WEEK 3: MODULI INTERPRETATION OF SHIMURA VARIETIES

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Abstract. Moduli interpretation.

Lecture 7 (Sep. 29, 2008) – continued

1. Introduction

Up to now we treated the Shimura varieties $X = \{X_U\}_U$ as quotients of symmetric spaces, which are not quite Shimura varieties yet — some people think that as long as we consider them over complex numbers, the X_U should be called the *arithmetic quotients of Hermitian symmetric domains*, because the methods involved are different. We chose to call them loosely as *Shimura varieties over* $\mathbb{C}$, as our presentation has been strongly motivated by the construction of Galois representations; we want to think of Betti cohomology and etale cohomology as two aspects of one and the same arithmetic object.

The topic of this week, the *moduli interpretation* of Shimura varieties, is where we start doing *algebraic geometry*, although this week it is still over complex numbers. Recall the definition of the modular curve X_U of level U, for an open compact subgroup $U \subset G(\mathbb{A}^\infty)$:

$$X_U := GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/(U \cdot U_\infty).$$

Our goal is to show that there is a natural bijection:

$$\left\{(A, \eta_{\text{et}}U)\right\}/\text{isogeny} \xrightarrow{\cong} X_U,$$

where A is an elliptic curve over $\mathbb{C}$ (yes, we have *abelian varieties* in mind), and $\eta_{\text{et}}U$ is a *level U structure* on A, which means the U-orbit of an isomorphism η_{et} of $\mathbb{A}^\infty$-modules:

$$\eta_{\text{et}} : V \otimes \mathbb{A}^\infty \xrightarrow{\cong} V_{\text{et}}A, \quad V_{\text{et}}A := TA \otimes \mathbb{Q}$$

where V is a fixed $\mathbb{Q}$-vector space of dimension 2, the *Tate module* $TA := \varprojlim A[N]$ is a free $\widehat{\mathbb{Z}}$-module of rank 2, defined as the inverse limit of groups $A[N]$ of N-torsion points of A. An element $g \in GL_2(\mathbb{A}^\infty)$ acts on the right on η_{et} via the action on $V \otimes \mathbb{A}^\infty$, i.e. $\eta \mapsto \eta \circ g$. Two elliptic curves A, A' are called *isogenous* if there is a finite surjective homomorphism $f : A \to A'$ (an *isogeny*), and it induces an isomorphism $f_* : V_{\text{et}}A \to V_{\text{et}}A'$ of $\mathbb{A}^\infty$-modules. An *isogeny* between the pairs $(A, \eta_{\text{et}}U)$ and $(A', \eta'_{\text{et}}U)$ means an isogeny $f : A \to A'$ such that $\eta'_{\text{et}}U = (f_* \circ \eta_{\text{et}})U$.

We will motivate all of these definitions in the course of this week; the upshot is that all of these data make sense not only over $\mathbb{C}$, but over *any ring*. Therefore, once the task of this week is done, this *moduli problem* will be naturally defined over $\mathbb{Q}$, over even $\mathbb{Z}$, and modulo *representability*, it immediately follows that there has to be an *algebraic variety* over $\mathbb{Q}$, whose set of $\mathbb{C}$-rational points is X_U.

Remembering our final goal of analyzing $H^*(X, \mathcal{L}_\xi)$ (see Week 2, §1), we will also need a moduli interpretation of locally constant sheaves $\mathcal{L}_\xi$ on X_U, where ξ is an (algebraic) finite dimensional representation $\xi : GL_2(\mathbb{Q}) \to V_\xi$ on a $\overline{\mathbb{Q}}$-vector space V_ξ. This will follow naturally from the moduli interpretation of X, and is realized as sheaves not on Zariski topology, but *etale topology* of algebraic varieties, and necessarily the cohomology theory would be the *ℓ-adic etale cohomology* for a prime ℓ, so we will be tensoring $\overline{\mathbb{Q}}_\ell$ on V_ξ; when thinking of Betti cohomology we can tensor $\mathbb{C}$ on V_ξ.

Through the moduli interpretation, most of what we saw already will acquire a new *meaning* in itself; what seemed to be a random interesting object will have a rigid structure, directions of morphisms and actions almost unchangeably determined, what seemed a loose array of concepts fit into a tight jigsaw puzzle without much room for our choice.

But there seems to be one choice in the convention; whether our algebraic group $GL_2/\mathbb{Q}$ is the automorphism group of the *homology group* $H_1(A, \mathbb{Q})$ or the *cohomology group* $H^1(A, \mathbb{Q})$ of the elliptic curve A. When defining the representation $\xi = \mathrm{Sym}^{k-2}(\mathrm{Sd})$ which gives the coefficient sheaf $\mathcal{L}_\xi$, we definitely want the *cohomology* sheaf of the universal family, so if Sd is the cohomology, then GL_2 is the automorphism group of the cohomology. But in the above moduli problem, the group GL_2 acted on our fixed module V, which is meant to rigidify the *homology*. When defining the level structure, it is easier and more natural to talk about the ones in homology, as they are just torsion points. This convention seems to have been chosen by Deligne in [De3] (after some changes from [De1], [De2]), and will result in seeing the *contragradient* of the Langlands correspondence in the cohomology of Shimura varieties, and we seem to live with it now. We will see how it goes.

Lecture 8 (Oct. 1, 2008)

2. The moduli interpretation: Classical setting

Symmetric spaces are moduli spaces of linear algebraic structures, or vector spaces with additional structures, like complex structure, bilinear form, etc. They are closely related to *linear algebraic groups*, which are automorphism groups of linear algebraic structures. Let us think of our basic case $G = GL_2/\mathbb{Q}$. It is the automorphism group of a two-dimensional $\mathbb{Q}$-vector space V, which we fix. We think of G as an *algebraic group* in the following way — it is a *functor* from the category of $\mathbb{Q}$-algebras to the category of groups, defined as:

$$G(R) := \mathrm{Aut}_R(V \otimes R)$$

for *any* $\mathbb{Q}$-algebra R. The term algebraic group is used synonymously to the term *group variety*, which is an algebraic variety with group structure, like Lie groups. As we prefer to think of algebraic varieties as *schemes*, which are functors from the category of rings to the category of sets, we think of algebraic groups as funtors too, the target category being the category of groups instead of sets.

The *moduli space* for a structure is a set whose points correspond bijectively to the isomorphism classes of structures we want to classify. It is usually a deeper question what kind of geometric structure this set carries, and in order to make this precise, we need to think of them as an object representing moduli functors — we will do this next week, but this week we only consider the moduli spaces as sets, and we ignore the geometric aspect of moduli spaces themselves.

2.1. **The double half plane and elliptic curves.** The double half plane $\mathcal{H}$ can be considered as the moduli space which classifies *complex structures* on $V \otimes \mathbb{R}$, i.e. the way we make $V \otimes \mathbb{R}$ into a 1-dimensional $\mathbb{C}$-vector space, or an isomorphism of $\mathbb{R}$-vector spaces:

$$h : \mathbb{C} \xrightarrow{\cong} V \otimes \mathbb{R},$$

and we call h, h' *equivalent* if they are $\mathbb{C}^\times$-multiple of one another, i.e. if there is an element $z \in \mathbb{C}^\times = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ such that $h' = h \circ z$. The set $\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R})$ of all such h has a left action of $G(\mathbb{R}) = \mathrm{Aut}(V \otimes \mathbb{R})$ and a right action of $\mathbb{C}^\times = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$:

$$h \mapsto g \circ h \quad (g \in G(\mathbb{R})),$$
$$h \mapsto h \circ z \quad (z \in \mathbb{C}^\times),$$

for all $h \in \mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R})$. Whenever we fix a particluar h, then:

$$G(\mathbb{R}) \ni g \longmapsto g \circ h \in \mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R})$$

is a bijection. We express this fact by saying that $\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R})$ is a left $G(\mathbb{R})$-*torsor* (or, *principal homogeneous space* under the left $G(\mathbb{R})$-action). In order to classify h up to equivalence, we take the quotient of $\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R})$ by the right $\mathbb{C}^\times$-action, i.e.:

$$\mathcal{H} := \mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R})/\mathbb{C}^\times,$$

which is our *symmetric space*, and $\mathcal{H}$ still retains the left $G(\mathbb{R})$-action. Whenever we choose a particular $h \bmod \mathbb{C}^\times \in \mathcal{H}$, its stabilizer is a subgroup U_∞ of $G(\mathbb{R})$ (which depends on the choice of $h \bmod \mathbb{C}^\times$), and we have a bijection of sets with left $G(\mathbb{R})$-action:

$$G(\mathbb{R})/U_\infty \ni g \bmod U_\infty \longmapsto g \circ h \bmod \mathbb{C}^\times \in \mathcal{H}.$$

Now the discrete subgroup. Let us take $\Gamma = GL_2(\mathbb{Z}) \subset G(\mathbb{R})$. This is realized as the automorphism group of a $\mathbb{Z}$-*lattice* Λ in V. As we saw in Week 1, a $\mathbb{Z}$-lattice in V means a $\mathbb{Z}$-submodule of V which is free of rank 2 and $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = V$. Let us fix such $\Lambda \subset V$, and define:

$$\Gamma := \mathrm{Aut}_{\mathbb{Z}}(\Lambda) \subset \mathrm{Aut}_{\mathbb{R}}(V \otimes \mathbb{R}) = G(\mathbb{R}).$$

How does the arithmetic quotient $X_\Gamma := \Gamma \backslash \mathcal{H}$ look in our new language? We are identifying two complex structures $h \bmod \mathbb{C}^\times$ and $h' \bmod \mathbb{C}^\times$, if and only if they are related to each other as $h' \bmod \mathbb{C}^\times = \gamma \circ h \bmod \mathbb{C}^\times$ for some $\gamma \in \Gamma = \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$. This means that there exists $z \in \mathbb{C}^\times$ such that $h' \circ z = \gamma \circ h$, hence inside $\mathbb{C}$:

$$h'^{-1}(\Lambda) = z \cdot (\gamma \circ h)^{-1}(\Lambda) = z \cdot h^{-1}(\Lambda).$$

Thus we are classifying the $\mathbb{Z}$-lattices $L = h^{-1}(\Lambda) \subset \mathbb{C}$ (which means the free rank 2 $\mathbb{Z}$-submodules $L \subset \mathbb{C}$ with $L \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}$) up to *homothety*, i.e. multiplication by $z \in \mathbb{C}^\times$. For each L, the quotient $A := \mathbb{C}/L$ is a compact Riemann surface which we call an *elliptic curve* over $\mathbb{C}$, and if $L' = zL$, then $z : \mathbb{C}/L \longrightarrow \mathbb{C}/L'$ gives an isomorphism between them. Thus we have a bijection:

$$\Gamma\backslash\mathcal{H} \xrightarrow{\cong} \{\text{ elliptic curves over } \mathbb{C} \}/\text{isomorphism.}$$
$$h \longmapsto A := \mathbb{C}/h^{-1}(\Lambda)$$

Here we can take the above description as definitions of elliptic curves and isomorphisms between them, but it turns out that the isomorphisms between elliptic curves as Riemann surfaces are always isomorphisms in the above sense. Moreover, we have an algebro-geometric definition of elliptic curves, where all the isomorphisms as Riemann surfaces turn out to be isomorphisms as algebraic varieties. We do not need to go into this for now.

2.2. **Modified interpretation I.** In the classical theory of modular curves, we extend the moduli interpretation of X_Γ to general *congruence subgroups*, smaller than $GL_2(\mathbb{Z})$, but we will not need that either, as we will take the adelic viewpoint from the beginning. But we will make a modification in the above construction — we considered the elliptic curves as quotients of $\mathbb{C}$, but this can be replaced with any 1-dimensional $\mathbb{C}$-vector space, which we denote by V_{dR} (this notation anticipates the interpretation of this vector space as the first de Rham homology of A, or the tangent space Lie A). In order to do this, we reconsider the left $G(\mathbb{R})$-torsor $\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R})$ by decomposing h into two isomorphisms, as the moduli space of triples

$$\big(V_{\mathrm{dR}},\ \eta_{\mathrm{dR}} : \mathbb{C} \to V_{\mathrm{dR}},\ c_{\mathrm{dR}} : V_{\mathrm{dR}} \to V \otimes \mathbb{R}\big),$$

where:

(i) V_{dR} is a 1-dimensional $\mathbb{C}$-vector space,
(ii) $\eta_{\mathrm{dR}} : \mathbb{C} \xrightarrow{\cong} V_{\mathrm{dR}}$ is an isomorphism of $\mathbb{C}$-vector spaces (the *de Rham rigidification*),
(iii) $c_{\mathrm{dR}} : V_{\mathrm{dR}} \xrightarrow{\cong} V \otimes \mathbb{R}$ is an isomorphism of $\mathbb{R}$-vector spaces.

Two triples $(V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR}})$ and $(V'_{\mathrm{dR}}, \eta'_{\mathrm{dR}}, c'_{\mathrm{dR}})$ are *equivalent* if there is an isomorphism $f : V_{\mathrm{dR}} \xrightarrow{\cong} V'_{\mathrm{dR}}$ of $\mathbb{C}$-vector spaces such that $\eta'_{\mathrm{dR}} = f \circ \eta_{\mathrm{dR}}$ and $c_{\mathrm{dR}} = c'_{\mathrm{dR}} \circ f$. We denote this equivalence relation by $\sim$.

This seems extremely pedantic at the moment; it does not change the space, because any such triple is equivalent to $(\mathbb{C},\ \mathrm{id},\ \mathbb{C} \to V \otimes \mathbb{R})$. But this formulation will clarify what we are doing here, and also the analogy with what we do for (etale) level structures. Now taking the quotient by $\mathbb{C}^\times$ is now considered as taking the quotient under the right $\mathbb{C}^\times$-action on η_{dR}, because $z \in \mathbb{C}^\times$ acts on this moduli space from the right by:

$$[z] : \big(V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR}}\big) \longmapsto \big(V_{\mathrm{dR}}, \eta_{\mathrm{dR}} \circ z, c_{\mathrm{dR}}\big).$$

Note that this action preserves the equivalence relation. Here observe that for fixed pair $(V_{\mathrm{dR}}, c_{\mathrm{dR}})$, two different choices $\eta_{\mathrm{dR}}, \eta'_{\mathrm{dR}}$ are always related by $\eta'_{\mathrm{dR}} = \eta_{\mathrm{dR}} \circ z$, i.e. the set of all choices of η_{dR} is a right $\mathbb{C}^\times$-torsor. Hence taking the quotient by the right $\mathbb{C}^\times$-action amounts to *forgetting the de Rham rigidification* η_{dR}, and the quotient $\mathcal{H}$ is now the moduli space of $(V_{\mathrm{dR}}, c_{\mathrm{dR}})$:

$$\mathcal{H} = \{(V_{\mathrm{dR}}, c_{\mathrm{dR}})\}/\sim,$$

where the equivalence is defined similarly as above. Note that we do not see $\mathbb{C}$, the "fixed" 1-dimensional $\mathbb{C}$-vector space, in this desrciption of $\mathcal{H}$.

Now the moduli interpretation of $\Gamma\backslash\mathcal{H}$ looks as follows:

$$\Gamma\backslash\mathcal{H} \xrightarrow{\cong} \{\text{ elliptic curves over } \mathbb{C}\ \}/\cong,$$
$$(V_{\mathrm{dR}}, c_{\mathrm{dR}}) \bmod \Gamma \longmapsto A := V_{\mathrm{dR}}/c_{\mathrm{dR}}^{-1}(\Lambda)$$

2.3. **Modified interpretation II.** To describe the inverse map, we will do one more modification, so that we can interpret the left quotient by Γ as forgetting a rigidification, in the same way as we did for the right quotient by $\mathbb{C}^\times$. We again consider $\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V\otimes\mathbb{R})$ by decomposing c_{dR} further, as the moduli space of 5-tuples:

$$\big(\Lambda_{\mathrm{B}},\ V_{\mathrm{dR}},\ \eta_{\mathrm{dR}},\ c_{\mathrm{dR,B}} : V_{\mathrm{dR}} \to \Lambda_{\mathrm{B}}\otimes\mathbb{R},\ \lambda_{\mathrm{B}} : \Lambda_{\mathrm{B}} \to \Lambda\big),$$

where:

(i) Λ_{B} is a free $\mathbb{Z}$-module of rank 2,
(ii) $V_{\mathrm{dR}}, \eta_{\mathrm{dR}}$ are as before,
(iii) $c_{\mathrm{dR,B}} : V_{\mathrm{dR}} \xrightarrow{\cong} \Lambda_{\mathrm{B}}\otimes\mathbb{R}$ is an isomorphism of $\mathbb{R}$-vector spaces (the *de Rham-Betti comparison isomorphism*).
(iv) $\lambda_{\mathrm{B}} : \Lambda_{\mathrm{B}} \xrightarrow{\cong} \Lambda$ is an isomorphism of $\mathbb{Z}$-modules (the $\mathbb{Z}$*-Betti rigidification*),

Observe that we decomposed c_{dR} as $c_{\mathrm{dR}} = (\lambda_{\mathrm{B}}\otimes\mathbb{R})\circ c_{\mathrm{dR,B}}$, and used $\Lambda\otimes_{\mathbb{Z}}\mathbb{R} = V\otimes_{\mathbb{Q}}\mathbb{R}$. We define $(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR,B}}, \lambda_{\mathrm{B}}) \sim (\Lambda'_{\mathrm{B}}, V'_{\mathrm{dR}}, \eta'_{\mathrm{dR}}, c'_{\mathrm{dR,B}}, \lambda'_{\mathrm{B}})$ if there is a pair $(f_{\mathrm{B}}, f_{\mathrm{dR}})$ of isomorphisms such that:

(i) $f_{\mathrm{B}} : \Lambda_{\mathrm{B}} \xrightarrow{\cong} \Lambda'_{\mathrm{B}}$ is an isomorphism of $\mathbb{Z}$-modules such that $\lambda_{\mathrm{B}} = \lambda'_{\mathrm{B}}\circ f_{\mathrm{B}}$,
(ii) $f_{\mathrm{dR}} : V_{\mathrm{dR}} \xrightarrow{\cong} V'_{\mathrm{dR}}$ is an isomorphism of $\mathbb{C}$-vector spaces such that $\eta'_{\mathrm{dR}} = f_{\mathrm{dR}}\circ\eta_{\mathrm{dR}}$,
(iii) $(f_{\mathrm{B}}\otimes\mathbb{R})\circ c_{\mathrm{dR,B}} = c'_{\mathrm{dR,B}}\circ f_{\mathrm{dR}}$.

As such 5-tuples are equivalent to $(\Lambda,\ \mathbb{C},\ \mathrm{id},\ h,\ \mathrm{id})$ for some $h : \mathbb{C}\to V\otimes\mathbb{R} = \Lambda\otimes\mathbb{R}$, the space does not change from $\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V\otimes\mathbb{R})$, but now the left $\Gamma = \mathrm{Aut}(\Lambda)$ action is given by $[\gamma] : \lambda_{\mathrm{B}} \longmapsto \gamma\circ\eta_B$, and taking the quotient by this action amounts to *forgetting the* $\mathbb{Z}$*-Betti rigidification* λ_{B}. We summarize this as follows:

$$\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V\otimes\mathbb{R}) = \{(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR,B}}, \lambda_{\mathrm{B}})\}/\sim,$$
$$\mathcal{H} = \{(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, \lambda_{\mathrm{B}})\}/\sim \quad \text{(quotient by right } \mathbb{C}^\times\text{-action on } \eta_{\mathrm{dR}}),$$
$$\Gamma\backslash\mathcal{H} = \{(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}})\}/\sim \quad \text{(quotient by left } \Gamma\text{-action on } \lambda_{\mathrm{B}}).$$

In this language, we can write the moduli interpretation of $\Gamma\backslash\mathcal{H}$ as follows:

Proposition 2.1. *Let Λ be a fixed free $\mathbb{Z}$-module of rank two, and let $\Gamma := \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$. The following is a bijection:*

$$\Gamma\backslash\mathcal{H} = \{(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}})\}/\sim \overset{1:1}{\longleftrightarrow} \{\text{ elliptic curves over } \mathbb{C} \}/\cong,$$
$$(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}) \longmapsto A := V_{\mathrm{dR}}/c_{\mathrm{dR,B}}^{-1}(\Lambda_B),$$
$$(H_1(A,\mathbb{Z}),\ \mathrm{Lie}\,A,\ c_A) \longleftarrow A,$$

where $c_A : \mathrm{Lie}\,A \xrightarrow{\cong} H_1(A,\mathbb{Z}) \otimes \mathbb{R}$ is the de Rham-Betti comparison isomorphism for A.

Note that in this proposition, all reference to Λ, V have disappeared. Actually this space is isomorphic to $\mathbb{C}$ by the *j-invariant*, and not much symmetry remains. Starting from this base space, we can work up to the moduli interpretations of $\mathcal{H}$ and $\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R})$ by putting the rigidifications back in. For an elliptic curve A over $\mathbb{C}$, we can consider:

(i) isomorphisms of $\mathbb{C}$-vector spaces $\eta_{\mathrm{dR}} : \mathbb{C} \xrightarrow{\cong} \mathrm{Lie}\,A$ (*de Rham rigidifications* of A),
(ii) isomorphisms of $\mathbb{Z}$-modules $\lambda_{\mathrm{B}} : H_1(A,\mathbb{Z}) \xrightarrow{\cong} \Lambda$ (*$\mathbb{Z}$-Betti rigidifications* of A).

An *isomorphism* between $(A, \eta_{\mathrm{dR}}, \lambda_{\mathrm{B}})$ and $(A', \eta'_{\mathrm{dR}}, \lambda'_{\mathrm{B}})$ is an isomorphism $f : A \to A'$ of elliptic curves such that $\eta'_{\mathrm{dR}} = f_* \circ \eta_{\mathrm{dR}}$ and $\lambda_{\mathrm{B}} = \eta'_B \circ f_*$, where $f_* : \mathrm{Lie}\,A \to \mathrm{Lie}\,A'$ and $f_* : H_1(A,\mathbb{Z}) \to H_1(A',\mathbb{Z})$ are the induced morphisms.

Proposition 2.2. *The following are bijections:*

$$\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R}) = \{(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR,B}}, \lambda_{\mathrm{B}})\}/\sim \overset{1:1}{\longleftrightarrow} \{(A, \eta_{\mathrm{dR}}, \lambda_{\mathrm{B}})\}/\cong,$$
$$\mathcal{H} = \{(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, \lambda_{\mathrm{B}})\}/\sim \overset{1:1}{\longleftrightarrow} \{(A, \lambda_{\mathrm{B}})\}/\cong,$$
$$\Gamma\backslash\mathcal{H} = \{(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}})\}/\sim \overset{1:1}{\longleftrightarrow} \{A\}/\cong,$$

where the A are elliptic curves over $\mathbb{C}$ on the RHS.

3. The moduli interpretation: Adelic setting

In the adelic setting, instead of the $\mathbb{Z}$-Betti rigidification $\lambda_{\mathrm{B}} : \Lambda_{\mathrm{B}} \xrightarrow{\cong} \Lambda$, we will use the *Betti rigidification*:

$$\eta_{\mathrm{B}} : V_{\mathrm{B}} \xrightarrow{\cong} V,$$

where V_{B} is a 2-dimensional $\mathbb{Q}$-vector space, because we are taking the left quotient by $G(\mathbb{Q}) = \mathrm{Aut}_{\mathbb{Q}}(V)$, not by $\Gamma = \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$. This will result in considering the elliptic curves A *up to isogeny*, but by what we have seen in §6 of Week 1, the *isogeny* classes of elliptic curves with level U-structure will correspond bijectively with the *isomorphism* classes of elliptic curves with level Γ-structure.

Lecture 9 (Oct. 3, 2008)

3.1. **Linear algebraic moduli interpretation of X_U.** As our Shimura varities will be quotients of the space $G(\mathbb{A}^\infty) \times \mathcal{H}$, we will start with the linear algebraic moduli interpretation of $G(\mathbb{A}^\infty) \times \mathcal{H}$, or even $G(\mathbb{A}^\infty) \times \mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R})$, which is a left $G(\mathbb{A})$-torsor, because $G(\mathbb{A}) = G(\mathbb{A}^\infty) \times G(\mathbb{R})$, and with a right action of $G(\mathbb{A}^\infty) \times \mathbb{C}^\times$.

The added data are the elements of $G(\mathbb{A}^\infty) = \mathrm{Aut}_{\mathbb{A}^\infty}(V \otimes \mathbb{A}^\infty)$, but when taking the double quotient, from the left by $G(\mathbb{Q})$ and the right by some $U \subset G(\mathbb{A}^\infty)$, we will want to consider the actions from each side separately. For this we use the same technique as we did for V_{dR} in §2.2, i.e. decomposing an isomorphism $V \otimes \mathbb{A}^\infty \longrightarrow V \otimes \mathbb{A}^\infty$ into two parts:

$$\eta_{\mathrm{et}} : V \otimes \mathbb{A}^\infty \to V_{\mathrm{et}}, \quad c_{\mathrm{et}} : V_{\mathrm{et}} \to V \otimes \mathbb{A}^\infty,$$

where V_{et} is a free $\mathbb{A}^\infty$-module of rank 2. Thus we go back to the notation of §2.2, and consider the moduli space of two triples:

$$\big(V_{\mathrm{dR}},\ \eta_{\mathrm{dR}} : \mathbb{C} \to V_{\mathrm{dR}},\ \ c_{\mathrm{dR}} : V_{\mathrm{dR}} \to V \otimes \mathbb{R},$$
$$V_{\mathrm{et}},\ \eta_{\mathrm{et}} : V \otimes \mathbb{A}^\infty \to V_{\mathrm{et}},\ c_{\mathrm{et}} : V_{\mathrm{et}} \to V \otimes \mathbb{A}^\infty\big),$$

where:

(i) $V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR}}$ are as in §2.2,
(ii) V_{et} is a free $\mathbb{A}^\infty$-module of rank 2,
(iii) $\eta_{\mathrm{et}} : V \otimes \mathbb{A}^\infty \xrightarrow{\cong} V_{\mathrm{et}}$ is an isomorphism of $\mathbb{A}^\infty$-modules (the *etale rigidification*),
(iv) $c_{\mathrm{et}} : V_{\mathrm{et}} \xrightarrow{\cong} V \otimes \mathbb{A}^\infty$ is an isomorphism of $\mathbb{A}^\infty$-modules.

This gives:

$$G(\mathbb{A}^\infty) \times \mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R}) \xrightarrow{\cong} \{(V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}, c_{\mathrm{et}})\}/\sim,$$

which is a left $G(\mathbb{A})$-torsor with a right $G(\mathbb{A}^\infty) \times \mathbb{C}^\times$-action. We will omit writing out the equivalence relation for now; it is similar to the one we saw in §2.2. Now we proceed to the procedure of §2.3. In order to take the quotient by the left $G(\mathbb{Q}) = \mathrm{Aut}_{\mathbb{Q}}(V)$-action, which acts diagonally on c_{dR} and c_{et}, we pull out the Betti rigidification from both of them by considering the 8-tuples:

$$\big(V_{\mathrm{B}},\ V_{\mathrm{dR}},\ \eta_{\mathrm{dR}} : \mathbb{C} \to V_{\mathrm{dR}},\ c_{\mathrm{dR,B}} : V_{\mathrm{dR}} \to V_{\mathrm{B}} \otimes \mathbb{R},$$
$$V_{\mathrm{et}},\ \eta_{\mathrm{et}} : V \otimes \mathbb{A}^\infty \to V_{\mathrm{et}},\ c_{\mathrm{et,B}} : V_{\mathrm{et}} \to V_{\mathrm{B}} \otimes \mathbb{A}^\infty,\ \eta_{\mathrm{B}} : V_{\mathrm{B}} \to V\big),$$

where:

(i) $V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}$ are as before,
(ii) $c_{\mathrm{et,B}} : V_{\mathrm{et}} \xrightarrow{\cong} V_{\mathrm{B}} \otimes \mathbb{A}^\infty$ is an isomorphism of $\mathbb{A}^\infty$-modules (the *etale-Betti comparison isomorphism*).
(iii) $c_{\mathrm{dR,B}} : V_{\mathrm{dR}} \xrightarrow{\cong} V_{\mathrm{B}} \otimes \mathbb{R}$ is an isomorphism of $\mathbb{R}$-vector spaces (the *de Rham-Betti comparison isomorphism*).
(iv) $\eta_{\mathrm{B}} : V_{\mathrm{B}} \xrightarrow{\cong} V$ is an isomorphism of $\mathbb{Q}$-vector spaces (the *Betti rigidification*).

We can summarize this in the following diagram:

$$V \xleftarrow{\eta_{\mathrm{B}}} V_{\mathrm{B}} \begin{cases} \otimes\mathbb{R} \xleftarrow{c_{\mathrm{dR,B}}} V_{\mathrm{dR}} \xleftarrow{\eta_{\mathrm{dR}}} \mathbb{C}\ , \\ \otimes\mathbb{A}^{\infty} \xleftarrow{c_{\mathrm{et,B}}} V_{\mathrm{et}} \xleftarrow{\eta_{\mathrm{et}}} V \otimes \mathbb{A}^{\infty}\ . \end{cases}$$

We will write out the equivalence relation for once. We define:

$$(V_{\mathrm{B}}, V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR,B}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}, c_{\mathrm{et,B}}, \eta_{\mathrm{B}}) \sim (V'_{\mathrm{B}}, V'_{\mathrm{dR}}, \eta'_{\mathrm{dR}}, c'_{\mathrm{dR,B}}, V'_{\mathrm{et}}, \eta'_{\mathrm{et}}, c'_{\mathrm{et,B}}, \eta'_{\mathrm{B}}),$$

if there exists a triple $(f_{\mathrm{B}}, f_{\mathrm{dR}}, f_{\mathrm{et}})$ of isomorphisms such that:

(i) $f_{\mathrm{B}} : V_{\mathrm{B}} \xrightarrow{\cong} V'_{\mathrm{B}}$ is an isomorphism of $\mathbb{Q}$-vector spaces such that $\eta_{\mathrm{B}} = \eta'_{\mathrm{B}} \circ f_{\mathrm{B}}$,
(ii) $f_{\mathrm{dR}} : V_{\mathrm{dR}} \xrightarrow{\cong} V'_{\mathrm{dR}}$ is an isomorphism of $\mathbb{C}$-vector spaces such that $\eta'_{\mathrm{dR}} = f_{\mathrm{dR}} \circ \eta_{\mathrm{dR}}$,
(iii) $f_{\mathrm{et}} : V_{\mathrm{et}} \xrightarrow{\cong} V'_{\mathrm{et}}$ is an isomorphism of $\mathbb{A}^{\infty}$-modules such that $\eta'_{\mathrm{et}} = f_{\mathrm{et}} \circ \eta_{\mathrm{et}}$,
(iv) $(f_{\mathrm{B}} \otimes \mathbb{R}) \circ c_{\mathrm{dR,B}} = c'_{\mathrm{dR,B}} \circ f_{\mathrm{dR}}$,
(v) $(f_{\mathrm{B}} \otimes \mathbb{A}^{\infty}) \circ c_{\mathrm{et,B}} = c'_{\mathrm{et,B}} \circ f_{\mathrm{et}}$.

This gives:

$$G(\mathbb{A}^{\infty}) \times \mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R}) \xrightarrow{\cong} \{(V_{\mathrm{B}}, V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR,B}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}, c_{\mathrm{et,B}}, \eta_{\mathrm{B}})\}/\sim,$$

with a right $\mathbb{C}^{\times}$-action on η_{dR}, a right $G(\mathbb{A}^{\infty})$-action on η_{et}, and a left $G(\mathbb{Q})$-action on η_{B}. Taking quotients by these groups (except that instead of $G(\mathbb{A}^{\infty})$ we take an open compact subgroup $U \subset G(\mathbb{A}^{\infty})$), we get:

$$\begin{aligned} G(\mathbb{A}^{\infty}) \times \mathcal{H} &\xrightarrow{\cong} \{(V_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}, c_{\mathrm{et,B}}, \eta_{\mathrm{B}})\}/\sim, \\ G(\mathbb{Q})\backslash\big(G(\mathbb{A}^{\infty}) \times \mathcal{H}\big) &\xrightarrow{\cong} \{(V_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}, c_{\mathrm{et,B}})\}/\sim \\ \big(G(\mathbb{A}^{\infty})/U\big) \times \mathcal{H} &\xrightarrow{\cong} \{(V_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}U, c_{\mathrm{et,B}}, \eta_{\mathrm{B}}\}/\sim, \\ X_U = G(\mathbb{Q})\backslash\Big(\big(G(\mathbb{A}^{\infty})/U\big) \times \mathcal{H}\Big) &\xrightarrow{\cong} \{(V_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}U, c_{\mathrm{et,B}})\}/\sim, \end{aligned}$$

where $\eta_{\mathrm{et}}U$ is the right U-orbit (right U-coset) of η_{et}. We call a right U-orbit $\eta_{\mathrm{et}}U$ of etale rigidifications an *(etale) level U structure* on V_{et}.

3.2. **Lattices versus $\mathbb{Q}$-vector spaces.** Let us make the dictionary between the $\mathbb{Z}$-Betti rigidification $(\Lambda_{\mathrm{B}}, \lambda_{\mathrm{B}})$ and the Betti rigidification $(V_{\mathrm{B}}, \eta_{\mathrm{B}})$. Fixing the $\mathbb{Z}$-lattice Λ in V, the obvious correspondence:

$$\begin{aligned} (\Lambda_{\mathrm{B}}, \lambda_{\mathrm{B}}) &\longmapsto \big(\Lambda_{\mathrm{B}} \otimes \mathbb{Q},\ \lambda_{\mathrm{B}} \otimes \mathbb{Q}\big) \\ \big(\eta_{\mathrm{B}}^{-1}(\Lambda),\ \eta_{\mathrm{B}}|_{\eta_{\mathrm{B}}^{-1}(\Lambda)}\big) &\longleftarrow (V_{\mathrm{B}}, \eta_{\mathrm{B}}) \end{aligned}$$

gives the bijections for $\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R})$ and $\mathcal{H}$:

$$\begin{aligned} \mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R}) = \{(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR,B}}, \lambda_{\mathrm{B}})\}/\sim &\xleftrightarrow{1:1} \{(V_{\mathrm{B}}, V_{\mathrm{dR}}, \eta_{\mathrm{dR}}, c_{\mathrm{dR,B}}, \eta_{\mathrm{B}})\}/\sim, \\ \mathcal{H} = \{(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, \lambda_{\mathrm{B}})\}/\sim &\xleftrightarrow{1:1} \{(V_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, \eta_{\mathrm{B}})\}/\sim\ . \end{aligned}$$

Now what about $\Gamma\backslash\mathcal{H}$? Recall from §6 of Week 1 that, if U is a finite index subgroup of $GL_2(\widehat{\mathbb{Z}})$ and $\det U = \widehat{\mathbb{Z}}^\times$, then we have:

$$\Gamma\backslash\mathcal{H} \xrightarrow{\cong} G(\mathbb{Q})\backslash\Big(\big(G(\mathbb{A}^\infty)/U\big)\times\mathcal{H}\Big) = X_U, \tag{3.2.1}$$

for $\Gamma = GL_2(\mathbb{Q})\cap U$. In particular, we can take $U = GL_2(\widehat{\mathbb{Z}})$ for $\Gamma = GL_2(\mathbb{Z})$. In our new language $\Gamma = G(\mathbb{Q})\cap U$, and if $U = \mathrm{Aut}_{\widehat{\mathbb{Z}}}(\Lambda\otimes\widehat{\mathbb{Z}}) \subset \mathrm{Aut}_{\mathbb{A}^\infty}(V\otimes\mathbb{A}^\infty) = G(\mathbb{A}^\infty)$, where $\Lambda\otimes\widehat{\mathbb{Z}}$ is the $\widehat{\mathbb{Z}}$-lattice in $V\otimes\mathbb{A}^\infty$, then we get $\Gamma = \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$. We set $\widehat{\Lambda} := \Lambda\otimes\widehat{\mathbb{Z}}$.

We will write out the linear algebraic moduli interpretation of both sides. So we start with a $\widehat{\mathbb{Z}}$-lattice $\widehat{\Lambda}$ in $V\otimes\mathbb{A}^\infty$, and set $U = \mathrm{Aut}_{\widehat{\mathbb{Z}}}(\widehat{\Lambda})$, which is an open compact subgroup of $G(\mathbb{A}^\infty)$. Then set $\Gamma = G(\mathbb{Q})\cap U$; we have $\Gamma = \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$ for $\Lambda := V\cap\widehat{\Lambda}$ (cf. Lemma 5.5 of Week 1). Recall that the isomorphism (3.2.1) was induced by the inclusion:

$$\mathcal{H} \ni z \longmapsto \big(1\cdot U,\ z\big) \in \big(G(\mathbb{A}^\infty)/U\big)\times\mathcal{H}. \tag{3.2.2}$$

One obvious way to decompose $1 = \mathrm{id} \in G(A^\infty) = \mathrm{Aut}_{\mathbb{A}^\infty}(V\otimes\mathbb{A}^\infty)$ is:

$$V\otimes\mathbb{A}^\infty \xleftarrow{\eta_{\mathrm{B}}\otimes\mathbb{A}^\infty} V_{\mathrm{B}}\otimes\mathbb{A}^\infty \xleftarrow{\mathrm{id}} V_{\mathrm{B}}\otimes\mathbb{A}^\infty \xleftarrow{(\eta_{\mathrm{B}}\otimes\mathbb{A}^\infty)^{-1}} V\otimes\mathbb{A}^\infty,$$

namely taking $V_{\mathrm{et}} := V_{\mathrm{B}}\otimes\mathbb{A}^\infty$, $c_{\mathrm{et,B}} = \mathrm{id}$ and $\eta_{\mathrm{et}} := (\eta_{\mathrm{B}}\otimes\mathbb{A}^\infty)^{-1}$. Therefore, using $V_{\mathrm{B}} := \Lambda_{\mathrm{B}}\otimes\mathbb{Q}$ and $\eta_{\mathrm{B}} := \lambda_{\mathrm{B}}\otimes\mathbb{Q}$, the inclusion (3.2.2) can be written as:

$$(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, \lambda_{\mathrm{B}}) \longmapsto (V_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}},\ V_{\mathrm{B}}\otimes\mathbb{A}^\infty,\ (\eta_{\mathrm{B}}\otimes\mathbb{A}^\infty)^{-1}U,\ \mathrm{id},\ \eta_{\mathrm{B}}).$$

Thus, forgetting λ_{B} on the LHS and η_{B} on the RHS, the isomorphism (3.2.1) for $U = \mathrm{Aut}_{\widehat{\mathbb{Z}}}(\widehat{\Lambda})$ and $\Gamma = \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$:

$$\Gamma\backslash\mathcal{H} \xrightarrow{\cong} X_U$$

$$\{(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}})\}/\sim \xleftrightarrow{1:1} \{(V_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}U, c_{\mathrm{et,B}})\}/\sim$$

is given by:

$$(\Lambda_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}) \longmapsto \big(V_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}},\ V_{\mathrm{B}}\otimes\mathbb{A}^\infty,\ (\eta_{\mathrm{B}}\otimes\mathbb{A}^\infty)^{-1}U,\ \mathrm{id}\big). \tag{3.2.3}$$

Note that we forgot η_{B}, but this description of the element in RHS by the data on the LHS involves $(\eta_{\mathrm{B}}\otimes\mathbb{A}^\infty)^{-1}U$, i.e. a partial remembrance of η_{B}. What amount of information does this level U structure $(\eta_{\mathrm{B}}\otimes\mathbb{A}^\infty)^{-1}U$ carry?

In general, the etale rigidification η_{et} is an isomorphism:

$$\eta_{\mathrm{et}} : V\otimes\mathbb{A}^\infty \longrightarrow V_{\mathrm{et}},$$

and U is the stabilizer of the $\widehat{\mathbb{Z}}$-lattice $\widehat{\Lambda}$ in $V\otimes\mathbb{A}^\infty$. Therefore the right U-coset $\eta_{\mathrm{et}}U$ cuts out a $\widehat{\mathbb{Z}}$-lattice in V_{et}, corresponding to $\widehat{\Lambda}$. In other words, we have a bijection:

$$\mathrm{Isom}_{\mathbb{A}^\infty}(V\otimes\mathbb{A}^\infty, V_{\mathrm{et}})/U \ni \eta_{\mathrm{et}}U \longmapsto \eta_{\mathrm{et}}(\widehat{\Lambda}) \in \{\ \widehat{\mathbb{Z}}\text{-lattices in } V_{\mathrm{et}}\ \},$$

so we will use $\eta_{\mathrm{et}}U$ and $\eta_{\mathrm{et}}(\widehat{\Lambda})$ interchangeably.

In our particular case of (3.2.3) where $V_{\text{et}} = V_{\text{B}} \otimes \mathbb{A}^\infty$, $\eta_{\text{et}} = (\eta_{\text{B}} \otimes \mathbb{A}^\infty)^{-1}$ and $(V_{\text{B}}, \eta_{\text{B}}) = (\Lambda_{\text{B}}, \lambda_{\text{B}}) \otimes \mathbb{Q}$, it cuts out the $\widehat{\mathbb{Z}}$-lattice:

$$\eta_{\text{et}}(\widehat{\Lambda}) = (\eta_{\text{B}} \otimes \mathbb{A}^\infty)^{-1}(\widehat{\Lambda}) = \lambda_{\text{B}}^{-1}(\Lambda) \otimes \widehat{\mathbb{Z}} = \Lambda_{\text{B}} \otimes \widehat{\mathbb{Z}},$$

inside $V_{\text{B}} \otimes \mathbb{A}^\infty$. So, what happened is that, even though we forgot η_{B}, we retained the information of the $\widehat{\mathbb{Z}}$-lattice $\Lambda_{\text{B}} \otimes \widehat{\mathbb{Z}}$ inside $V_{\text{B}} \otimes \mathbb{A}^\infty$. That is why we can recover the LHS from the RHS in (3.2.3) by:

$$\Lambda_{\text{B}} = V_{\text{B}} \cap (\Lambda_{\text{B}} \otimes \widehat{\mathbb{Z}}) = V_{\text{B}} \cap (\eta_{\text{B}} \otimes \mathbb{A}^\infty)^{-1}(\widehat{\Lambda})$$

(cf. Lemma 5.5 of Week 1, again). Thus we can rewrite (3.2.3), using our convention of $\eta_{\text{et}} U \leftrightarrow \eta_{\text{et}}(\widehat{\Lambda})$, as

$$(\Lambda_{\text{B}}, V_{\text{dR}}, c_{\text{dR,B}}) \longmapsto \big(V_{\text{B}}, V_{\text{dR}}, c_{\text{dR,B}},\ V_{\text{B}} \otimes \mathbb{A}^\infty,\ \Lambda_{\text{B}} \otimes \widehat{\mathbb{Z}},\ \text{id}\big). \tag{3.2.4}$$

3.3. Moduli interpretation of X_U.

Now we will relate what we saw to the elliptic curves. Recall Proposition 2.1, which says:

$$\begin{aligned} \{A\}/\cong \stackrel{1:1}{\longleftrightarrow} \{(\Lambda_{\text{B}}, V_{\text{dR}}, c_{\text{dR,B}})\}/\sim . \\ A \longmapsto (H_1(A, \mathbb{Z}),\ \text{Lie}\, A,\ c_A) \end{aligned}$$

Let us compose this with (3.2.4). Recalling the universal coefficient theorem which tells you that $H_1(A, R) = H_1(A, \mathbb{Z}) \otimes R$ for any ring R:

$$\begin{aligned} V_{\text{B}} = \Lambda_{\text{B}} \otimes \mathbb{Q} = H_1(A, \mathbb{Z}) \otimes \mathbb{Q} = H_1(A, \mathbb{Q}), \\ V_{\text{et}} = V_{\text{B}} \otimes \mathbb{A}^\infty = H_1(A, \mathbb{Q}) \otimes \mathbb{A}^\infty = H_1(A, \mathbb{A}^\infty), \\ \Lambda_{\text{B}} \otimes \widehat{\mathbb{Z}} = H_1(A, \mathbb{Z}) \otimes \widehat{\mathbb{Z}} = H_1(A, \widehat{\mathbb{Z}}). \end{aligned}$$

In order to work out the inverse, we use

$$\Lambda_{\text{B}} = V_{\text{B}} \cap c_{\text{et,B}}(\eta_{\text{et}}(\widehat{\Lambda})),$$

and $A = V_{\text{dR}} / c_{\text{dR,B}}^{-1}(\Lambda_{\text{B}})$ (Proposition 2.1). Thus we conclude:

Proposition 3.1. *Let $\widehat{\Lambda}$ be a fixed $\widehat{\mathbb{Z}}$-lattice in $V \otimes \mathbb{A}^\infty$, and let $U := \text{Aut}_{\widehat{\mathbb{Z}}}(\widehat{\Lambda}) \subset G(\mathbb{A}^\infty)$. The following is a bijection:*

$$X_U = \{(V_{\text{B}}, V_{\text{dR}}, c_{\text{dR,B}}, V_{\text{et}}, \eta_{\text{et}} U, c_{\text{et,B}})\}/\sim \stackrel{1:1}{\longleftrightarrow} \{\text{ elliptic curves over } \mathbb{C}\ \}/\cong,$$

$$(V_{\text{B}}, V_{\text{dR}}, c_{\text{dR,B}}, V_{\text{et}}, \eta_{\text{et}}(\widehat{\Lambda}), c_{\text{et,B}}) \longmapsto A := V_{\text{dR}}\ /\ c_{\text{dR,B}}^{-1}\big(V_{\text{B}} \cap c_{\text{et,B}}(\eta_{\text{et}}(\widehat{\Lambda}))\big),$$

$$\big(H_1(A, \mathbb{Q}),\ \text{Lie}\, A,\ c_A,\ H_1(A, \mathbb{A}^\infty),\ H_1(A, \widehat{\mathbb{Z}}),\ \text{id} \otimes \mathbb{A}^\infty\big) \longleftarrow A,$$

where $c_A : \text{Lie}\, A \xrightarrow{\cong} H_1(A, \mathbb{R})$ is the de Rham-Betti comparison isomorphism for A, and we expressed the level U-structure $\eta_{\text{et}} U$ on V_{et} by the $\widehat{\mathbb{Z}}$-lattice $\eta_{\text{et}}(\widehat{\Lambda})$ in V_{et}.

Now, the equivalence relation on the LHS, i.e. a triple $(f_{\text{B}}, f_{\text{dR}}, f_{\text{et}})$ of $\mathbb{Q}$-, $\mathbb{C}$- and $\mathbb{A}^\infty$-module isomorphisms, does not necessarily seem to come from an isomorphism betwen elliptic curves, because any *isogeny* $f : A \to A'$ between elliptic curves, i.e. a homomorphism with finite kernel, will induce such a triple of isomorphisms. But the condition that f_{et} :

$V_{\mathrm{et}} \to V'_{\mathrm{et}}$ is compatible with the level U-structure means that f_{et} has to send the $\widehat{\mathbb{Z}}$-lattice $\eta_{\mathrm{et}}(\widehat{\Lambda})$ to the other $\widehat{\mathbb{Z}}$-lattice $\eta'_{\mathrm{et}}(\widehat{\Lambda})$, forces that it has to come from an *isomorphism* between elliptic curves. Here the key fact is that $V_{\mathrm{et}} = H_1(A, \mathbb{A}^\infty)$ coming from an elliptic curve A comes with a *canonical* $\widehat{\mathbb{Z}}$*-lattice* $H_1(A, \widehat{\mathbb{Z}})$ *inside it*, hence has a canonical level U-structure for a stabilizer $U = \mathrm{Aut}_{\widehat{\mathbb{Z}}}(\widehat{\Lambda})$ of any $\widehat{\mathbb{Z}}$-lattice $\widehat{\Lambda}$ in $V \otimes \mathbb{A}^\infty$, i.e. any maximal compact subgroup U of $G(\mathbb{A}^\infty)$. So, the RHS of Proposition 3.1, the set of isomorphism classes of elliptic curves A, is also the set of *isogeny* classes of pairs $\big(A, H_1(A, \widehat{\mathbb{Z}})\big)$, where $H_1(A, \widehat{\mathbb{Z}})$ can be considered as a level U-structure for any maximal U. This seems to be a redundant procedure, because *once we fix a maximal* U, all isogenies between $\big(A, H_1(A, \widehat{\mathbb{Z}})\big)$ and $\big(A', H_1(A', \widehat{\mathbb{Z}})\big)$ automatically comes from an isomorphism $f : A \to A'$.

But this suggests the natural substitute for RHS when we consider *arbitrary* open compact subgroup $U \subset G(\mathbb{A}^\infty)$, which is:

$$\{(A, \eta_{\mathrm{et}} U)\}/\text{isogeny},$$

where $\eta_{\mathrm{et}} : V \otimes \mathbb{A}^\infty \to H_1(A, \mathbb{A}^\infty)$ is an *etale rigidification* of A, and we call its right U-coset $\eta_{\mathrm{et}} U$ a *level* U*-structure* on A. Here note that a general U can be contained in several different maximal compact subgroups — in other words, the U can fix several different lattices $\widehat{\Lambda}$ in $V \otimes \mathbb{A}^\infty$, and depending on $\widehat{\Lambda}$, the construction of Proposition 3.1 results in *different but isogenous* A. This issue does not arise if we require that (i) the U is always contained in a fixed maximal compact subgroup $\mathrm{Aut}_{\widehat{\mathbb{Z}}}(\widehat{\Lambda})$ of $G(\mathbb{A}^\infty)$ and (ii) fix a particular $\widehat{\Lambda}$ throughout the moduli interpretation, but we do not want that; because (i) we want all U's, not just the ones contained in a particular maximal compact subgroup, in order to treat the full Hecke action of $G(\mathbb{A}^\infty)$ as isomorphisms and not algebraic correspondences (see §1 of Week 2), and (ii) we want each X_U, hence its moduli interpretation as well, to be canonically associated to U without a choice of maximal compact subgroup containing it.

Let us write out this situation for the set of isogeny classes of pairs (A, η_{et}), where A is an elliptic curves and η_{et} is an etale rigidification of A. We have a natural map from this set to the set which we saw at the end of §3.1:

$$\{(A, \eta_{\mathrm{et}})\}/\sim \xrightarrow{\cong} \{(V_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}, c_{\mathrm{et,B}})\}/\sim\; = G(\mathbb{Q})\backslash\big(G(\mathbb{A}^\infty) \times \mathcal{H}\big),$$
$$(A, \eta_{\mathrm{et}}) \longmapsto \big(H_1(A, \mathbb{Q}),\ \mathrm{Lie}\, A,\ c_A,\ H_1(A, \mathbb{A}^\infty),\ \eta_{\mathrm{et}},\ \mathrm{id} \otimes \mathbb{A}^\infty\big)$$

where $(A, \eta_{\mathrm{et}}) \sim (A', \eta'_{\mathrm{et}})$ on the LHS if there is an isogeny $f : A \to A'$ such that the induced isomorphism $f_* : H_1(A, \mathbb{A}^\infty) \to H_1(A', \mathbb{A}^\infty)$ is compatible with the etale rigidification, i.e. $\eta'_{\mathrm{et}} = f_{\mathrm{et}} \circ \eta_{\mathrm{et}}$. Note that for each pair (A, η_{et}), the inverse image $\widehat{\Lambda}$ of $H_1(A, \widehat{\mathbb{Z}})$ is a $\widehat{\mathbb{Z}}$-lattice in $V \otimes \mathbb{A}^\infty$, canonically associated to the pair (A, η_{et}), but an isogenous $(A', \eta'_{\mathrm{et}})$ is associated to a possibly different $\widehat{\Lambda}'$. We construct the inverse map as follows. Choose any $\widehat{\mathbb{Z}}$-lattice $\widehat{\Lambda}$ in $V \otimes \mathbb{A}^\infty$, and make the construction in Proposition 3.1:

$$(V_{\mathrm{B}}, V_{\mathrm{dR}}, c_{\mathrm{dR,B}}, V_{\mathrm{et}}, \eta_{\mathrm{et}}, c_{\mathrm{et,B}}) \longmapsto \big(A := V_{\mathrm{dR}} \;/\; c_{\mathrm{dR,B}}^{-1}\big(V_{\mathrm{B}} \cap c_{\mathrm{et,B}}(\eta_{\mathrm{et}}(\widehat{\Lambda}))\big), \eta_{\mathrm{et}}\big),$$

(note that canonically $H_1(A, \widehat{\mathbb{Z}}) = \eta_{\mathrm{et}}(\widehat{\Lambda})$, hence $H_1(A, \mathbb{A}^\infty) = V_{\mathrm{et}}$ and η_{et} becomes an etale rigidification of A). Here a different choice of $\widehat{\Lambda}'$ results in different but *isogenous* $(A', \eta'_{\mathrm{et}})$.

Thus we proved the third bjiection of the following theorem, and we easily get the other three by putting $\eta_{\mathrm{B}}, \eta_{\mathrm{dR}}$ back in and taking the quotient by U.

Theorem 3.2. *The following are bijections:*

$$G(\mathbb{A}^\infty) \times \mathrm{Isom}_{\mathbb{R}}(\mathbb{C}, V \otimes \mathbb{R}) \overset{1:1}{\longleftrightarrow} \{(A, \eta_{\mathrm{dR}}, \eta_{\mathrm{et}}, \eta_{\mathrm{B}})\}/\sim,$$
$$G(\mathbb{A}^\infty) \times \mathcal{H} \overset{1:1}{\longleftrightarrow} \{(A, \eta_{\mathrm{et}}, \eta_{\mathrm{B}})\}/\sim,$$
$$G(\mathbb{Q})\backslash\big(G(\mathbb{A}^\infty) \times \mathcal{H}\big) \overset{1:1}{\longleftrightarrow} \{(A, \eta_{\mathrm{et}})\}/\sim,$$
$$X_U \overset{1:1}{\longleftrightarrow} \{(A, \eta_{\mathrm{et}}U)\}/\sim$$

where the A are elliptic curves over $\mathbb{C}$ and $\sim$ denotes the isogeny on the RHS.

We will not write out the correspondences, which are obvious extensions of what we saw in the course of proving the theorem. The last bijection of this theorem is what we have promised in the introduction.

4. Towards algebraic geometry

4.1. **Algebraic definitions.** Introducing a small amount of notation, we summarize the data appearing in above correspondences, associated to an elliptic curve A over $\mathbb{C}$.

(i) $V_{\mathrm{B}}A := H_1(A, \mathbb{Q})$ — a 2-dimensional $\mathbb{Q}$-vector space.
(ii) $V_{\mathrm{dR}}A := \mathrm{Lie}\, A$ — a 1-dimensional $\mathbb{C}$-vector space.
(iii) $V_{\mathrm{et}}A := H_1(A, \mathbb{A}^\infty)$ — a free rank 2 $\mathbb{A}^\infty$-module.
(iv) $\eta_{\mathrm{B}} : V_{\mathrm{B}}A \xrightarrow{\cong} V$ — a Betti rigidification of A.
(v) $\eta_{\mathrm{dR}} : \mathbb{C} \xrightarrow{\cong} V_{\mathrm{dR}}A$ — a de Rham rigidification of A.
(vi) $\eta_{\mathrm{et}} : V \otimes \mathbb{A}^\infty \xrightarrow{\cong} V_{\mathrm{et}}A$ — an etale rigidification of A.
(vii) $c_{\mathrm{dR,B}}(A) : V_{\mathrm{dR}}A \xrightarrow{\cong} V_{\mathrm{B}}A \otimes \mathbb{R}$ — the de Rham-Betti comparison for A.
(viii) $c_{\mathrm{et,B}}(A) : V_{\mathrm{et}}A \xrightarrow{\cong} V_{\mathrm{B}}A \otimes \mathbb{A}^\infty$ — the (trivial) etale-Betti comparison for A.

Algebraic definition. $\ker^1(G, \mathbb{Q})$-issue.

4.2. **More general case.** Polarization. Endomorphism.

5. Notes on the Literature

References

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