Sets})$ is called an *algebraic space* if the following two conditions are satisfied:

(i) It is a *etale sheaf* on $(\mathcal{O}\text{-alg})$, i.e. for any affine scheme $\operatorname{Spec} R$ and its etale covering $\{ \operatorname{Spec} R'_i \to \operatorname{Spec} R \}_{i \in I}$ by affine schemes, the sequence of maps is exact:
$$\mathcal{F}(R) \longrightarrow \prod_i \mathcal{F}(R'_i) \rightrightarrows \prod_{i,j} \mathcal{F}(R'_i \otimes_R R'_j) \ .$$

(ii) It is *etale locally representable*, i.e. there is an etale covering $\{ \operatorname{Spec} A'_i \to \mathcal{F} \}_{i \in I}$ of $\mathcal{F}$ by affine schemes.

The condition (ii) is equivalent to say that *every geometric point $\xi \in \mathcal{F}(k)$ has an affine etale neighborhood*, because then we can take I as the set of all geometric points of $\mathcal{F}$.

Lecture 12 (Oct. 20, 2008)

3. Local Rings and Deformation Rings

3.1. **Etale neighborhoods and local rings.** Let us first recall the notion of Zariski/etale neighborhood. For a point $\xi \in \mathcal{F}(k)$, an (affine) *Zariski* (resp. *etale*) *neighborhood* of ξ is a triple (A, ξ', f) which makes the following diagram commute:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[dr, "\xi"'] & \operatorname{Spec} A \arrow[d, "f\ :\ \text{open immersion (resp. etale)}"] \\ & \mathcal{F} \end{tikzcd}$$

The Zariski neighborhoods and open immersions between them (resp. etale neighborhoods and etale morphisms between them) make a direct system, and the *Zariski* (resp. *etale*) *local ring* of $\mathcal{F}$ at ξ is defined as:
$$\mathcal{O}_\xi := \varinjlim A, \qquad \big(\text{resp. } \mathcal{O}_\xi^{\text{et}} := \varinjlim A, \big)$$
where the direct limit is taken over all Zariski (resp. etale) neighborhoods (A, ξ', f) of ξ. Precisely speaking, it is a triple $(\mathcal{O}_\xi, \xi_0, f_0)$ (resp. $(\mathcal{O}_\xi^{\text{et}}, \xi_0^{\text{et}}, f_0^{\text{et}})$) as follows:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi_0"] \arrow[dr, "\xi"'] & \operatorname{Spec} \mathcal{O}_\xi \arrow[d, "f_0"] \\ & \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi_0^{\text{et}}"] \arrow[dr, "\xi"'] & \operatorname{Spec} \mathcal{O}_\xi^{\text{et}} \arrow[d, "f_0^{\text{et}}"] \\ & \mathcal{F} \end{tikzcd}$$

but this is *not* necessarily a Zariski (resp. etale) neighborhood of ξ any more, because the morphism f_0 (resp. f_0^{et}) is not locally of finite presentation any more, but a limit of them. We might think of this morphism as being *formally an open immersion*, e.g. $\operatorname{Spec} \mathbb{Z}_{(p)} \to \operatorname{Spec} \mathbb{Z}$ (resp. we will show that f_0^{et} is formally etale at ξ_0^{et}).

The Zariski (resp. etale) local ring is determined by any Zariski (resp. etale) neighborhood of the point, i.e. if (A, ξ', f) is a Zariski (resp. etale) neighborhood of ξ, then the Zariski (resp. etale) neighborhoods of $\xi' \in \operatorname{Spec} A(k)$ are cofinal in the direct system of all Zariski (resp. etale) neighborhoods of ξ, thus $\mathcal{O}_{\xi'} \xrightarrow{\cong} \mathcal{O}_\xi$ (resp. $\mathcal{O}^{\mathrm{et}}_{\xi'} \xrightarrow{\cong} \mathcal{O}^{\mathrm{et}}_{\xi}$).

Lemma 3.1. *Let R be a local ring with maximal ideal $\mathfrak{m}$, and $\iota \in \operatorname{Spec} R(k)$ be such that $\iota : R/\mathfrak{m} \xrightarrow{\cong} k$.*

(i) *If $(\mathcal{F}', \xi', f)$ is a Zariski neighborhood of ι, then f is an isomorphism.*
(ii) *We have $R \xrightarrow{\cong} \mathcal{O}_\iota$. (There is no Zariski neighborhood of ι other than (R, ι, id).)*

Proof. (i): We have $f : \mathcal{F}' \xrightarrow{\cong} \mathcal{U}_E$ for some $E \subset R$, i.e.

$$f_* : \mathcal{F}'(R') \ni \xi \overset{\cong}{\longmapsto} \xi \circ f \in \mathcal{U}_E(R') = \left\{\varphi \in \operatorname{Hom}_{\mathcal{O}}(R, R') \mid (\varphi(E)) \ni 1\right\}$$

for all $R' \in (\mathcal{O}\text{-alg})$. Taking $R' = k$, by $\xi' \mapsto \iota \in \mathcal{U}_E(k)$ we have $(\iota(E)) \ni 1$, or $E \cap R^\times = E \setminus \mathfrak{m} \neq \emptyset$. Therefore $\mathcal{U}_E = \operatorname{Spec} R$.

(ii): Every Zariski neighborhood (A, ξ', f) of ι is isomorphic to (R, ι, id) by (i). $\square$

Corollary 3.2. *Open immersions are etale. (Hence Zariski coverings are etale coverings, and etale sheaves are Zariski sheaves.)*

Proof. Open immersions are locally of finite presentation by Example 2.4. For an open immersion f and a diagram as below with an Artinian R and $\iota : R/\mathfrak{m} \xrightarrow{\cong} k$, we can take the fiber product:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[d, "\iota"'] & \mathcal{F}' \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "\eta"'] \arrow[ur, dotted, "\eta'"] & \mathcal{F} \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[d, "\iota"'] & \operatorname{Spec} R \times_{\mathcal{F}} \mathcal{F}' \arrow[d, "f' \,:\, \text{open imm.}"] \\ \operatorname{Spec} R \arrow[r, equal] \arrow[ur, dotted, "\eta'"] & \operatorname{Spec} R \end{tikzcd}$$

The unique existence of η' is clear because f' is an isomorphism by Lemma 3.1(i). $\square$

This corollary shows that, for a point $\xi \in \mathcal{F}(k)$, the point ξ_0 of the Zariski local ring $\mathcal{O}_\xi$ factors through the etale local ring $\mathcal{O}^{\mathrm{et}}_\xi$:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi_0^{\mathrm{et}}"] \arrow[dr, "\xi"'] & \operatorname{Spec} \mathcal{O}^{\mathrm{et}}_\xi \arrow[r] \arrow[d, "f_0^{\mathrm{et}}"] & \operatorname{Spec} \mathcal{O}_\xi \arrow[dl, "f_0"] \\ & \mathcal{F} & \end{tikzcd}$$

Exercise 3.3. More generally, if $\mathcal{F} = \operatorname{Spec} A$ is an affine scheme, then every point $\xi \in \operatorname{Spec} A(k)$ factors through $\operatorname{Spec} \kappa(\mathfrak{p})$ for a unique prime ideal $\mathfrak{p}$ of A, and $\mathcal{O}_\xi$ is canonically isomorphic to the localization $A_{\mathfrak{p}}$, and $\mathcal{O}_\xi/\mathfrak{m} \cong \kappa(\mathfrak{p})$.

Remark 3.4. Similarly, for $\mathcal{F} = \operatorname{Spec} A$ and $\xi \in \operatorname{Spec} A(k)$ factoring through $\kappa(\mathfrak{p})$ such that $\kappa(\mathfrak{p}) \cong k$, the etale local ring $\mathcal{O}_\xi^{\mathrm{et}}$ is canonically isomorphic to a local ring called the *henselization* of $A_\mathfrak{p}$. If $\xi \cong \operatorname{Spec} A(k)$ is a geometric point with $\kappa(\mathfrak{p})^{\mathrm{sep}} \cong k$, then $\mathcal{O}_\xi^{\mathrm{et}}$ is canonically isomorphic to the *strict henselization* of $A_\mathfrak{p}$.

3.2. **Deformation rings.**

Definition 3.5. Let $k \in (\mathcal{O}\text{-alg})$ be a field. The category $\mathscr{C}_k$ is defined as the category of all pairs (R, ι) where R is a complete noetherian local $\mathcal{O}$-algebra (with maximal ideal $\mathfrak{m}$), and an $\mathcal{O}$-homomorphism $\iota \in \operatorname{Hom}_\mathcal{O}(R, k)$ which induces an isomorphism $R/\mathfrak{m} \cong k$. A morphism $f : (R, \iota) \to (R', \iota')$ of $\mathscr{C}_k$ is a local homomorphism $f : R \to R'$ with $\iota' \circ f = \iota$. We denote by $\mathscr{C}_k^0$ the full subcategory of pairs (R, ι) with Artinian R.

For a point $\xi \in \mathcal{F}(k)$, a *deformation* of ξ is a triple (R, ι, η) with $(R, \iota) \in \mathscr{C}_k$ which makes the following diagram commute:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[d, "\iota"'] \arrow[dr, "\xi"] & \\
\operatorname{Spec} R \arrow[r, "\eta"'] & \mathcal{F}
\end{tikzcd}$$

We say that $\eta \in \mathcal{F}(R)$ is a deformation of ξ to (R, ι). We denote the set of all deformations of ξ to (R, ι) by $\operatorname{Def}_\xi \mathcal{F}(R, \iota)$. We call the functor

$$\begin{aligned}
\operatorname{Def}_\xi \mathcal{F} : \mathscr{C}_k &\longrightarrow (\mathrm{Sets}) \\
(R, \iota) &\longmapsto \operatorname{Def}_\xi \mathcal{F}(R, \iota) := \big\{ \eta \in \mathcal{F}(R) \mid \eta \circ \iota = \xi \big\}
\end{aligned}$$

the *deformation functor* of $\mathcal{F}$ at ξ. We denote the restriction of $\operatorname{Def}_\xi \mathcal{F}$ to $\mathscr{C}_k^0$ by $\operatorname{Def}_\xi^0 \mathcal{F}$.

Lemma 3.6. *Let $A \in (\mathcal{O}\text{-alg})$ be a noetherian local ring with the maximal ideal $\mathfrak{m}$ and $\xi : A/\mathfrak{m} \xrightarrow{\cong} k$. Then $\operatorname{Def}_\xi(\operatorname{Spec} A)$ is represented by $(\widehat{A}, \widehat{\xi}) \in \mathscr{C}_k$, where $\widehat{\xi} : \widehat{A}/\widehat{\mathfrak{m}} \xrightarrow{\cong} k$.*

Proof. For $(R, \iota) \in \mathscr{C}_k$, we have

$$\operatorname{Def}_\xi(\operatorname{Spec} A)(R, \iota) = \big\{ \eta \in \operatorname{Hom}_\mathcal{O}(A, R) \mid \iota \circ \eta = \xi \big\}.$$

As R is complete, every local morphism η factors through the completion $\widehat{A}$, hence

$$\operatorname{Def}_\xi(\operatorname{Spec} A)(R, \iota) = \operatorname{Hom}_{\mathscr{C}_k}\big((\widehat{A}, \widehat{\xi}), (R, \iota) \big),$$

i.e. the functor $\operatorname{Def}_\xi(\operatorname{Spec} A)$ is represented by $(\widehat{A}, \widehat{\xi})$. □

Let $f : \mathcal{F}' \to \mathcal{F}$ be a morphism. Let $\xi' \in \mathcal{F}'(k)$ be a point of $\mathcal{F}'$, and $\xi \in \mathcal{F}(k)$ be its image. Then f induces a morphism of functors by $\eta' \mapsto f \circ \eta'$:

$$f_* : \operatorname{Def}_{\xi'} \mathcal{F}' \longrightarrow \operatorname{Def}_\xi \mathcal{F}, \qquad f_* : \operatorname{Def}_{\xi'}^0 \mathcal{F}' \longrightarrow \operatorname{Def}_\xi^0 \mathcal{F}.$$

Note that f is formally etale at ξ' if and only if the latter morphism is an isomorphism: for any $(R,\iota) \in \mathscr{C}_k^0$ and a commutative diagram of the following form (the left one):

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[d, "\iota"'] \arrow[dr, "\xi"] & \mathcal{F}' \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "\eta"'] & \mathcal{F} \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[d, "\iota"'] & \mathcal{F}' \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "\eta"'] \arrow[ur, dotted, "\eta'"] & \mathcal{F} \end{tikzcd}$$

there is a *unique* η' making the right diagram commute, i.e. we have the bijection:

$$\operatorname{Def}^0_{\xi'} \mathcal{F}'(R,\iota) \ni \eta' \longmapsto \eta \in \operatorname{Def}^0_{\xi} \mathcal{F}(R,\iota).$$

Lemma 3.7. *Let $\xi \in \mathcal{F}(k)$, and let $(\mathcal{O}^{\mathrm{et}}_{\xi}, \xi^{\mathrm{et}}_0, f^{\mathrm{et}}_0), (\mathcal{O}_{\xi}, \xi_0, f_0)$ be its etale/Zariski local ring. Then f^{et}_0, f_0 is formally etale at $\xi^{\mathrm{et}}_0, \xi_0$, i.e. we have: Then we have:*

$$f^{\mathrm{et}}_{0*} : \operatorname{Def}^0_{\xi^{\mathrm{et}}_0}(\operatorname{Spec} \mathcal{O}^{\mathrm{et}}_{\xi}) \xrightarrow{\cong} \operatorname{Def}^0_{\xi_0}(\operatorname{Spec} \mathcal{O}_{\xi}) \xrightarrow{\cong} \operatorname{Def}^0_{\xi} \mathcal{F}.$$

Proof. Let (A, ξ', f) be an etale neighborhood of ξ; we have $\operatorname{Spec} \mathcal{O}^{\mathrm{et}}_{\xi} = \varprojlim \operatorname{Spec} A$.

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[dr, "\xi"'] & \operatorname{Spec} A \arrow[d, "f"] \\ & \mathcal{F} \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi^{\mathrm{et}}_0"] \arrow[dr, "\xi"'] & \operatorname{Spec} \mathcal{O}^{\mathrm{et}}_{\xi} \arrow[r] \arrow[d, "f^{\mathrm{et}}_0"] & \operatorname{Spec} \mathcal{O}_{\xi} \arrow[dl, "f_0"] \\ & \mathcal{F} & \end{tikzcd}$$

By the formal etaleness of f, we have the unique η' in the left diagram for all $(R,\iota) \in \mathscr{C}_k^0$:

(3.2.1)
$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[d, "\iota"'] & \operatorname{Spec} A \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "\eta"'] \arrow[ur, dotted, "\eta'"] & \mathcal{F} \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi^{\mathrm{et}}_0"] \arrow[d, "\iota"'] & \operatorname{Spec} \mathcal{O}^{\mathrm{et}}_{\xi} \arrow[r] \arrow[d, "f^{\mathrm{et}}_0"] & \operatorname{Spec} \mathcal{O}_{\xi} \arrow[dl, "f_0"] \\ \operatorname{Spec} R \arrow[r, "\eta"'] \arrow[ur, dotted, "\eta^{\mathrm{et}}_0"] & \mathcal{F} & \end{tikzcd}$$

and hence the unique η^{et}_0 for the right diagram; then $\eta \mapsto \eta^{\mathrm{et}}_0$ gives the inverse to f^{et}_{0*}. $\square$

We will show that, for these f and f_0, the induced morphisms between Def (not just Def^0) are isomorphisms.

Lecture 13 (Oct. 22, 2008)

Lemma 3.8. *Let k be a field, and $(R,\iota) \in \mathscr{C}_k$. Let $\mathfrak{m}$ be the maximal ideal of R.*

(i) *Let $f : \operatorname{Spec} A \to \operatorname{Spec} R$ be a morphism and $\xi' \in \operatorname{Spec} A(k)$ be a point such that $f(\xi') = \iota$. If f is formally etale at ξ', then there is a unique $\eta' : \operatorname{Spec} R \to \operatorname{Spec} A$ such that $\eta' \circ \iota = \xi'$ and $f \circ \eta' = \mathrm{id}$.*

(ii) *If $(R,\iota) \in \mathscr{C}_k$, then $R \xrightarrow{\cong} \mathcal{O}^{\mathrm{et}}_{\iota}$. (There is no "connected" etale neighborhood of ι other than (R,ι).))*

Proof. (i): As $R/\mathfrak{m}^n$ is Artinian for all $n \in \mathbb{Z}_{>0}$, the formal etaleness of f implies that there is a unique η'_n making the following diagram commute:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[d, "\iota_n"'] & \operatorname{Spec} A \arrow[d, "f"] \\ \operatorname{Spec} R/\mathfrak{m}^n \arrow[r] \arrow[ur, dotted, "\eta'_n"] & \operatorname{Spec} R \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[d, "\iota"'] & \operatorname{Spec} A \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, equal] \arrow[ur, dotted, "\eta'"] & \operatorname{Spec} R \end{tikzcd}$$

Thus we obtain $\eta' := \varprojlim \eta'_n \in \varprojlim \operatorname{Hom}_{\mathcal{O}}(A, R/\mathfrak{m}^n) = \operatorname{Hom}_{\mathcal{O}}(A, R) = \operatorname{Spec} A(R)$.

(ii): Among the etale neighborhoods (A, ξ', f) of ι, the triple (R, ι, id) is final by (i). $\square$

Proposition 3.9. *Let k be a field.*

(i) *Let $f : \mathcal{F}' \to \mathcal{F}$ be an etale morphism. Let $\xi' \in \mathcal{F}'(k)$ and $\xi \in \mathcal{F}(k)$ be its image. Then the following morphism of functors is an isomorphism:*

$$f_* : \operatorname{Def}_{\xi'} \mathcal{F}' \xrightarrow{\cong} \operatorname{Def}_{\xi} \mathcal{F}.$$

(ii) *Let $\xi \in \mathcal{F}(k)$, and let $(\mathcal{O}^{\mathrm{et}}_{\xi}, \xi^{\mathrm{et}}_0, f^{\mathrm{et}}_0), (\mathcal{O}_{\xi}, \xi_0, f_0)$ be its etale/Zariski local ring. Then we have:*

$$f^{\mathrm{et}}_{0*} : \operatorname{Def}_{\xi^{\mathrm{et}}_0}(\operatorname{Spec} \mathcal{O}^{\mathrm{et}}_{\xi}) \xrightarrow{\cong} \operatorname{Def}_{\xi_0}(\operatorname{Spec} \mathcal{O}_{\xi}) \xrightarrow{\cong} \operatorname{Def}_{\xi} \mathcal{F}.$$

Proof. (i): Let $(R, \iota) \in \mathscr{C}_k$ and $\eta \in \operatorname{Def}_{\xi} \mathcal{F}(R, \iota)$ as in the right diagram:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[drr, "\iota"'] & \operatorname{Spec} A \arrow[r, "\text{open}"] & \operatorname{Spec} R \times_{\mathcal{F}} \mathcal{F}' \arrow[r] \arrow[d, "f' \,:\, \text{etale}"] & \mathcal{F}' \arrow[d, "f \,:\, \text{etale}"] \\ & & \operatorname{Spec} R \arrow[r, "\eta"'] \arrow[ul, dotted, "\eta'"'] & \mathcal{F} \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[d, "\iota"'] & \mathcal{F}' \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "\eta"'] \arrow[ur, dotted, "\eta'"] & \mathcal{F} \end{tikzcd}$$

We took the fiber product of η and f in the left diagram. As f is locally of finite presentation, the functor $\operatorname{Spec} R \times_{\mathcal{F}} \mathcal{F}'$ has an affine Zariski covering. Hence the point ξ' factors through a Zariski neighborhood $\operatorname{Spec} A$. As this A gives an etale neighborhood of ι, we have a unique morphism $\eta' : \operatorname{Spec} R \to \operatorname{Spec} A$ making the left diagram commute by Lemma 3.8(i), thus a unique $\eta' : \operatorname{Spec} R \to \mathcal{F}'$ making the right diagram commute. This shows the bijection $\operatorname{Def}_{\xi'} \mathcal{F}'(R, \iota) \ni \eta' \longmapsto \eta \in \operatorname{Def}_{\xi} \mathcal{F}(R, \iota)$.

(ii): By (i), the implication of (3.2.1) now holds for all $(R, \iota) \in \mathscr{C}_k$. $\square$

3.3. **Noetherian case.**

Definition 3.10. A scheme $\mathcal{F}$ is called a *locally noetherian scheme* if every point has an affine Zariski neighborhood (A, ξ', f) where A is a noetherian $\mathcal{O}$-algebra. An algebraic space $\mathcal{F}$ is called a *locally noetherian algebraic space* if every geometric point has an affine etale neighborhood (A, ξ', f) where A is a noetherian $\mathcal{O}$-algebra.

By Exercise 3.3, every Zariski local ring $\mathcal{O}_\xi$ of a locally noetherian scheme is a noetherian local ring. (We will see the etale counterpart of this statement in Proposition 4.5(ii).)

Proposition 3.11. (i) *Let $\mathcal{F}$ be a locally noetherian scheme, and $\xi \in \mathcal{F}(k)$. If $\xi_0 : \mathcal{O}_\xi/\mathfrak{m} \xrightarrow{\cong} k$, then the functor $\mathrm{Def}_\xi \mathcal{F}$ is represented by $(\widehat{\mathcal{O}}_\xi, \widehat{\xi}_0) \in \mathscr{C}_k$.*

(ii) *Let $f : \operatorname{Spec} A' \to \operatorname{Spec} A$ be an etale morphism, where A, A' are noetherian $\mathcal{O}$-algebras. Let $\xi' \in \operatorname{Spec} A'(k)$, and its image be $\xi \in \operatorname{Spec} A(k)$. If we have $\xi_0 : \mathcal{O}_\xi/\mathfrak{m} \xrightarrow{\cong} \mathcal{O}_{\xi'}/\mathfrak{m}' \xrightarrow{\cong} k$, then $\widehat{\mathcal{O}}_\xi \xrightarrow{\cong} \widehat{\mathcal{O}}_{\xi'}$.*

Proof. (i): As $\mathcal{O}_\xi$ is noetherian, use Lemma 3.6.

(ii): Combine (i) with Proposition 3.9(i). $\square$

Proposition 3.12. *For noetherian $\mathcal{O}$-algebras A and A', an etale morphism $f : \operatorname{Spec} A' \to \operatorname{Spec} A$ is flat.*

Proof. For every point $\xi' \in \operatorname{Spec} A'(k)$ of $\operatorname{Spec} A'$ and its image $\xi \in \operatorname{Spec} A$, there is an induced morphism $f : \operatorname{Spec} \mathcal{O}_{\xi'} \to \operatorname{Spec} \mathcal{O}_\xi$. It is enough to show this morphism is flat at ξ'_0 for all ξ'. By replacing k by $\mathcal{O}_{\xi'}/\mathfrak{m}'$, we can assume $\xi'_0 : \mathcal{O}_{\xi'}/\mathfrak{m}' \xrightarrow{\cong} k$. (**Reduce** here to the case where $\mathcal{O}_\xi/\mathfrak{m} \cong \mathcal{O}_{\xi'}/\mathfrak{m}'$ by faithfully flat descent.) By Lemma 3.7, we have the isomorphisms of deformation functors:

$$\begin{tikzcd}
\mathrm{Def}^0_{\xi'_0}(\operatorname{Spec} \mathcal{O}_{\xi'}) \arrow[r, "f_{0*}", "\cong"'] \arrow[d, "f_*"', "\cong"] & \mathrm{Def}^0_{\xi'} \mathcal{F}' \arrow[d, "f_*"', "\cong"] \\
\mathrm{Def}^0_{\xi_0}(\operatorname{Spec} \mathcal{O}_\xi) \arrow[r, "f_{0*}", "\cong"'] & \mathrm{Def}^0_\xi \mathcal{F}
\end{tikzcd}$$

Thus $f : \operatorname{Spec} \mathcal{O}_{\xi'} \to \operatorname{Spec} \mathcal{O}_\xi$ is formally etale at ξ'_0. By Proposition 3.11(ii), the induced morphism $\widehat{\mathcal{O}}_\xi \to \widehat{\mathcal{O}}_{\xi'}$ is an isomorphism. We have a diagram:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "\widehat{\xi}'_0"] & \operatorname{Spec} \widehat{\mathcal{O}}_{\xi'} \arrow[r] \arrow[dr, "\cong"'] & \operatorname{Spec}(\widehat{\mathcal{O}}_\xi \otimes_{\mathcal{O}_\xi} \mathcal{O}_{\xi'}) \arrow[r] \arrow[d, shift left, "f' \,:\, \text{fm. et. at } \widehat{\xi}'_0"] & \operatorname{Spec} \mathcal{O}_{\xi'} \arrow[d, "f \,:\, \text{fm. et. at } \xi'_0"] \\
 & & \operatorname{Spec} \widehat{\mathcal{O}}_\xi \arrow[r] \arrow[u, dotted, shift left, "\eta'"] & \operatorname{Spec} \mathcal{O}_\xi
\end{tikzcd}$$

where the right square is the fiber product, hence f' is formally etale at $\widehat{\xi}'_0$. Therefore by Lemma 3.8(i), there is a unique η' making the triangle commute and $f' \circ \eta' = \mathrm{id}$. This shows that $(\widehat{\mathcal{O}}_\xi)_{\widehat{\xi}_0} \xrightarrow{\cong} (\widehat{\mathcal{O}}_\xi \otimes \mathcal{O}_{\xi'})_{\widehat{\xi}'_0}$, hence f' is flat at $\widehat{\xi}'_0$. As $\mathcal{O}_\xi \to \widehat{\mathcal{O}}_\xi$ is faithfully flat, we deduce that f is flat at ξ'_0. $\square$

3.4. **Faithfully flat descent.**

Proposition 3.13. *Affine schemes are etale sheaves. In particular, affine schemes are schemes, and schemes are algebraic spaces.*

4. Representability Theorem

4.1. **Functors locally of finite presentation.** For the representability theorem, we consider the functors that are *locally of finite presentation*, i.e. it commutes with every filtered direct limits:

$$\mathcal{F}\big(\varinjlim_i R_i\big) = \varinjlim_i \mathcal{F}(R_i).$$

This definition is motivated by the fact that every $R \in (\mathcal{O}\text{-alg})$ is a direct limit of filtered direct system $\{R_i\}$ of $\mathcal{O}$-algebras of finite presentation (you adjoin the generators and the relations one by one). Thus a functor locally of finite presentation is determined by the sets $\mathcal{F}(R)$ for $\mathcal{O}$-algebras R of finite presentation. If we think of an element of $\mathcal{F}(R)$ (an R-valued point of $\mathcal{F}$) as a morphism $\operatorname{Spec} R \to \mathcal{F}$, then we can rephrase this definition as: *every R-valued point factors through an R'-valued point for some $\mathcal{O}$-algebra R' of finite presentation.*

We call a point $\xi : \operatorname{Spec} k \to \mathcal{F}$ a point *of finite presentation* if it factors through $\operatorname{Spec} k'$ where k' is a field which is of finitely presentation over $\mathcal{O}$. (If $\mathcal{F}$ is locally of finite presentation, then every point $\xi \in \mathcal{F}(k)$ of $\mathcal{F}$ factors through $\operatorname{Spec} A$ for an $\mathcal{O}$-algebra A of finite presentaion, but it is not necessarily a point of finite presentation unless it factors through $k' = A/\mathfrak{m}$ for a finitely generated maximal ideal $\mathfrak{m}$ of A.)

Lemma 4.1. *Let $\mathcal{F} : (\mathcal{O}\text{-alg}) \to (\text{Sets})$ be locally of finite presentation.*

(i) *If $\mathcal{F}$ is an affine scheme* $\operatorname{Spec} A$*, then A is an $\mathcal{O}$-algebra of finite presentation.*
(ii) *A collection $\{ \mathcal{F}'_i \to \mathcal{F} \}$ of etale morphisms (resp. open immersions) is an etale (resp. Zariski) covering if all geometric points (resp. points)* of finite presentation *of $\mathcal{F}$ lifts to a point of $\mathcal{F}'_i$ for some i.*

Proof. (i): Write $A = \varinjlim A_i$ with $\mathcal{O}$-algebras A_i of finite presentation. As $\mathrm{id} \in \operatorname{Hom}_{\mathcal{O}}(A, A) = \mathcal{F}(A) = \varinjlim \mathcal{F}(A_i) = \varinjlim \operatorname{Hom}_{\mathcal{O}}(A, A_i)$, the id factors though A_i for some i, i.e. $A = A_i$.

(ii): Any point $\xi \in \mathcal{F}(k)$ of $\mathcal{F}$ factors through $\operatorname{Spec} A$ for an $\mathcal{O}$-algebra A of finite presentaion. For a morphism $\mathcal{F}' \to \mathcal{F}$ locally of finite presentation, the functor $\operatorname{Spec} A \times_{\mathcal{F}} \mathcal{F}'$ has a Zariski covering by affine schemes $\operatorname{Spec} A'_i$, where A'_i are A-algebras of finite presentation:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, dotted, "\xi''"] \arrow[dr, dotted, "\xi'"] \arrow[ddr, "\xi"'] & \operatorname{Spec} A'_i \arrow[d, "\text{open}"] & \\
& \operatorname{Spec} A \times_{\mathcal{F}} \mathcal{F}' \arrow[r] \arrow[d, "f' \,:\, \text{loc. f.p.}"] & \mathcal{F}' \arrow[d, "f \,:\, \text{loc. f.p.}"] \\
& \operatorname{Spec} A \arrow[r] & \mathcal{F}
\end{tikzcd}$$

Thus if ξ lifts to a point of $\mathcal{F}'$, then it lifts to a point $\xi'' \in \operatorname{Spec} A_i'(k)$ for some i. Therefore we have reduced the question to the case where $\mathcal{F} = \operatorname{Spec} A$ and $\mathcal{F}_i = \operatorname{Spec} A_i'$ are affine schemes and A, A_i' are $\mathcal{O}$-algebras of finite presentation. $\square$

Lecture 14 (Oct. 24, 2008)

4.2. **Patching procedure.** The two defining properties of schemes/algebraic spaces $\mathcal{F}$, the sheaf property and the local representability, interact when we try to construct an R-valued point using the affince covering of $\mathcal{F}$. Suppose we have an R-valued point $\xi \in \mathcal{F}(R)$, and $\mathcal{F}$ has a (Zariski or etale) covering by affine schemes $\operatorname{Spec} A_i'$. Then we can form the fiber product to get a covering of $\operatorname{Spec} R$:

$$\begin{tikzcd} \amalg_i \operatorname{Spec} R \times_{\mathcal{F}'} \operatorname{Spec} A_i' \arrow[r] \arrow[d, "\text{cov.}"] & \amalg_i \operatorname{Spec} A_i' \arrow[d, "\text{cov.}"] \\ \operatorname{Spec} R \arrow[r, "\xi"] & \mathcal{F} \end{tikzcd}$$

As the morphism $\operatorname{Spec} A_i' \to \mathcal{F}$ is locally of finite presentation, the functor $\operatorname{Spec} R \times_{\mathcal{F}'} \operatorname{Spec} A_i'$ has a Zariski covering by affine schemes $\operatorname{Spec} R_{ij}'$, where R_{ij}' are R-algebras of finite presentation. Now we have a Zariski covering $\{ \operatorname{Spec} R_{ij}' \to \operatorname{Spec} R \}$, but as affine schemes are quasi-compact, we can take a *finite* subcovering. By renaming the indices and adding multiple copies of $\operatorname{Spec} A_i'$, we can make the following commutative diagram:

$$\begin{tikzcd} \amalg_i \operatorname{Spec} R_i' \arrow[r, "(\xi_i')"] \arrow[d, "\text{fin. cov.}"] & \amalg_i \operatorname{Spec} A_i' \arrow[d, "\text{cov.}"] \\ \operatorname{Spec} R \arrow[r, "\xi"] & \mathcal{F} \end{tikzcd}$$

Where the index set of the upper left corner is a finite subset of the index set of the upper right corner. Now the Zariski sheaf property of $\mathcal{F}$ says that we can recover ξ from the (ξ_i') and their pull back to the self fiber products. To see this, form the fiber products:

$$\begin{tikzcd} \amalg_{i,j} \operatorname{Spec}(R_i' \otimes_R R_j') \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \amalg_{i,j} \operatorname{Spec} A_i' \times_{\mathcal{F}} \operatorname{Spec} A_j' \arrow[d, shift left] \arrow[d, shift right] \\ \amalg_i \operatorname{Spec} R_i' \arrow[r] & \amalg_i \operatorname{Spec} A_i' \end{tikzcd}$$

Again because $\operatorname{Spec} A_j' \to \mathcal{F}$ is locally of finite presentation, the functor $\operatorname{Spec} A_i' \times_{\mathcal{F}} \operatorname{Spec} A_j'$ has a Zariski covering by $\operatorname{Spec} A_{ijk}'$ with A_i'-algebras A_{ijk}' of finite presentation. Again doing the similar thing as before, i.e. forming the fiber product and taking the affine Zariski covering and choosing a finite subcovering, we get the commutative diagram:

$$\begin{tikzcd} \amalg_k \operatorname{Spec} R'_{ijk} \arrow[r] \arrow[d, "\text{fin. cov.}"] & \amalg_k \operatorname{Spec} A'_{ijk} \arrow[d, "\text{cov.}"] \\ \operatorname{Spec}(R'_i \otimes_R R'_j) \arrow[r] & \operatorname{Spec} A'_i \times_{\mathcal{F}'} \operatorname{Spec} A'_j \end{tikzcd}$$

where R'_{ijk} are $(R'_i \otimes_R R'_j)$-algebras of finite presentation, hence R-algebras of finite presentation. The upshot is the commutative diagram:

$$\begin{tikzcd} \amalg_{i,j,k} \operatorname{Spec} R'_{ijk} \arrow[r, "(\xi'_{ijk})"] \arrow[d, shift left] \arrow[d, shift right] & \amalg_{i,j,k} \operatorname{Spec} A'_{ijk} \arrow[d, shift left] \arrow[d, shift right] \\ \amalg_i \operatorname{Spec} R'_i \arrow[r, "(\xi'_i)"] \arrow[d, "\text{fin. cov.}"] & \amalg_i \operatorname{Spec} A'_i \arrow[d] \\ \operatorname{Spec} R \arrow[r, dotted, "\xi"] & \mathcal{F} \end{tikzcd}$$

and the Zariski sheaf property of $\mathcal{F}$ says that having ξ is equivalent to having the finite set of morphisms $(\xi'_i), (\xi'_{ijk})$ (which are $\mathcal{O}$-algebra homomorphisms) which make the above diagram commute. Let us see one example of how this principle can be used (Proposition 4.3).

Lemma 4.2. *Let $R = \varinjlim R_\alpha$ with $\mathcal{O}$-algebras R_α.*

(i) *Let R' be an R-algebra of finite presentation. For a big enough α, there is R_α-algebra R'_α of finite presentation with $R' = R \otimes_{R_\alpha} R'_\alpha$, and $R' = \varinjlim R'_\alpha$ for such α.*

(ii) *Let $\{ \operatorname{Spec} R'_i \to \operatorname{Spec} R' \}$ be a finite Zariski covering where R' and R'_i are all R-algebras of finite presentation, and choose $R'_\alpha, R'_{i,\alpha}$ as in* (i). *Then $\{ \operatorname{Spec} R'_{i,\alpha} \to \operatorname{Spec} R'_\alpha \}$ is a Zariski covering for a big enough α.*

Proof. (i): If $R' \cong R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$, then choose α so that all coefficients of f_i lie in R_α. Then $R' = R \otimes_{R_\alpha} R'_\alpha$ for $R'_\alpha \cong R_\alpha[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$, and $R' = \varinjlim R'_\alpha$ for such α.

(ii): For each i, we have $\operatorname{Spec} R'_i = \mathcal{U}_{E_i}$ for a *finite* $E_i \subset R'$. The fact that $\{ \mathcal{U}_{E_i} \to \operatorname{Spec} R' \}$ is a Zariski covering translates to the fact that there is an equality $1 = \sum_j a_j t_j$ for $a_j \in R'$ and $t_j \in \bigcup E_i$. Therefore, choosing α so that *all* $a_j \in R'_\alpha$ and *all* $E_i \subset R'_\alpha$, we have $\mathcal{U}_{E_i} = \operatorname{Spec} R'_{i,\alpha}$ and $\{ \operatorname{Spec} R'_{i,\alpha} \to \operatorname{Spec} R'_\alpha \}$ is a Zariski covering. □

Proposition 4.3. *If $\mathcal{F}, \mathcal{F}'$ are Zariski sheaves and $\mathcal{F}' \to \mathcal{F}$ is a morphism locally of finite presentation, then $\mathcal{F}'$ is also of locally of finite presentation.*

Proof. Take any R-valued point $\xi' \in \mathcal{F}'(R)$ of $\mathcal{F}'$. As $\mathcal{F}$ is locally of finite presentation, the image $\xi = f(\xi') \in \mathcal{F}(R)$ factors through $\operatorname{Spec} A$ for an $\mathcal{O}$-algebra A of finite presentation, we have a similar diagram as above, and we can replace $\mathcal{F}, \mathcal{F}'$ with $\operatorname{Spec} A$ and $\operatorname{Spec} A \times_{\mathcal{F}} \mathcal{F}'$, which is a *scheme* by Remark 2.5; thus we are reduced to the case where $\mathcal{F}'$ is a scheme, locally of finite presentation over $\operatorname{Spec} A$.

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r, "\xi'"] \arrow[dr, "\xi"'] & \mathcal{F}' \arrow[d, "f\,:\,\text{loc. f.p.}"] \\
& \operatorname{Spec} A
\end{tikzcd}$$

As f is locally of finite presentation, the functor $\mathcal{F}'$ has a Zariski covering $\operatorname{Spec} A'_i \to \mathcal{F}'$ by affine schemes $\operatorname{Spec} A'_i$ where A'_i are A-algebras of finite presentation. By the construction we saw above, we get a commutative diagram:

$$\begin{tikzcd}
\amalg_{i,j,k} \operatorname{Spec} R'_{ijk} \arrow[r, "(\xi'_{ijk})"] \arrow[d, shift left] \arrow[d, shift right] & \amalg_{i,j,k} \operatorname{Spec} A'_{ijk} \arrow[d, shift left] \arrow[d, shift right] \\
\amalg_i \operatorname{Spec} R'_i \arrow[r, "(\xi'_i)"] \arrow[d, "\text{fin. cov.}"] & \amalg_i \operatorname{Spec} A'_i \arrow[d] \\
\operatorname{Spec} R \arrow[r, "\xi'"] & \mathcal{F}'
\end{tikzcd}$$

where A'_{ijk} are the A'_i-algebras of finite presentation (thus A'_i and A'_{ijk} are $\mathcal{O}$-algebras of finite presentation), and R'_i and R'_{ijk} are R-algebras of finite presentation.

Now let $R = \varinjlim R_\alpha$ with $\mathcal{O}$-algebras R_α of finite presentation. If R' is an R-algebra of finite presentation, we can write $R' = \varinjlim R'_\alpha$ as in Lemma 4.2(i) where R'_α are R_α-algebras of finite presentation, and every morphism $\operatorname{Spec} R' \to \operatorname{Spec} A'$, where A' is an $\mathcal{O}$-algebra of finite presentation, will factor through some R'_α. Apply this procedure to obtain $R'_{i,\alpha}$ and $R'_{ijk,\alpha}$ for all i, j, k. As there are only *finitely* many of them, we can choose an α so that:

(i) $\{\ \operatorname{Spec} R'_{i,\alpha} \to \operatorname{Spec} R_\alpha\ \}$ is a Zariski covering,
(ii) $\operatorname{Spec} R'_i \to \operatorname{Spec} A'_i$ factors through $\operatorname{Spec} R'_{i,\alpha}$ for all i,
(iii) $\{\ \operatorname{Spec} R'_{ijk,\alpha} \to \operatorname{Spec}(R'_{i,\alpha} \otimes_{R_\alpha} R'_{j,\alpha})\ \}$ is a Zariski covering for all i, j (note here that $R'_i \otimes_R R'_j = R \otimes_{R_\alpha} \big(R'_{i,\alpha} \otimes_{R_\alpha} R'_{j,\alpha}\big) = \varinjlim\big(R'_{i,\alpha} \otimes_{R_\alpha} R'_{j,\alpha}\big)$), and
(iv) $\operatorname{Spec} R'_{ijk} \to \operatorname{Spec} A'_{ijk}$ factors through $\operatorname{Spec} R'_{ijk,\alpha}$ for all i, j, k.

Then we have a diagram:

$$\begin{tikzcd}
\amalg_{i,j,k} \operatorname{Spec} R'_{ijk} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \amalg_{i,j,k} \operatorname{Spec} R'_{ijk,\alpha} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \amalg_{i,j,k} \operatorname{Spec} A'_{ijk} \arrow[d, shift left] \arrow[d, shift right] \\
\amalg_i \operatorname{Spec} R'_i \arrow[r] \arrow[d, "\text{fin. cov.}"] & \amalg_i \operatorname{Spec} R'_{i,\alpha} \arrow[r] \arrow[d, "\text{fin. cov.}"] & \amalg_i \operatorname{Spec} A'_i \arrow[d] \\
\operatorname{Spec} R \arrow[r] & \operatorname{Spec} R_\alpha \arrow[r, dotted] & \mathcal{F}'
\end{tikzcd}$$

where the left two squares are cartesian by Lemma 4.2(i). The upper right square is not necessarily commutative, but it is commutative when composed from the upper left corner. In terms of A-algebra homomorphisms, there are two homomorphisms $A'_i \to R'_{ijk,\alpha}$ which coincide when composed with $R'_{ijk,\alpha} \to R'_{ijk}$. We can make the original homomorphisms coincide by making α bigger, because they are determined by the image of the set of generators of A'_i, which is of finite type over A. We can do this for all (finitely many) i that are involved. Once we have the commutativity of the upper right square, the Zariski sheaf property of $\mathcal{F}'$ gives the dotted arrow. $\square$

Corollary 4.4. *Let $\mathcal{F}$ be an etale (resp. Zariski) sheaf, locally of finite presentation. If (A, ξ, f) is an etale (resp. Zariski) neighborhood of $\mathcal{F}$, then A is an $\mathcal{O}$-algebra of finite presentation. In particular, if $\mathcal{O}$ is a noetherian ring, then A is also noetherian.*

Proposition 4.5. *Let $\mathcal{O}$ be a noetherian ring and $\mathcal{F}$ be locally of finite presentation.*

- (i) *If $\mathcal{F}$ is a scheme, then for every point $\xi \in \mathcal{F}(k)$, the Zariski local ring $\mathcal{O}_\xi$ is a noetherian local ring, and* $\operatorname{Def}_\xi \mathcal{F}$ *is represented by $(\widehat{\mathcal{O}}_\xi, \widehat{\xi}_0) \in \mathscr{C}_k$.*
- (ii) *If $\mathcal{F}$ is an algebraic space, then for every point $\xi \in \mathcal{F}(k)$, the etale local ring $\mathcal{O}_\xi^{\mathrm{et}}$ is a noetherian local ring, and* $\operatorname{Def}_\xi \mathcal{F}$ *is represented by $(\widehat{\mathcal{O}}_\xi^{\mathrm{et}}, \widehat{\xi}_0^{\mathrm{et}}) \in \mathscr{C}_k$.*

Proof. (i): The previous corollary shows that $\mathcal{F}$ is a locally noetherian scheme.

(ii): By the previous corollary, the etale local ring is a direct limit of noetherian local rings with local homomorphisms that are etale. May add the details later – see EGA $0_{\mathbf{III}}$, (10.3.1.3). The second assertion follows from Proposition 3.9(iii) and Lemma 3.6. $\square$

4.3. **Separatedness.** As schemes/algebraic spaces are Zariski/etale locally representable, they are obtained by patching affine schemes together — let us think of this as starting with the disjoint union of affine pieces and taking the quotient by an *equivalence relation*.

Let $\mathcal{F}$ be a scheme (resp. algebraic space), and $\{ f_i : \operatorname{Spec} A'_i \to \mathcal{F} \}_{i \in I}$ be its affine Zariski (resp. etale) covering. We set:

$$U_i := \operatorname{Spec} A'_i, \quad U := \coprod_{i \in I} U_i, \quad f := \amalg_i f_i : U \to \mathcal{F}.$$

Form a fiber product:

$$V := U \times_{\mathcal{F}} U \rightrightarrows U \xrightarrow{f} \mathcal{F}$$

and consider the morphism $r_f : V \to U \times U$ corresponding to the two arrows. For each $R \in (\mathcal{O}\text{-alg})$, the map $r_f : V(R) \to U(R) \times U(R)$ gives an equivalence relation $\sim_f$ on the set $U(R)$:

$$x \sim_f y \iff (x, y) \in \operatorname{Im} r_f \quad (\forall x, y \in U(R)).$$

In particular, the diagonal morphism $\Delta : U \to U \times U$ factors through r_f. Note that $\mathrm{pr}_i \circ r_f : V \to U$ $(i = 1, 2)$ are the morphisms obtained by base change from $f : U \to \mathcal{F}$, hence are open immersions (resp. etale).

Remark 4.6. Although we have a cartesian square of maps

$$\begin{tikzcd} V(R) \arrow[r] \arrow[d] & U(R) \arrow[d, "f"'] \\ U(R) \arrow[r, "f"] & \mathcal{F}(R) \end{tikzcd}$$

for all $R \in (\mathcal{O}\text{-alg})$, we cannot recover the set of R-valued points of $\mathcal{F}$ from $U(R)$, i.e. the inclusion of $U(R)/\sim_f$ into $\mathcal{F}(R)$ is *not* surjective for all R. That is why we needed to (1) require that $\mathcal{F}$ is a *sheaf*, and (2) take a covering of $\operatorname{Spec} R$ and do the patching procedure to get an R-valued point of $\mathcal{F}$ in the last section.

Definition 4.7. We say that $\mathcal{F}$ is *separated* (resp. *locally separated*) if there exist a covering f of $\mathcal{F}$ by affine schemes such that $r_f : U \times_{\mathcal{F}} U \to U \times U$ is a closed immersion (resp. factors as $r_f = j \circ i$ where j is a closed immersion and i is an open immersion).

4.4. Representability theorem.

Theorem 4.8. *A functor* $\mathcal{F} : (\mathcal{O}\text{-alg}) \to (\text{Sets})$*, locally of finite presentation, is a separated algebraic space if the following conditions are satisfied:*

(i) *It is an etale sheaf.*
(ii) *For all field* $k \in (\mathcal{O}\text{-alg})$ *and* $\xi \in \mathcal{F}(k)$*, the deformation functor* $\operatorname{Def}_\xi F$ *is representable by a normal ring of fixed Krull dimension* n*.*
(iii) *It is* relatively representable*, i.e. for any two morphisms* $\operatorname{Spec} A \rightrightarrows \mathcal{F}$*, its kernel is a closed subfunctor of* $\operatorname{Spec} A$ *(hence an affine scheme* $\operatorname{Spec} A/(E)$ *for some* E*).*

4.5. **Approximation and algebrization.** For a geometric point $\xi \in \mathcal{F}(k)$, an (affine) *algebraic neighborhood* of ξ is a triple (A, ξ', f) which makes the following diagram commute:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\xi'"] \arrow[dr, "\xi"'] & \operatorname{Spec} A \arrow[d, "f\,:\,\text{locally of finite presentation}"] \\ & \mathcal{F} \end{tikzcd}$$

Theorem 4.9. (Artin's algebrization) *Let* $\mathcal{F} : (\mathcal{O}\text{-alg}) \to (\text{Sets})$ *be locally of finite presentation, and* $\xi \in \mathcal{F}(k)$ *be a geometric point. If* $\operatorname{Def}_\xi \mathcal{F}$ *is representable, then there exists an algebraic neighborhood* (A, ξ', f) *of* ξ *such that* $f_* : \operatorname{Def}_{\xi'}(\operatorname{Spec} A) \longrightarrow \operatorname{Def}_\xi \mathcal{F}$ *is an isomorphism.*

5. Notes on the Literature

References

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