WEEK 2: BETTI COHOMOLOGY OF SHIMURA VARIETIES — THE MATSUSHIMA FORMULA

TERUYOSHI YOSHIDA

ABSTRACT. Rough exposition of the goal of this course. Hecke actions on Shimura varieties and their cohomology groups. The Matsushima formula in the classical setting; relative Lie algebra cohomology, spectral decomposition on $L^2(\Gamma\backslash G)$. The Matsushima formula in the adelic setting — automorphic representations appear in the Betti cohomology of Shimura varieties. Admissible $(\mathfrak{g}, U)$-modules and cohomological representations, Hodge theory. Back to GL_2-case; discussion of the Eichler-Shimura isomorphism.

Lecture 4 (Sep. 22, 2008) – continued

1. Where we are going

We defined the automorphic representations of $GL_2(\mathbb{A}^\infty)$ as the representations appearing in the space of automorphic forms $\mathcal{A}_k$, or $\mathcal{A}_k^0$, which were in turn the space of sections of certain holomorphic line bundle $\mathcal{L}_k$ on the compactifications of a modular curve

$$X_U := GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/(U \cdot U_\infty)$$

for some open compact subgroup $U \subset GL_2(\mathbb{A}^\infty)$ (i.e. the subgroups conjugate to a finite index subgroup $U \subset GL_2(\widehat{\mathbb{Z}})$). The goal of our lecture is to associate a 2-dimensional *ℓ-adic Galois representation* to each cuspidal automorphic representation of $GL_2(\mathbb{A}^\infty)$ (although we will turn to compact unitary Shimura curves over CM fields at some point).

1.1. **Hecke action.** How do we find the Galois representations? They do not seem to appear in the space $\mathcal{A}_k$ of automorphic forms. But being representations, automorphic representations are more mobile than automorphic forms – they can appear in different vector spaces with a left $GL_2(\mathbb{A}^\infty)$-action. In fact, we will look for various *cohomology groups* associated to X_U, because they will give rise to vector spaces on which $GL_2(\mathbb{A}^\infty)$ acts (in a smooth admissible way), and will have more structures as well. This happens because the spaces X_U have the right action of $GL_2(\mathbb{A}^\infty)$, in a certain sense. Let us make it precise.

Let us denote by X the *inverse system* of (non-connected) Riemann surfaces $\{X_U\}_U$ for each open compact subgroup $U \subset GL_2(\mathbb{A}^\infty)$ (which eventually turns out to be algebraic varieties, when we give the moduli interpretation which descends to $\mathbb{Q}$), with the canonical surjections, which are finite coverings with no ramification points (*finite etale* morphisms) for small enough U, U':

$$[\mathrm{can}] : X_{U'} \longrightarrow X_U \quad \text{whenever } U' \subset U.$$

As the line bundles $\mathcal{L}_k$ on X_U descended from $GL_2(\mathbb{A}^\infty) \times \mathcal{H}$, they live compatibly on each X_U, i.e. $[\mathrm{can}]^*\mathcal{L}_k = \mathcal{L}_k$; we think of $\mathcal{L}_k$ as a line bundle over X. Now we think of X as having a right $GL_2(\mathbb{A}^\infty)$-action, coming from the right action on the space

$$GL_2(\mathbb{Q})\backslash\big(GL_2(\mathbb{A}^\infty) \times \mathcal{H}\big), \quad \text{or even} \quad GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}).$$

What does this mean concretely, in terms of each X_U in $X = \{X_U\}_U$? It means that, for each $g \in GL_2(\mathbb{A}^\infty)$, we have an isomorphism:

$$[g] : X_{gUg^{-1}} \ni h \cdot gUg^{-1} \overset{\cong}{\longmapsto} hg \cdot U \in X_U$$

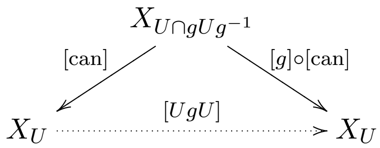
for each U, which we loosely call *Hecke action* of $GL_2(\mathbb{A}^\infty)$. Compare this with what we saw in §7 of Week 1 for the smooth representation V of $GL_2(\mathbb{A}^\infty)$:

$$g : V^U \ni v \overset{\cong}{\longmapsto} gv \in V^{gUg^{-1}}.$$

Also observe that when $U' \lhd U$, then $g \in U$ gives isomorphisms $[g] : X_{U'} \to X_{U'}$, which depends only on gU'; in other words, the finite group U/U' acts on $X_{U'}$ from the right by automorphisms. And as long as U, U' are small enough, the map $[\mathrm{can}] : X_{U'} \to X_U$ should identify X_U as the *quotient* space of $X_{U'}$ by the right action of U/U'. Thus we can think of X_U as the "quotient" of X by the right action of U.

Well, each X_U *is* the quotient of $GL_2(\mathbb{Q})\backslash\big(GL_2(\mathbb{A}^\infty) \times \mathcal{H}\big)$ by U, and it helps to think of X as $GL_2(\mathbb{Q})\backslash\big(GL_2(\mathbb{A}^\infty) \times \mathcal{H}\big)$, a sort of inverse limit of $\{X_U\}_U$. At least it helps our intuition during this week, where we treat X_U as Riemann surfaces — but later, although each X_U turns out to be an algebraic variety (hence the term *Shimura variety* instead of *arithmetic quotient of a symmetric space*), there is no way to think of $GL_2(\mathbb{Q})\backslash\big(GL_2(\mathbb{A}^\infty) \times \mathcal{H}\big)$ as an algebraic variety...

In some algebro-geometric setting that we will see later (in particular with the "classical" level structure on moduli problems), we need to restrict our definition of X_U to the case where $U \subset GL_2(\widehat{\mathbb{Z}})$, hence we can only describe the isomorphism $[g]$ when $gUg^{-1} \subset GL_2(\widehat{\mathbb{Z}})$. To describe the action $[g]$ for all $g \in GL_2(\mathbb{A}^\infty)$ (for a fixed U), traditionally we think of them as *Hecke correspondences*, which are *algebraic correspondences* attached to double U-cosets as follows. For g and U as above, the intersection $U \cap gUg^{-1}$ is another open compact subgroup (this time contained in $GL_2(\widehat{\mathbb{Z}})$), hence there are two maps going down to X_U, namely the simple [can] and the composite $[g] \circ [\mathrm{can}] : X_{U \cap gUg^{-1}} \to X_{gUg^{-1}} \to X_U$:



The dotted arrow in the bottom row is not a map but one-to-many *correspondence*, in the sense that for each point $x \in X_U$, take the inverse image under [can], which is a finite set, and then send them back to X_U by $[g] \circ [\mathrm{can}]$. In algebraic geometry, an *algebraic correspondence* from X to Y is simply defined as a cycle Z in $X \times Y$ having the right dimension, generalizing the graph of a map. (A correspondence, as a cycle in $X \times Y$, may not be lying nicely over each components of $X \times Y$ as it is in our case, but it still induces a map between cohomology.) The notation $[UgU]$ is naturally explained when we see the induced endomorphism on the cohomology groups, which we will do later.

1.2. **Cohomology groups.** Now that we have formulated what we mean by $GL_2(\mathbb{A}^\infty)$ acting on $X = \{X_U\}_U$ from the right, we should expect that any *cohomology groups* associated to X will have the corresponding left $GL_2(\mathbb{A}^\infty)$-action, and possibly some additional structure — hence a good place to look for automorphic representations, coupled with other associated structures (namely, Hodge structures, Galois representations,...).

By a *cohomology theory*, we mean a contravariant functor from the category of varieties or manifolds to the category of vector spaces over a field of characteristic 0, with some reasonable list of properties that we do not formulate here. Specifically, we have two theories in mind; the *Betti (singular) cohomology*, which are $\mathbb{C}$-vector spaces $H^*(X, \mathbb{C})$ for complex manifolds X, and the *ℓ-adic etale cohomology*, which are $\overline{\mathbb{Q}}_\ell$-vector spaces $H^*(X, \overline{\mathbb{Q}}_\ell)$ for algebraic varieties X. We generally use $*$ to denote arbitrary degree. But we haste to add that, whenever we fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$ of fields, we have a *comparison isomorphism* for any algebraic variety X over $\mathbb{C}$, also considered as a complex manifold:

$$H^*(X, \overline{\mathbb{Q}}_\ell) \otimes_\iota \mathbb{C} \xrightarrow{\cong} H^*(X, \mathbb{C}).$$

Eventually we will find the Galois representations (ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) in the ℓ-adic etale cohomology of X (more precisely $X_U \otimes \overline{\mathbb{Q}}$, once we know that X_U are varieties over $\mathbb{Q}$), but this week the Betti cohomology of X (as complex manifolds) will be the main object of study.

Let us formulate the common structure in the scheme of things. In each theory, we will consider the locally constant sheaves $\mathcal{L}_\xi$ on X, where ξ will correspond to the weight k of automorphic representations. By this we mean that we will have a locally constant sheaf $\mathcal{L}_\xi$ on X_U for each U compatibily, i.e. $[\mathrm{can}]^*\mathcal{L}_\xi = \mathcal{L}_\xi$. Then by the contravariance, we have the direct system of vector spaces $H^*(X_U, \mathcal{L}_\xi)$, and we take the inductive limit:

$$H^*(X, \mathcal{L}_\xi) := \varinjlim_U H^*(X_U, \mathcal{L}_\xi).$$

Now any reasonable cohomology theory (in particular the above two) would have the following property: (i) the spaces $H^*(X_U, \mathcal{L}_\xi)$ are finite dimensional, and (ii) when $[\mathrm{can}] : X_{U'} \to X_U$ is a finite etale map which makes X_U into a quotient of $X_{U'}$ by the right action of U/U', then $H^*(X_U, \mathcal{L}_\xi) = H^*(X_{U'}, \mathcal{L}_\xi)^{U/U'}$, the subspace of U/U'-invariant vectors. These properties show that:

(i) the $H^*(X, \mathcal{L}_\xi)$ is a smooth admissible representation of $GL_2(\mathbb{A}^\infty)$,
(ii) and we have $H^*(X_U, \mathcal{L}_\xi) = H^*(X, \mathcal{L}_\xi)^U$, the space of U-fixed vectors.

We also note that once the locally constant sheaf $\mathcal{L}_\xi$ is defined algebraically (as a *smooth ℓ-adic etale sheaf*), the comparison isomorphisms extend to the cohomology groups with coefficients in $\mathcal{L}_\xi$, and are functorial, hence equivariant under the left action of $GL_2(\mathbb{A}^\infty)$.

Thus, apart from $\mathcal{A}_k$, we have another smooth admissible representation $H^*(X, \mathcal{L}_\xi)$ of $GL_2(\mathbb{A}^\infty)$, for each $\mathcal{L}_\xi$ (corresponding to the weight k), canonically associated to $X = \{X_U\}$. And we can ask ourselves if the representations we find in $H^*(X, \mathcal{L}_\xi)$ are automorphic representations, and the answer is yes.

Roughly the picture we see would be the following:

$$\begin{array}{ccc} H^*_{\text{Betti}}(X, \mathcal{L}_\xi) & \xrightarrow{\cong} & \bigoplus_{\pi^\infty} \pi^\infty \otimes \big(\text{ 2-dim. } \mathbb{C}\text{-vector sp. associated to } \pi^\infty \big) \\ \cong \uparrow \iota & & \cong \uparrow \iota \\ H^*_{\text{etale}}(X, \mathcal{L}_\xi) & \xrightarrow{\cong} & \bigoplus_{\pi^\infty} \pi^\infty \otimes \big(\text{ 2-dim. } \ell\text{-adic Galois rep. associated to } \pi^\infty \big). \end{array}$$

Here the direct sum runs through the cuspidal automorphic representations of $GL_2(\mathbb{A}^\infty)$. Unfortunately, due to the fact that X_U are not compact, we need to modify the cohomology groups. But we will eventually be reducing things to the *compact* Shimura varieties by considering different reductive groups over $\mathbb{Q}$ than GL_2, so we will systematically ignore the issues related to the boundary. The goal is to show that the Galois representation we see in the lower-right bracket, denote by $R_\ell(\pi^\infty)$, is the Galois representation which correspond to π^∞ by the *global Langlands correspondence.* This correspondence is characterized by the property that the local component π_p of π^∞ should correspond to $R_\ell(\pi^\infty)|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ by the *local Langlands correspondence*, so once we construct $R_\ell(\pi^\infty)$, the remaining work would be the analysis of $R_\ell(\pi^\infty)|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ at each p, which is largely arithmetic geometry over $\mathbb{Z}_p$.

Lecture 5 (Sep. 24, 2008)

2. The Matsushima formula: Classical setting

As we have only defined X as complex manifolds (Riemann surfaces), we will talk about the Betti cohomology of X during this week. Especially, we will treat the top horizontal isomorphism of above diagram, which explains (via comparison isomorphism) the fact that the Galois representations $R_\ell(\pi^\infty)$ that we will get later are 2-dimensional. The 2-dimensional $\mathbb{C}$-vector space in the top-left of above diagram are the *relative Lie algebra cohomology groups* $H^*(\mathfrak{g}, U_\infty, \pi_\infty \otimes \xi)$ for certain *discrete series* (i.e. square integrable modulo center) representations π_∞ of $GL_2(\mathbb{R})$, and the said isomorphism:

$$H^*_{\text{Betti}}(X, \mathcal{L}_\xi) \xrightarrow{\cong} \bigoplus_{\pi^\infty} \pi^\infty \otimes H^*(\mathfrak{g}, U_\infty, \pi_\infty \otimes \xi)$$

(we will make it more precise later) is traditionally called *the Matsushima formula*, as it originated in the pioneering work of Matsushima (and Murakami) in the 1960's (see e.g. [MaMu]). Canonical reference is [BoWa], Chapter VII, and it is customary that the translation into adelic language is left to the reader.

In the basic case of modular curves, this will be equivalent to what is traditionally called the *Eichler-Shimura isomorphism* for classical modular forms, as was used in [De]. We will explain the dictionary between the two languages in the last section. The advantage of this formulation of the Matsushima formula is that it readily generalizes to the Shimura varieties for more general groups; in fact it is almost completely real-analytic in nature and works for general locally symmetric spaces, the main geometric ingredient being the *de Rham theorem*, namely the resolution of a locally constant sheaf $\mathcal{L}$ by sheaves of C^∞-differential forms with values in $\mathcal{L}$.

In this section, let us use the notation:

$$G := GL_2(\mathbb{R}), \quad U := U_\infty = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right\} \subset G, \quad \mathcal{H} = G/U.$$

This U is different from our usual U, only in this section and §4. Note that our U is a *maximal connected compact modulo center* subgroup of G; the maximal compact subgroup is $O_2(\mathbb{R})$, the maximal connected compact subgroup is $SO_2(\mathbb{R})$, and our U is $\mathbb{R}^\times_{>0} \cdot SO_2(\mathbb{R})$. The argument of this section presumably works with $G = \mathcal{G}(\mathbb{R})$ for any connected reductive algebraic group $\mathcal{G}/\mathbb{R}$.

2.1. **De Rham theorem: relative Lie algebra cohomology.** When we start looking at the differential forms on symmetric spaces, we will quickly find what is called the *Lie algebra cohomology.* Thinking of G as a real C^∞-manifold, its tangent space at the identity $T(G)_1$ is the *Lie algebra* $\mathfrak{g} := \operatorname{Lie} G$. For every $g \in G$, the tangent space at g is isomorphic to $\mathfrak{g}$ by $g : T(G)_1 \xrightarrow{\cong} T(G)_g$. Therefore, for a finite dimensional $\mathbb{C}$-vector space V, we can think of the $\mathbb{C}$-vector space $C^i(G, V)$ of C^∞ *differential i-forms on G with values in V* as:

$$C^i(G, V) = \operatorname{Hom}_{\mathbb{R}}\big(\wedge^i \mathfrak{g},\ C^\infty(G, V)\big) =: C^i\big(\mathfrak{g}, C^\infty(G, V)\big), \tag{2.1.1}$$

where $C^\infty(G, V) = C^\infty(G) \otimes V$ is the space of C^∞-functions of G with values in V. The exterior differentials on the i-forms have the corresponding expression for $C^i(\mathfrak{g}, C^\infty(G, V))$, involving the Lie brackets (see [BoWa], Chapter I).

In order to define these differentials, we need the left $\mathfrak{g}$-module structure on $C^\infty(G, V)$; we are thinking of the elements of $\mathfrak{g}$ as differential operators on the functions on G. More precisely, the $C^\infty(G, V)$ has the left G-action by the regular representation coming from the right G-action on G itself, i.e. $g\phi := (h \mapsto \phi(hg))$ for $g \in G$ and $\phi \in C^\infty(G, V)$, and by differentiability it naturally induces the left $\mathfrak{g}$-action. Actually the complex $C^*(\mathfrak{g}, W)$ of vector spaces is defined for general (left) $\mathfrak{g}$-modules, and the cohomology groups $H^*(\mathfrak{g}, W)$ of this complex are called the *Lie algebra cohomology.* This is the right derived functor of the left exact functor

$$\big(\ \mathfrak{g}\text{-modules over } \mathbb{C}\ \big) \ni W \longmapsto W^{\mathfrak{g}} := \{v \in W \mid Xv = 0\ (\forall X \in \mathfrak{g})\} \in \big(\ \mathbb{C}\text{-vector spaces}\ \big).$$

The $\mathfrak{g}$-modules are same as modules over the *universal enveloping algebra* $U(\mathfrak{g})$, which is a non-commutative but usual (unital and associative) $\mathbb{C}$-algebra. Then the above functor is

$$\big(\ U(\mathfrak{g})\text{-modules over } \mathbb{C}\ \big) \ni W \longmapsto W^{\mathfrak{g}} := \operatorname{Hom}_{U(\mathfrak{g})}(\mathbb{C}, W) \in \big(\ \mathbb{C}\text{-vector spaces}\ \big),$$

therefore we have $H^*(\mathfrak{g}, W) = \operatorname{Ext}_{U(\mathfrak{g})}(\mathbb{C}, W)$ under this identification.

Now let Γ be a discrete subgroup of G which acts freely on $\mathcal{H}$ (hence $\Gamma\backslash X$ will be smooth), and assume that V is equipped with a left action of Γ, i.e. it is a finite dimensional representation of Γ. Then this V defines a locally constant sheaf $\mathcal{L}$ of $\mathbb{C}$-vector spaces (or, a local system of $\mathbb{C}$-vector spaces) on $\Gamma\backslash G$ as follows:

$$\mathcal{L} := \Gamma\backslash(G \times V) \longrightarrow \Gamma\backslash G,$$

where we made Γ act diagonally on $G \times V$. The pull back of $\mathcal{L}$ by the canonical surjection $G \to \Gamma\backslash G$ is the constant sheaf associated to the vector space V. We can think of the C^∞-sections of $\mathcal{L}$ as the Γ-equivariant functions in $C^\infty(G, V)$:

$$C^\infty(\Gamma\backslash G, \mathcal{L}) = \big\{\phi \in C^\infty(G, V) \bigm| \phi(\gamma g) = \gamma\phi(g)\big\}, \tag{2.1.2}$$

and the i-forms on $\Gamma\backslash G$ with values in $\mathcal{L}$ are:

$$C^i(\Gamma\backslash G, \mathcal{L}) = \mathrm{Hom}_{\mathbb{R}}\big(\wedge^i\mathfrak{g},\ C^\infty(\Gamma\backslash G, \mathcal{L})\big) = C^i\big(\mathfrak{g}, C^\infty(\Gamma\backslash G, \mathcal{L})\big).$$

Note that the tangent space of $\Gamma\backslash G$ is still $\mathfrak{g}$, because Γ is discrete and $G \to \Gamma\backslash G$ is locally isomorphic. This gives the isomorphism:

$$H^*(\Gamma\backslash G, \mathcal{L}) = H^*\big(\mathfrak{g}, C^\infty(\Gamma\backslash G, \mathcal{L})\big)$$

by the de Rham theorem (i.e. the complex $C^i(\Gamma\backslash G, \mathcal{L})$ is a fine resolution of $\mathcal{L}$, hence computes the singular cohomology of $\mathcal{L}$).

We carry out the last step: taking the quotient of $\Gamma\backslash G$ by U from the right, to get $X_\Gamma := \Gamma\backslash G/U = \Gamma\backslash\mathcal{H}$. We will pass from Lie algebra cohomolgy to the $(\mathfrak{g}, U)$-*cohomology*, or *relative Lie algebra cohomology.* We start from the constant sheaf on $\mathcal{H} := G/U$ associated to the finite dimensional $\mathbb{C}$-vector space V with a left Γ-action. This defines a locally constant sheaf $\mathcal{L}$ on $X_\Gamma = \Gamma\backslash\mathcal{H}$ as above. The diffential i-forms on $\mathcal{H}$ (resp. X_Γ) are the ones on G (resp. $\Gamma\backslash G$) that are invariant under the right U-action. If we set $\mathfrak{k} := \mathrm{Lie}\, U$, then it is a subalgebra of $\mathfrak{g}$, and the tangent spaces of $\mathcal{H}$ (or X_Γ) are isomorphic to $\mathfrak{g}/\mathfrak{k}$, and we modify (2.1.1) to:

$$C^i(\mathcal{H}, V) = \mathrm{Hom}_U\big(\wedge^i(\mathfrak{g}/\mathfrak{k}),\ C^\infty(G, V)\big) =: C^i\big(\mathfrak{g}, U; C^\infty(G, V)\big), \tag{2.1.3}$$

where the action of U on $\wedge^i(\mathfrak{g}/\mathfrak{k})$ is induced by the adjoint representation. Note that we have not changed the coefficient space $C^\infty(G, V)$ from (2.1.1). We are taking the left U-invariant forms by taking the Hom_U space, and there we use the left action of U on $C^\infty(G, V)$ as well as the $\mathfrak{g}$-module structure. The exterior diffentials of our new complex $C^i\big(\mathfrak{g}, U; C^\infty(G, V)\big)$ are the restriction of the old one, and we denote the cohomology groups by $H^*\big(\mathfrak{g}, U; C^\infty(G, V)\big)$.

Thus the domain of our new cohomology theory (the *relative Lie algebra cohomology*) is the category of $(\mathfrak{g}, U)$-*modules* (over $\mathbb{C}$), a $\mathfrak{g}$-module on which U acts differentiably and the induced $\mathfrak{k}$-actions (from $\mathfrak{g}$ and $\mathfrak{k}$) coincide[1]. We need to require some finiteness conditions for the definition of $(\mathfrak{g}, U)$-modules, but we postpone discussing it for now. Note that in our $C^\infty(G, V)$, its $(\mathfrak{g}, U)$-module structure was induced by the G-module structure (we can say

[1] Actually the homological algebra (e.g. interpretation as Ext functor etc.) works better if we consider the cohomology functor on the category of $(\mathfrak{g}, \mathfrak{k})$-*modules* over $\mathbb{C}$, similarly defined with U replaced by $\mathfrak{k}$. In our case U is connected, and for a $(\mathfrak{g}, U)$-module W, we have $H^*(\mathfrak{g}, U; W) = H^*(\mathfrak{g}, \mathfrak{k}; W)$.)

we are forgetting a part of G-module structure); the G-modules are the obvious abundant source of $(\mathfrak{g}, U)$-modules. Similarly we have:

$$C^i(X_\Gamma, \mathcal{L}) = \mathrm{Hom}_U\big(\wedge^i(\mathfrak{g}/\mathfrak{k}),\ C^\infty(\Gamma\backslash G, \mathcal{L})\big) = C^i\big(\mathfrak{g}, U; C^\infty(\Gamma\backslash G, \mathcal{L})\big),$$

and taking the cohomology groups, we conclude by the de Rham theorem:

Proposition 2.1. $H^*(X_\Gamma, \mathcal{L}) = H^*\big(\mathfrak{g}, U; C^\infty(\Gamma\backslash G, \mathcal{L})\big)$.

2.2. **Trivialization of $\mathcal{L}$.** Now we put another important assumption; the action of Γ on V actually extends to the action of G on V, namely a finite dimensional representation $\xi : G \to \mathrm{Aut}(V)$. We will write $\mathcal{L}_\xi$ for $\mathcal{L}$, because the locally constant sheaf $\mathcal{L}$ is determined by ξ (or more precisely, its restriction $\xi|_\Gamma$). Then the $\mathcal{L}_\xi$ becomes isomorphic to the constant sheaf on $\Gamma\backslash G$ defined by V as follows:

$$\begin{aligned}(\Gamma\backslash G) \times V &\xrightarrow{\cong} \Gamma\backslash(G \times V) =: \mathcal{L}_\xi,\\ \big(h \bmod \Gamma, v\big) &\longmapsto (h, hv) \bmod \Gamma\end{aligned}$$

where $h \in G$, thus we have the isomorphism between the spaces of their sections:

$$\begin{aligned}C^\infty(\Gamma\backslash G) \otimes V &\xrightarrow{\cong} C^\infty(\Gamma\backslash G, \mathcal{L}_\xi),\\ \phi \otimes v &\longmapsto (h \mapsto \phi(h) \cdot hv)\end{aligned}$$

where we consider $C^\infty(\Gamma\backslash G, \mathcal{L}_\xi) \subset C^\infty(G, V)$ as in (2.1.2); note that $h \mapsto \phi(h) \cdot hv$ is left Γ-*equivariant*, because the function ϕ is left Γ-*invariant*.

Now this isomorphism is not just an isomorphism of vector spaces. We consider the regular representation of G on the RHS — so far we have not used the right G action on the base space $\Gamma\backslash G$. So $g \in G$ acts on $C^\infty(\Gamma\backslash G, \mathcal{L}_\xi)$ from the left by $g\psi := (h \mapsto \psi(hg))$. Consider $\psi = (h \mapsto \phi(h) \cdot hv)$ for $\phi \otimes v$ as in the above isomorphism. Then $g\psi = (h \mapsto \phi(hg) \cdot hgv)$ corresponds to $g\phi \otimes gv$ for the usual regular representation $g\phi := (h \mapsto \phi(gh))$ of G on $\phi \in C^\infty(\Gamma\backslash G)$. Thus it was actually an isomorphism of left G-representations:

$$C^\infty(\Gamma\backslash G) \otimes \xi \xrightarrow{\cong} C^\infty(\Gamma\backslash G, \mathcal{L}_\xi),$$

where the tensor product of two representations of G is defined by the action $g(\phi \otimes v) := g\phi \otimes gv$ for all $g \in G$, and we wrote ξ instead of V to remember that we are thinking of the tensor product of vector spaces as the tensor product of representations. As the $(\mathfrak{g}, U)$-module structure is induced from the G-module structure, it is naturally the tensor product of $(\mathfrak{g}, U)$-modules. Therefore we can rewrite Proposition 2.1 as:

Proposition 2.2. *Let $\Gamma \subset G$ be a discrete subgroup which acts freely on $\mathcal{H}$, and $\mathcal{L}_\xi$ be a locally constant sheaf on $X_\Gamma := \Gamma\backslash\mathcal{H}$ defined by a finite dimensional representation $\xi : G \to$ $\mathrm{Aut}(V)$. Then:*

$$H^*(X_\Gamma, \mathcal{L}_\xi) = H^*\big(\mathfrak{g}, U; C^\infty(\Gamma\backslash G) \otimes \xi\big).$$

This proposition reduces the computation of Betti cohomology groups of X_Γ (which are of global nature) to the computation of $(\mathfrak{g}, U)$-cohomology of $C^\infty(\Gamma\backslash G) \otimes \xi$, and all the global information must be now contained in the decomposition of $C^\infty(\Gamma\backslash G)$ as $(\mathfrak{g}, U)$-modules, which in turn is induced from its decomposition as a G-representation.

2.3. **Decomposition of** $C^\infty(\Gamma\backslash G)$**.** We first need to discuss the finiteness properties of $(\mathfrak{g}, U)$-modules and their cohomology. Getting back to say (2.1.3), our coefficient module $C^\infty(G, V)$ is a huge space, but the U-invariant cochains are of manageable size, because any vector in the image of a U-equivariant homomorphism from $\wedge^i(\mathfrak{g}/\mathfrak{k})$ (which is finite dimensional over $\mathbb{C}$) is *U-finite*, i.e. its U-span is finite dimensional over $\mathbb{C}$. Thus when we compute the cohomology groups $H^*(\mathfrak{g}, U; W)$, we can replace W by its subspace consisting of all U-finite vectors in W (*the* $(\mathfrak{g}, U)$*-cohomology only notices the U-finite vectors*).

We usually require in the definition of $(\mathfrak{g}, U)$-module W that, as a U-module, it is (i) semisimple, and (ii) all vectors are U-finite. When we think of $(\mathfrak{g}, U)$-cohomology with a coefficient module that does not satisfy (ii), we understand that we are taking its subspace of all U-finite vectors.

Now how do we analyze the $(\mathfrak{g}, U)$-module $C^\infty(\Gamma\backslash G)$ appearing in Proposition 2.2? For this, we first enlarge the space into the space $L^2(\Gamma\backslash G)$ of square integrable functions on $\Gamma\backslash G$, and appeal to its spectral decomposition. For this, we would like to assume that $X_\Gamma = \Gamma\backslash G/U$ is *compact* — this is not the case for the modular curves, but there are large classes of compact Shimura varieties (we will discuss this later). Recall that $U = ZK$, where Z is the connected component of the center of G and K is a maximal connected compact subgroup (in our case, $Z = \mathbb{R}^\times_{>0}$ and $K = SO_2(\mathbb{R})$). To say that X_Γ is compact is equivalent to say that $\Gamma\backslash G$ is *compact modulo center*, by which we mean that $\Gamma\backslash G/Z$ is compact.

As we are appealing to the harmonic analysis on $\Gamma\backslash G$, we are moving to the framework of *unitary representations*, where the essential part of the theory is for the *semisimple groups* which have finite center. As we do want to work with reductive groups G, we will fix the central charcter, or more precisely the character of the connected component of the center (which we denote by Z), and think of a representation that are unitary modulo Z. We say that a representation of G is *unitary modulo center* with the *central character* $\omega : Z \to \mathbb{C}^\times$, if its restriction to the derived group of G, in this case $SL_2(\mathbb{R})$, is unitary and Z acts through ω (note that $SL_2(\mathbb{R}) \cong G/Z$). For each ω, we write:

$$C^\infty(\Gamma\backslash G)[\omega] := \big\{\phi \in C^\infty(\Gamma\backslash G) \,\big|\, \phi(xz) = \omega(z)\phi(x)\ (\forall z \in Z)\big\},$$

and similarly for $L^2(\Gamma\backslash G)[\omega]$. This $L^2(\Gamma\backslash G)$ is a unitary modulo center representation of G with the central character ω.

Theorem 2.3. (Gelfand-Piatetski-Shapiro [GGPS], Chapter 1, §2.3) *Let $\Gamma\backslash G$ be compact modulo center, and fix $\omega : Z \to \mathbb{C}^\times$. Then the space $L^2(\Gamma\backslash G)[\omega]$ decomposes into a discrete Hilbert direct sum of irreducible representations of G with finite multiplicities:*

$$L^2(\Gamma\backslash G)[\omega] = \widehat{\bigoplus_\pi} m(\pi, \Gamma)\pi.$$

Now we get back to Proposition 2.2. As Z acts trivially on $\wedge^i(\mathfrak{g}/\mathfrak{k})$, the space of cochains $C^i(\mathfrak{g}, U; W) = \mathrm{Hom}_U(\wedge^i(\mathfrak{g}/\mathfrak{k}), W)$ is 0 unless Z acts trivially on W. Applying this to $W = C^\infty(\Gamma\backslash G) \otimes \xi$, if the central character of ξ is ω^{-1}, then only the cochains that land into $C^\infty(\Gamma\backslash G)[\omega] \otimes \xi$ will survive, hence:

$$H^*\big(\mathfrak{g}, U; C^\infty(\Gamma\backslash G) \otimes \xi\big) = H^*\big(\mathfrak{g}, U; C^\infty(\Gamma\backslash G)[\omega] \otimes \xi\big).$$

(As the association $\xi \mapsto \mathcal{L}_\xi$ is functorial, we can restrict our consideration to the case where ξ is *irreducible* and therefore has a central character.) Then observe that $C^\infty(\Gamma\backslash G)[\omega]$ is the dense subspace in $L^2(\Gamma\backslash G)[\omega]$ consisting of the C^∞-vectors. Now the argument in p.143-144 of [BoWa] tells us that:

Theorem 2.4. (The Matsushima formula: classical setting, [BoWa] VII.5.2) *Let $\Gamma \subset G$ be a discrete subgroup such that* (i) *$\Gamma\backslash G$ is compact modulo center, and* (ii) *acts freely on $\mathcal{H}$. Let $\mathcal{L}_\xi$ be a locally constant sheaf on $X_\Gamma := \Gamma\backslash\mathcal{H}$ defined by a finite dimensional representation $\xi : G \to \mathrm{Aut}(V)$. Then:*

$$H^*(X_\Gamma, \mathcal{L}_\xi) = \bigoplus_\pi m(\pi,\Gamma) H^*\big(\mathfrak{g}, U; \pi\otimes\xi\big),$$

where π runs through the irreducible unitary modulo center representations of G. If ξ has the central character $\omega^{-1} : Z \to \mathbb{C}^\times$, then all of π have the central character ω.

3. The Matsushima formula: Adelic setting

As we would like to stick to the compact setting, it will be false to deal with the $GL_2(\mathbb{A}^\infty)$-case any longer. We quickly introduce a more general setting without delving into the details of necessary conditions. Let $G/\mathbb{Q}$ be a connected reductive algebraic group, and let U_∞ be the maximal connected compact modulo center subgroup of $G(\mathbb{R})$, and set $\mathcal{H} := G(\mathbb{R})/U_\infty$. For an open compact subgroup $U \subset G(\mathbb{A}^\infty)$, we define:

$$X_U := G(\mathbb{Q})\backslash\Big(\big(G(\mathbb{A}^\infty)/U\big)\times\mathcal{H}\Big) = G(\mathbb{Q})\backslash G(\mathbb{A})/(U\cdot U_\infty).$$

By the strong approximation theorem for G (generalizing what we saw last week), this X_U is a disjoint union of quotients $\Gamma_\sigma\backslash\mathcal{H}$ of $\mathcal{H}$ by discrete subgroup $\Gamma_\sigma := G(\mathbb{Q}) \cap \sigma U \sigma^{-1}$ for finitely many $\sigma \in G(\mathbb{A}^\infty)$. This gives an inverse system of comlex manifolds indexed by U, with the obvious surjections $[\mathrm{can}] : X_{U'} \to X_U$ whenever $U \subset U'$, and we write $X := \{X_U\}_U$. It has the Hecke action of $G(\mathbb{A}^\infty)$, as described in §1.1.

For a finite dimensional representation $\xi : G(\mathbb{R}) \to \mathrm{Aut}(V_\xi)$ on a $\mathbb{C}$-vector space V_ξ, we start with the $G(\mathbb{Q})$-equivariant constant sheaf $\mathcal{L}_\xi$ on $\big(G(\mathbb{A}^\infty)/U\big)\times\mathcal{H}$ associated to V_ξ, and descend to X_U to define a locally constant sheaf $\mathcal{L}_\xi$ on X_U,

$$\mathcal{L}_\xi := G(\mathbb{Q})\backslash\Big(\big(G(\mathbb{A}^\infty)/U\big)\times\mathcal{H}\times V_\xi\Big),$$

in the usual way. (Eventually we will only consider the *algebraic* representation $\xi : G \to \mathrm{Aut}(V_\xi)$ for a $\overline{\mathbb{Q}}$-vector space V_ξ, but for now we think of V_ξ as a $\mathbb{C}$-vector space, as we only deal with the Betti cohomology.) As $[\mathrm{can}]^*\mathcal{L}_\xi = \mathcal{L}_\xi$, we have the direct system of Betti cohomology groups $H^*(X_U, \mathcal{L}_\xi)$ for each degree $*$. We write:

$$H^*(X, \mathcal{L}_\xi) := \varinjlim_U H^*(X_U, \mathcal{L}_\xi),$$

which is a smooth admissible representation of $G(\mathbb{A}^\infty)$ by what we discussed in §1.2. As each X_U is none other than a finite disjoint union of $X_{\Gamma_\sigma} := \Gamma_\sigma\backslash\mathcal{H}$, Proposition 2.2 reads as follows. Assume that U is small enough so that each Γ_σ acts freely on $\mathcal{H}$. Then:

$$H^*(X_U, \mathcal{L}_\xi) = H^*\big(\mathfrak{g}, U; C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/U) \otimes \xi\big),$$

and taking the limit with respect to U, we get:

$$H^*(X, \mathcal{L}_\xi) = H^*\big(\mathfrak{g}, U; C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \xi\big),$$

where $C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$ denotes the space of smooth functions in a sense that it is U-invariant for some open compact subgroup $U \subset G(\mathbb{A}^\infty)$.

Now we ensure that each X_U is a *compact* complex manifold by assuming that the derived group G^{der} has $\mathbb{Q}$-rank 0 (this implies that G has no proper parabolic subgroup defined over $\mathbb{Q}$, thus no boundaries).

Example 3.1. For example, if we take the multiplicative group G of an indefinite quaternion algebra over $\mathbb{Q}$, then we have an example of such G with $G(\mathbb{R}) = GL_2(\mathbb{R})$. Then the Riemann surfaces X_Γ for discrete subgroups $\Gamma = G(\mathbb{Q}) \cap U$ for $U \subset G(\mathbb{A}^\infty)$ give abundant examples for Theorem 2.4, which we call *(quaternionic) Shimura curves over* $\mathbb{Q}$.

Now we need to use the spectral decomposition of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Fixing a character $\omega_\infty : Z \to \mathbb{C}^\times$ of the connected component Z of the center of $G(\mathbb{R})$, we have:

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))[\omega_\infty] = \widehat{\bigoplus_\pi} m(\pi)\pi,$$

where π runs through irreducible representations of $G(\mathbb{A})$. This follows from Theorem 2.3 as follows. For each U, we have:

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/U)[\omega_\infty] = \widehat{\bigoplus_{\pi_\infty}} m(\pi_\infty, U)\pi_\infty,$$

where π_∞ runs through the irreducible representations of $G(\mathbb{R})$. On the other hand, for each π_∞, the spaces

$$\mathrm{Hom}_{G(\mathbb{R})}\big(\pi_\infty, L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/U)\big)$$

are finite dimensional for each U, hence they pile up into a smooth admissible representation of $G(\mathbb{A}^\infty)$. By showing that they become Hilbert direct sum of irreducible admissible representations π^∞ with finite multiplicities, we obtain the claimed decomposition into $\pi = \pi^\infty \otimes \pi_\infty$. By the same argument as in the classical setting, we conclude:

Theorem 3.2. (The Matsushima formula: adelic setting) *Let $G/\mathbb{Q}$ be a connected reductive group such that $\mathbb{Q}$-rank of G^{der} is 0. Let $\mathcal{L}_\xi$ be a locally constant sheaf on X defined by an irreducible finite dimensional representation $\xi : G(\mathbb{R}) \to \mathrm{Aut}(V_\xi)$. Then we have:*

$$H^*(X, \mathcal{L}_\xi) = \bigoplus_\pi m(\pi)\pi^\infty \otimes H^*\big(\mathfrak{g}, U_\infty; \pi_\infty \otimes \xi\big),$$

as admissible representations of $G(\mathbb{A}^\infty)$, where $\pi = \pi^\infty \otimes \pi_\infty$ runs through the automorphic representations of G, and $m(\pi)$ is the multiplicity of π in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))[\omega_\infty]$, where $\omega_\infty : Z \to \mathbb{C}^\times$ is the inverse of the central character of ξ.

Lecture 6 (Sep. 26, 2008)

4. The Matsushima formula: Complex setting

In this section, we return to the notation in §2. The significance of the Matsushima formula (Theorem 2.4) is that on the RHS, the "global information" of Γ, or X_Γ, on the LHS is all subsumed in the integers $m(\pi, \Gamma)$, and the remaining parts $H^*\big(\mathfrak{g}, U; \pi \otimes \xi\big)$ have nothing to do with Γ any more — they are objects living in the representation theory of G. Actually we can narrow down the representations π that occur in the formula, simply by looking at ξ.

Definition 4.1. Let $\xi : G \to \mathrm{Aut}(V)$ be a finite dimensional representation. An irreducible representation π of G is called *cohomological for* ξ if $H^*\big(\mathfrak{g}, U; \pi \otimes \xi\big) \neq 0$ for some $*$.

4.1. **Finiteness.** We take a small detour on the finiteness of $(\mathfrak{g}, U)$-cohomology. As the singular cohomology of compact space X_Γ is finite dimensional, we see from the Matsushima formula (Theorem 2.4) that the vector spaces $H^*\big(\mathfrak{g}, U; \pi \otimes \xi\big)$ appearing in the formula are finite dimensional. This is a general fact for irreducible π. As the cochains $\mathrm{Hom}_U(\wedge^i(\mathfrak{g}/\mathfrak{k}), W)$ which computes the $(\mathfrak{g}, U)$-cohomology for a $(\mathfrak{g}, U)$-module W is only looking at the representation of U (we need the action of $\mathfrak{g}$ for the differential), the cochains are finite dimensional as long as each irreducible representaion χ of U appears in W with finite multiplicity. This property is what ensures the finiteness of $(\mathfrak{g}, U)$-cohomology. Recall that we require that $(\mathfrak{g}, U)$-module is always semisimple as a representation of U and all the vectors are U-finite.

Definition 4.2. Let W be a $(\mathfrak{g}, U)$-module. An irreducible representation χ of U is called a *U-type* of W if it appears in the decomposition of W as a representation of U. If each U-type appears with finite multiplicity, then W is called *admissible*. By the above argument, if W is an admissible $(\mathfrak{g}, U)$-module, then $C^*(\mathfrak{g}, U; W)$ (hence also $H^*(\mathfrak{g}, U; W)$) are finite dimensional $\mathbb{C}$-vector spaces.

It was Harish-Chandra who first studied the $(\mathfrak{g}, U)$-modules associated to the irreducible representations of G. Recall that when we think of a representation of G as a $(\mathfrak{g}, U)$-modules, then we take its subspace consisting of all U-finite vectors.

Proposition 4.3. *If π is a irreducible unitary modulo center representation of G, then its associated $(\mathfrak{g}, U)$-module is an admissible $(\mathfrak{g}, U)$-module, hence $H^*(\mathfrak{g}, U; \pi \otimes \xi)$ is finite dimensional for any finite dimensional representaiton $\xi : G \to \mathrm{Aut}(V)$.*

Actually Harish-Chandra showed that, in the context of semisimple G and U is a maximal compact subgroup of U, the associated $(\mathfrak{g}, U)$-module is irreducible and determines the original π up to unitary equivalence. Sometimes the associated $(\mathfrak{g}, U)$-module is called *Harish-Chandra module* of π.

4.2. **Hodge theory.** Now the finite dimensionality of the cochains $C^i(\mathfrak{g}, U; W)$ opens up the possibility of doing *Hodge theory* for these cochains. The original Hodge theory (on compact Kähler manifolds; see [KM]) used the *Laplacian*, a second order differential operator, to define the *harmonic cochains* which represents all the cohomology classes. We have a similar structure for the cochains $C^i(\mathfrak{g}, U; W)$ when W is a *unitary* $(\mathfrak{g}, U)$-module, where the *Casimir element* in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ plays the role of the Laplacian — which is (presumably) a translation of the Hodge theory on symmetric spaces into $(\mathfrak{g}, U)$-cohomology, but it is simpler because the cochains are of local nature ([BoWa], II-2,3).

At this point we should assume that $\mathfrak{k}$ is associated to a *Cartan involution* of G, so that the decomposition of $\mathfrak{g}$ into ± 1-eigenspaces is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{p} \cong \mathfrak{g}/\mathfrak{k} \cong \text{the tangent space of } \mathcal{H},$$

and there is a G-invariant and θ-invariant symmetric bilinear form B on $\mathfrak{g}$, which is negative on $\mathfrak{k}$ and positive on $\mathfrak{p}$. We call the trace form of B, i.e. for a basis $\{e_i\}$ of $\mathfrak{g}$ and its dual basis $\{e_i^*\}$ with respect to B, the degree two element

$$\mathcal{C} := \sum_{1 \leq i \leq \dim \mathfrak{g}} e_i \cdot e_i^* \in U(\mathfrak{g}),$$

the *Casimir element* of $U(\mathfrak{g})$. It is well-defined up to a scalar multiple and is contained in the center $\mathcal{Z}$ of $U(\mathfrak{g})$. We do not go into the definitions of harmonic cochains $\mathcal{H}^i(W)$ in $C^i(\mathfrak{g}, U; W)$, but the cohomology classes come in bijection with harmonic cochains, as:

$$\begin{aligned} C^i(\mathfrak{g}, U; W) &= \mathcal{H}^i(W) \oplus dC^{i-1}(W) \oplus \delta C^{i+1}(W), \\ H^i(\mathfrak{g}, U; W) &\xrightarrow{\cong} \mathcal{H}^i(W). \end{aligned}$$

What makes it more desirable is that, as the Casimir element is in the center $\mathcal{Z}$, it acts by a scalar on any irreducible $\mathfrak{g}$-module. Thus when the coefficient is of the form $\pi \otimes \xi$, the harmonic cochains are, if they exist at all, all of the cochians. The upshot is:

Proposition 4.4. ([BoWa], II.3.1) *Let π be an irreducible unitary modulo center representation of G, and $\xi : G \to \mathrm{Aut}(V)$ be a finite dimensional representation. Let λ_π, λ_ξ be the eigenvalues of the Casimir element in π, ξ respectively. Then:*

$$H^i(\mathfrak{g}, U; \pi \otimes \xi) = \begin{cases} C^i(\mathfrak{g}, U; \pi \otimes \xi) = \mathrm{Hom}_U(\wedge^i \mathfrak{p}, \pi \otimes \xi) & (\lambda_\pi + \lambda_\xi = 0), \\ 0 & (\lambda_\pi + \lambda_\xi \neq 0). \end{cases}$$

Thus we have narrowed down the representations which are cohomological for ξ.

Corollary 4.5. *Let π be an irreducible unitary modulo center representation of G, and $\xi : G \to \mathrm{Aut}(V)$ be an irreducible finite dimensional representation. Let ω_π, ω_ξ be the central characters, λ_π, λ_ξ be the eigenvalues of the Casimir element, respectively in π, ξ. If π is cohomological for ξ, then $\omega_\pi \omega_\xi = \mathbf{1}$ and $\lambda_\pi + \lambda_\xi = 0$ (the $\mathbf{1}$ is the trivial character of Z), and $H^i(\mathfrak{g}, U; \pi \otimes \xi) = \mathrm{Hom}_U(\wedge^i \mathfrak{p}, \pi \otimes \xi)$.*

Moreover, we reduced the computation of $H^i(\mathfrak{g}, U; \pi \otimes \xi)$ to that of $\mathrm{Hom}_U(\wedge^i \mathfrak{p}, \pi \otimes \xi)$, which only sees the U-types of $\pi \otimes \xi$. This makes the analysis of U-types of representations all-important. For example, when π is square integrable modulo center, then the only non-trivial $H^*(\mathfrak{g}, U; W)$ appears when $i = \dim G/U$, and it is 1-dimensional, coming from the tensor of the lowest weight of ξ (the highest weight of $\xi^\vee$) and the lowest U-type of π ([BoWa], II.5).

4.3. **Complex structure.** Now assume that $\mathcal{H} = G/U$ carries a complex structure. This is when an element z_0 in the center of $\mathfrak{k}$ gives an invariant almost complex structure $J = \mathrm{ad}(z_0)|_{\mathfrak{p}}$ on $\mathfrak{p}$, in which case the complexification $\mathfrak{p}_{\mathbb{C}} := \mathfrak{p} \otimes \mathbb{C}$ decomposes into the $\pm i$-eigenspaces of $\mathrm{ad}(z_0)$ as:

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

which are two commutative subalgebras of $\mathfrak{g}_{\mathbb{C}}$, normalized by $\mathfrak{k}_{\mathbb{C}}$, and complex conjugate of each other. Note that $\mathfrak{p} \cong \mathfrak{p}^+$ is the tangent space of $\mathcal{H}$ as a complex manifold. In this case, the decomposition:

$$\wedge^i(\mathfrak{p}_{\mathbb{C}}) = \bigoplus_{p+q=i} \wedge^p \mathfrak{p}^+ \otimes \wedge^q \mathfrak{p}^-$$

gives a bigrading on the cochain complex for a $(\mathfrak{g}, U)$-module W by:

$$C^{p,q}(\mathfrak{g}, U; W) := \mathrm{Hom}_U(\wedge^p \mathfrak{p}^+ \otimes \wedge^q \mathfrak{p}^-, W), \quad C^{q,p} = (C^{p,q})^c \quad (c : \text{complex conjugation})$$

$$C^i(\mathfrak{g}, U; W) = \mathrm{Hom}_{U,\mathbb{R}}(\wedge^i \mathfrak{p}, W) = \mathrm{Hom}_{U,\mathbb{C}}(\wedge^i(\mathfrak{p}_{\mathbb{C}}), W) = \bigoplus_{p+q=i} C^{p,q}(\mathfrak{g}, U; W).$$

Thus, using the fact that cohomology classes are represented by the harmonic cochains and the Laplacian preserves the bidegree, we have a corresponding *Hodge decomposition* of the cohomology groups.

Proposition 4.6. *Assume that $\mathcal{H} = G/U$ carries a complex structure. For an admissible $(\mathfrak{g}, U)$-module W, we have the Hodge decomposition (we write c for the complex conjugation):*

$$H^i(\mathfrak{g}, U; W) = \bigoplus_{p+q=i} H^{p,q}(\mathfrak{g}, U; W), \quad H^{q,p} = (H^{p,q})^c$$

where $H^{p,q}(\mathfrak{g}, U; W)$ is the cohomology classes represented by the elements of $C^{p,q}(\mathfrak{g}, U; W)$. In particular, when $W = \pi \otimes \xi$ as in Proposition 4.4, we have:

$$H^{p,q}(\mathfrak{g}, U; \pi \otimes \xi) = C^{p,q}(\mathfrak{g}, U; \pi \otimes \xi) := \mathrm{Hom}_U\big(\wedge^p \mathfrak{p}^+ \otimes \wedge^q \mathfrak{p}^-, \pi \otimes \xi\big).$$

This Hodge decomposition of $(\mathfrak{g}, U)$-cohomology compares in the obvious way with the Hodge theory of the complex manifolds $X_\Gamma \cong \Gamma\backslash\mathcal{H}$ via the Matsushima formula. The Hodge decomposition of singular cohomology groups of complex manifolds is defined through the decomposition of differential i-forms into (p, q)-*forms*, which gives:

$$H^i(X_\Gamma, \mathcal{L}_\xi) = \bigoplus_{p+q=i} H^{p,q}(X_\Gamma, \mathcal{L}_\xi),$$

where $H^{p,q}(X_\Gamma, \mathcal{L}_\xi)$ is the cohomology classes represented by the (p, q)-forms. Moreover, as the complex $C^{p,*}(X_\Gamma, \mathcal{L}_\xi)$ with the $\overline{\partial}$-differentials gives the *Dolbeault resolution* ([KM], 2.6) of the holomorphic vector bundle $\mathcal{L}_\xi \otimes \Omega^p$, where Ω^p is the sheaf of *holomorphic p-forms* (and $\Omega^0 = \mathcal{O}$, the sheaf of holomorphic functions), we have:

$$H^i(X_\Gamma, \mathcal{L}_\xi) \cong \bigoplus_{p+q=i} H^q(X_\Gamma, \mathcal{L}_\xi \otimes \Omega^p).$$

As the (p,q)-forms on X_Γ corresponds to the (p,q)-cochains in the $(\mathfrak{g}, U)$-cohomology, we have the following refinement of the Matsushima formula:

Theorem 4.7. (The Matsushima formula: complex setting) *Let G, Γ, ξ as in Theorem 2.4, and assume that $\mathcal{H} = G/U$ carries a complex structure. Then, for each $p, q \in \mathbb{Z}_{\geq 0}$:*

$$H^q(X_\Gamma, \mathcal{L}_\xi \otimes \Omega^p) = \bigoplus_\pi m(\pi, \Gamma) \operatorname{Hom}_U\big(\wedge^p \mathfrak{p}^+ \otimes \wedge^q \mathfrak{p}^-, \pi \otimes \xi\big),$$

where π runs through the irreducible unitary modulo center representations of G that are cohomological for ξ (hence satisfies the conditions of Corollary 4.5). Similarly, if G, ξ are as in Theorem 3.2 and $\mathcal{H} = G/U$ carries a complex structure, then for each $p, q \in \mathbb{Z}_{\geq 0}$:

$$H^q(X, \mathcal{L}_\xi \otimes \Omega^p) = \bigoplus_\pi m(\pi)\pi^\infty \otimes \operatorname{Hom}_U\big(\wedge^p \mathfrak{p}^+ \otimes \wedge^q \mathfrak{p}^-, \pi_\infty \otimes \xi\big),$$

as admissible representations of $G(\mathbb{A}^\infty)$, where $\pi = \pi^\infty \otimes \pi_\infty$ runs through the automorphic representations of G, and $m(\pi)$ is as in Theorem 3.2.

Lecture 7 (Sep. 29, 2008)

5. Eichler-Shimura isomorphism

Now let us go back to the upper half plane setting to see how it recovers the *Eichler-Shimura isomorphism*, modulo the cusp issue. We start by relating the line bundles $\mathcal{V}_k$ in the Week 1, whose sections were the *weight k* of classical modular forms, with the holomorphic vector bundle $\mathcal{L}_\xi \otimes \mathcal{O}$ for certain finite dimensional representation $\xi : GL_2(\mathbb{R}) \to \operatorname{Aut}(V_\xi)$. This is done by defining a *Hodge filtration* on $\mathcal{L}_\xi \otimes \mathcal{O}$, by realizing $\mathcal{L}_\xi \otimes \mathcal{O}$ over $\mathcal{H}$ as a pull back from a vector bundle $\mathcal{V}_\xi$ over a flag variety (in this case $\mathbb{P}^1(\mathbb{C})$).

On of the most important thing about the Eichler-Shimura isomorphism was that it is equivariant under the Hecke action. This will be seen readily when we use the adelic formulation and vary the level U, and apply the Matsushima formula as identity between the smooth admissible representations of $G(\mathbb{A}^\infty)$.

5.1. **Holomorphic vector bundle $\mathcal{L}_\xi \otimes \mathcal{O}$.** We give an independent construction of the trivial (but left $GL_2(\mathbb{R})$-equivariant) holomorphic vector bundle $\mathcal{L}_\xi \otimes \mathcal{O}$ on $\mathcal{H}$, using the flag variety. Let us write $G := GL_2$ for brevity, and let:

$$Q(\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, b, d \in \mathbb{C} \right\} \subset G(\mathbb{C})$$

be the upper triangular parabolic subgroup of $G(\mathbb{C})$, so that we have $G(\mathbb{C})/Q(\mathbb{C}) \cong \mathbb{P}^1(\mathbb{C})$, as $\mathbb{C}$-points of the quotient of algebraic varieties $G/Q \cong \mathbb{P}^1$. We consider an embedding

$$j : G(\mathbb{R}) \ni g \longmapsto AgA^{-1} \in G(\mathbb{C}), \quad A := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},$$

so that $j^{-1}(Q(\mathbb{C})) = U_\infty$, and:

$$j : U_\infty \ni \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \longmapsto \begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix} \in Q(\mathbb{C}). \tag{5.1.1}$$

Thus we have an induced embedding, compatible with the complex structure:

$$j : \mathcal{H} = G(\mathbb{R})/U_\infty \longrightarrow G(\mathbb{C})/Q(\mathbb{C}) = \mathbb{P}^1(\mathbb{C}).$$

Now let $\xi : Q(\mathbb{C}) \to \mathrm{Aut}(V_\xi)$ be any (algebraic) finite dimensional representation of $Q(\mathbb{C})$ on a $\mathbb{C}$-vector space V_ξ, and define a *holomorphic* vector bundle $\mathcal{V}_\xi$ on $\mathbb{P}^1(\mathbb{C})$ with (by now) the usual construction, except that as we take the right quotient, we make $Q(\mathbb{C})$ act on V_ξ from the right. So define the right action of $q \in Q(\mathbb{C})$ on $G(\mathbb{C}) \times V_\xi$ as $(h, v) \mapsto (hq, q^{-1}v)$, and take the quotient to define the vector bundle $\mathcal{V}_\xi$ on $\mathbb{P}^1(\mathbb{C})$:

$$\mathcal{V}_\xi := (G(\mathbb{C}) \times V_\xi)/Q(\mathbb{C}) \longrightarrow G(\mathbb{C})/Q(\mathbb{C}) = \mathbb{P}^1(\mathbb{C}).$$

This $\mathcal{V}_\xi$ is a left $G(\mathbb{C})$-equivariant vector bundle via *trivial* action of $G(\mathbb{C})$ on V_ξ, i.e. $g \in G(\mathbb{C})$ acts from the left by $(h, v) \bmod Q \mapsto (gh, v) \bmod Q$ (we already used the left $G(\mathbb{C})$-action on V_ξ, interpreted as a right action, to define $\mathcal{V}_\xi$, so no action left!). By the left $G(\mathbb{C})$-equivariance, its restriction $\mathcal{V}_\xi|_{\mathcal{H}}$ under j is left $G(\mathbb{R})$-equivariant, hence descends to a holomorphic vector bundle $\mathcal{V}_\xi$ over $X_\Gamma := \Gamma\backslash\mathcal{H}$ for any discrete subgroup $\Gamma \subset G(\mathbb{R})$ which acts freely on $\mathcal{H}$. We can consider $\mathcal{V}_\xi$ on $\big(G(\mathbb{A}^\infty)/U\big) \times \mathcal{H}$, and then take the left quotient by $G(\mathbb{Q})$, for a small enough open compact subgroup $U \subset G(\mathbb{A}^\infty)$, to get $\mathcal{V}_\xi$ on X_U.

Now when ξ is actually a representation of $G(\mathbb{C})$, not just $Q(\mathbb{C})$, this $\mathcal{V}_\xi$ is trivialized as in §2.2, i.e. isomorphic to the constant vector bundle $\mathbb{P}^1(\mathbb{C}) \times V_\xi$ (left $G(\mathbb{C})$-equivariant by the true left action ξ on V_ξ), as follows:

$$\begin{aligned} \mathbb{P}^1(\mathbb{C}) \times V_\xi = (G(\mathbb{C})/Q(\mathbb{C})) \times V_\xi &\xrightarrow{\cong} (G(\mathbb{C}) \times V_\xi)/Q(\mathbb{C}) = \mathcal{V}_\xi, \\ \big(h \bmod Q, v\big) &\longmapsto (h, h^{-1}v) \bmod Q \end{aligned}$$

for all $h \in G$. Note that this trivialization is left $G(\mathbb{C})$-equivariant, because $\big(gh \bmod Q, gv\big) \mapsto (gh, h^{-1}v) \bmod Q$. Restricting this isomorphism to $\mathcal{H}$ by the embedding j we obtain the left $G(\mathbb{R})$-equivariant trivialization of $\mathcal{V}_\xi|_{\mathcal{H}}$:

$$\mathcal{H} \times V_\xi \xrightarrow{\cong} (G(\mathbb{R}) \times V_\xi)/U_\infty = \mathcal{V}_\xi|_{\mathcal{H}}.$$

The holomorphic vector bundle on the RHS is isomorphic to $\mathcal{L}_\xi \otimes \mathcal{O}$ by definition (the $\mathcal{L}_\xi$ is a constant sheaf on $\mathcal{H}$ associated to V_ξ, and $\mathcal{O}$ is the sheaf of holomorphic functions on $\mathcal{H}$). Moreover, as this trivialization is left $G(\mathbb{R})$-equivariant, it descends to X_Γ for every Γ acting freeely on $\mathcal{H}$. Thus we conclude:

$$\mathcal{L}_\xi \otimes \mathcal{O} \xrightarrow{\cong} \mathcal{V}_\xi \quad \text{as holomorphic vector bundles on } X_\Gamma. \tag{5.1.2}$$

We can play the same trivialization on $\mathcal{V}_\xi$ over $\big(G(\mathbb{A}^\infty)/U\big) \times \mathcal{H}$, and take the quotient by $G(\mathbb{Q})$ to get $\mathcal{L}_\xi \otimes \mathcal{O} \cong \mathcal{V}_\xi$ on X_U for small $U \subset G(\mathbb{A}^\infty)$, compatibly when we vary U.

5.2. **The vector bundle $\mathcal{V}_\xi$ containing $\mathcal{V}_k$.** We will look for the right ξ so that $\mathcal{V}_\xi$ contains the line bundles in the Week 1. The basic building block of finite dimensional representations of $GL_2(\mathbb{C})$ are the *standard representation* of $GL_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{C}^2)$ on $\mathbb{C}^2$, which we denote by Sd. For $k \in \mathbb{Z}_{\geq 2}$, we will first consider the $(k-2)$-th symmetric product $\mathrm{Sym}^{k-2}(\mathrm{Sd})$, which is an irreducible $(k-1)$-dimensional representation of $GL_2(\mathbb{C})$. Note that $\mathrm{Sym}^0(\mathrm{Sd}) = \mathbf{1}$ is a trivial representation on $\mathbb{C}$ by definition. For this ξ, the corresponding holomorphic vector bundle $\mathcal{V}_\xi$ on $\mathcal{H}$ (hence on X_Γ or X_U) will contain the line bundle $\mathcal{V}_{k-2}$ that we defined in Week 1, as we shall see.

For this, we look at the U_∞-types of ξ, where we consider $U_\infty \subset GL_2(\mathbb{C})$ by the embedding j, as in (5.1.1), and writing $z = a + bi \in \mathbb{C}$, we have:

$$U_\infty = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^c \end{pmatrix} \right\} \subset T(\mathbb{C}) = \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \right\} \subset Q(\mathbb{C}),$$

where T is a maximal torus in GL_2. We write its algebraic characters as:

$$(r, s) : T(\mathbb{C}) \ni \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \longmapsto z_1^r z_2^s \in \mathbb{C}^\times \qquad (r, s \in \mathbb{Z}),$$

and we denote its restriction to U_∞ by (r, s) too. We also think of (r, s) as characters of $Q(\mathbb{C})$, by the natural surjection $Q \to T$, taking the quotient by the subgroup of unipotent matrices. Then we have:

$$\begin{aligned} \mathrm{Sd}\,|_{T(\mathbb{C})} &= (1, 0) \oplus (0, 1), \\ \xi = \mathrm{Sym}^{k-2}(\mathrm{Sd})|_{T(\mathbb{C})} &= (k-2, 0) \oplus (k-3, 1) \oplus \cdots \oplus (0, k-2). \end{aligned}$$

As representation of $Q(\mathbb{C})$ (not $T(\mathbb{C})$), this is not a decomposition but a *filtration* by subrepresentations of ξ, whose graded pieces are representations of $Q(\mathbb{C})$ factoring through $T(\mathbb{C})$. This gives the *Hodge filtration* on $\mathcal{V}_\xi$, a filtration by left $G(\mathbb{C})$-equivariant subbundles of $\mathcal{V}_\xi$ with the "trivial action on the fiber" as we saw in the previous subsection. We look at the bottom filtration, i.e. the smallest subrepresentation. It is a line bundle $\mathcal{V}_{(k-2,0)}$ on $\mathbb{P}^1(\mathbb{C})$ corresponding to the character $(k-2, 0)$ of $Q(\mathbb{C})$, defined as in the previous subsection; namely $\mathcal{V}_{(k-2,0)} := (G(\mathbb{C}) \times \mathbb{C})/Q(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, where $q \in Q(\mathbb{C})$ acts on $G(\mathbb{C}) \times \mathbb{C}$ by (recall that we made Q act from the right via inverse):

$$(h, v) \mapsto \big(hq, z^{-(k-2)}v\big) \ \text{ if } q = \begin{pmatrix} z & * \\ 0 & * \end{pmatrix}.$$

By restricting $(k-2, 0)$ from $Q(\mathbb{C})$ to U_∞, we see that this $\mathcal{V}_{(k-2,0)}$ on $\mathbb{P}^1(\mathbb{C})$ restricts to the line bundle $\mathcal{V}_{k-2}$ on $\mathcal{H}$ that we saw in Week 1. (Incidentally, we *proved* that the line bundles $\mathcal{V}_k$ defined in Week 1 are indeed holomorphic.) So the left $G(\mathbb{C})$-equivariant embedding $\mathcal{V}_{(k-2,0)} \to \mathcal{V}_\xi$ on $\mathbb{P}^1(\mathbb{C})$, which restricts to a left $G(\mathbb{R})$-equivariant embedding $\mathcal{V}_{k-2} \to \mathcal{V}_\xi$ on $\mathcal{H}$. Thus we have:

(5.2.1) an inclusion $\mathcal{V}_{k-2} \to \mathcal{V}_\xi$ of holomorphic vector bundles on X_Γ,

or on X_U, compatibly when we vary U.

The reason why we considered the $\mathcal{V}_\xi$ containing $\mathcal{V}_{k-2}$ is because we will relate $\mathcal{V}_k$ with the $\mathcal{V}_\xi \otimes \Omega^1 \cong \mathcal{L}_\xi \otimes \Omega^1$, the coefficient in the $(p,q) = (1,0)$ case of Theorem 4.7. For this we use the following left $G(\mathbb{C})$-equivariant isomorphism of holomorphic line bundles on $\mathbb{P}^1(\mathbb{C})$:

$$\mathcal{V}_{(2,0)} \xrightarrow{\cong} \mathcal{V}_{\det} \otimes \Omega^1.$$

Recall that the tangent space of $\mathcal{H}$ is identified with the subalgebra $\mathfrak{p}^+$ of $\mathfrak{g}_\mathbb{C}$ (see §4.3), and we can compute the adjoint action of $Q(\mathbb{C})$ on them:

$$\mathfrak{p}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbb{C}, \quad \begin{pmatrix} z_1 & * \\ 0 & z_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z_1^{-1} & * \\ 0 & z_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & z_1 z_2^{-1} \\ 0 & 0 \end{pmatrix},$$

which shows that $Q(\mathbb{C})$ acts on $\mathfrak{p}^+$ by the character $(1,-1)$ in our notation. The holomorphic line bundle $\Omega^1 := \mathrm{Hom}(\mathfrak{p}^+, \mathcal{O})$ of holomorphic differentials on $\mathbb{P}^1(\mathbb{C})$ can be written as

$$\Omega^1 = (G(\mathbb{C}) \times (\mathfrak{p}^+)^\vee)/Q(\mathbb{C}), \quad (\mathfrak{p}^+)^\vee = \mathrm{Hom}(\mathfrak{p}^+, \mathbb{C}),$$

and $Q(\mathbb{C})$ acts on $(\mathfrak{p}^+)^\vee$ by the character $(-1,1)$, thus Ω^1 is the holomorphic line bundle $\mathcal{V}_{(1,-1)}$ on $\mathbb{P}^1(\mathbb{C})$ (remember the inverse in the definition of $\mathcal{V}_\xi$!). As $\det|_{Q(\mathbb{C})} = (1,1)$, the $\mathcal{V}_{\det}$ is the holomorphic line bundle associated to the character $(1,1)$ (as det is a representation of $G(\mathbb{C})$, this line bundle is equivariantly trivialized as in the previous subsection). Thus, as left $G(\mathbb{C})$-equivariant holomorphic line bundles on $\mathbb{P}^1(\mathbb{C})$, we have:

$$\mathcal{V}_{\det} \otimes \Omega^1 = \mathcal{V}_{(1,1)} \otimes \mathcal{V}_{(1,-1)} = \mathcal{V}_{(2,0)}.$$

Restricting to $\mathcal{H}$, we get

(5.2.2) $$\mathcal{V}_2 \xrightarrow{\cong} \mathcal{V}_{\det} \otimes \Omega^1 \quad (\textit{the Kodaira-Spencer isomorphism}),$$

again left $G(\mathbb{R})$-equivariant, and descends to X_Γ and X_U, compatibly when we vary U. The name comes from the moduli interpretation and deformation theory.

Combining (5.2.1) and (5.2.2), and now redefining $\xi := \mathrm{Sym}^{k-2}(\mathrm{Sd}) \otimes \det$, we obtain a left $G(\mathbb{R})$-equivariant inclusion:

(5.2.3) $$\mathcal{V}_k = \mathcal{V}_{k-2} \otimes \mathcal{V}_2 \longrightarrow \mathcal{V}_\xi \otimes \Omega^1 \cong \mathcal{L}_\xi \otimes \Omega^1,$$

on $\mathcal{H}$, or X_Γ for a discrete subgroup $\Gamma \subset G(\mathbb{R})$ acting freely on $\mathcal{H}$, and we have an injection of $\mathbb{C}$-vector spaces of global sections

(5.2.4) $$H^0(X_\Gamma, \mathcal{V}_k) \longrightarrow H^0(X_\Gamma, \mathcal{L}_\xi \otimes \Omega^1),$$

where $H^0(X_\Gamma, \mathcal{V}_k)$ contains the space $\mathcal{A}_k(\Gamma)$ of *modular forms* of weight k with respect to Γ (and it *is* the space $\mathcal{A}_k(\Gamma)$ if X_Γ is compact). We record this as a proposition in the adelic setting, which takes into account the Hecke action of $G(\mathbb{A}^\infty)$ — observe that everything we did in this section are $G(\mathbb{A}^\infty)$-equivariant when we vary U, as we did not touch the $G(\mathbb{A}^\infty)$-component:

Proposition 5.1. *Let G be such that $G(\mathbb{R}) = GL_2(\mathbb{R})$ and let $\xi = \mathrm{Sym}^{k-2}(\mathrm{Sd}) \otimes \det$. We have an inclusion $\mathcal{V}_k \to \mathcal{L}_\xi \otimes \Omega^1$ of holomorphic vector bundles on X_U for small open compact subgroups $U \subset G(\mathbb{A}^\infty)$, compatibly when we vary U. Thus we have an injection:*

$$H^0(X, \mathcal{V}_k) \longrightarrow H^0(X, \mathcal{L}_\xi \otimes \Omega^1)$$

of smooth $G(\mathbb{A}^\infty)$-representations (clearly admissible when all X_U are compact).

Note that when the quotients X_U are compact, the space $H^0(X, \mathcal{V}_k)$, which we will write $\mathcal{A}_k(G)$, are what we call the space of *automorphic forms* of weight k on $G(\mathbb{A}^\infty)\times\mathcal{H}$, analogue of what we defined for $G = GL_2$ in Week 1. As the spaces are compact, all automorphic forms are cusp forms. We call the irreducible smooth admissible representations in $\mathcal{A}_k(G)$ the *automorphic representations* of $G(\mathbb{A}^\infty)$ of weight k (and all of them are *cuspidal*).

5.3. **The Eichler-Shimura isomorphism.** The Proposition 5.1 relates the space of automorphic forms with the LHS of Theorem 4.7, and it remains to analyze the $(\mathfrak{g}, U)$-cohomology for the representations of $GL_2(\mathbb{R})$ that are cohomological for ξ_k. Actually, these representations π_k exhaust the list of discrete series (i.e. square-integrable modulo center) representations of $GL_2(\mathbb{R})$.

Proposition 5.2. *Let $k \in \mathbb{Z}_{\geq 2}$. For $\xi_k := \mathrm{Sym}^{k-2}(\mathrm{Sd})$, there is a unique discrete series representation π_k of $GL_2(\mathbb{R})$ which is cohomological for ξ_k, and we have:*

$$\dim_{\mathbb{C}} H^*(\mathfrak{g}, U; \pi_k \otimes \xi_k) = \begin{cases} 2 & (* = 1) \\ 0 & (* \neq 1) \end{cases}$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\mathfrak{p}^+, \pi_k \otimes \xi_k) = \dim_C \mathrm{Hom}_U(\mathfrak{p}^-, \pi_k \otimes \xi_k) = 1.$$

Now the Matsushima formula (Theorem 4.7), *if X_Γ is compact*, reads as follows, where we wrote $\Omega := \Omega^1$:

$$H^1(X_\Gamma, \mathcal{L}_{\xi_k}) = H^0(X_\Gamma, \mathcal{L}_{\xi_k} \otimes \Omega) \oplus H^1(X_\Gamma, \mathcal{L}_{\xi_k} \otimes \mathcal{O}),$$

$$H^0(X_\Gamma, \mathcal{L}_{\xi_k} \otimes \Omega) = \bigoplus_{\pi} m(\pi, \Gamma) \,\mathrm{Hom}_U(\mathfrak{p}^+, \pi_k \otimes \xi_k),$$

where $\mathrm{Hom}_U(\mathfrak{p}^+, \pi_k \otimes \xi_k)$ is the 1-dimensional $\mathbb{C}$-vector space. In the adelic setting, it reads:

$$H^1(X, \mathcal{L}_{\xi_k}) = H^0(X, \mathcal{L}_{\xi_k} \otimes \Omega) \oplus H^1(X, \mathcal{L}_{\xi_k} \otimes \mathcal{O}),$$

$$H^0(X, \mathcal{L}_{\xi_k} \otimes \Omega) = \bigoplus_{\pi} m(\pi)\pi^\infty \otimes \mathrm{Hom}_U(\mathfrak{p}^+, \pi_k \otimes \xi_k),$$

Theorem 5.3. *Let $k \in \mathbb{Z}_{\geq 2}$, and $\xi_k := \mathrm{Sym}^{k-2}(\mathrm{Sd})$.*

(i) (Eichler) *Let $\Gamma \subset GL_2(\mathbb{R})$ be as in Theorem 2.4. The injection (5.2.4):*

$$\mathcal{A}_k(\Gamma) = H^0(X_\Gamma, \mathcal{V}_k) \longrightarrow H^0(X_\Gamma, \mathcal{L}_{\xi_k} \otimes \Omega)$$

is an isomorphism, thus by Hodge decomposition we have an isomorphism:

$$H^1(X_\Gamma, \mathcal{L}_{\xi_k}) \xleftarrow{\cong} \mathcal{A}_k(\Gamma) \oplus \mathcal{A}_k(\Gamma)^c.$$

Adelically speaking, let G be as in Thoerem 3.2, and $G(\mathbb{R}) = GL_2(\mathbb{R})$. We have an isomorphism of smooth admissible representations of $G(\mathbb{A}^\infty)$:

$$H^1(X, \mathcal{L}_{\xi_k}) \xleftarrow{\cong} \mathcal{A}_k(G) \oplus \mathcal{A}_k(G)^c.$$

(ii) (Gelfand-Piatetskii-Shapiro) *We have $\dim_{\mathbb{C}} \mathcal{A}_k(\Gamma) = m(\pi_k, \Gamma)$, where $m(\pi_k, \Gamma)$ is the multiplicity of π_k in $L^2(\Gamma\backslash G)[\omega_k]$ where ω_k is the central character of π_k.*

Unfortunately, for the modular curves, i.e. when $\Gamma = SL_2(\mathbb{Z}) \cap U$ for $U \subset GL_2(\mathbb{A}^\infty)$, the complex manifolds X_Γ are not compact. Shimura showed that by replacing the LHS of Theorem 5.3 with the image of compact support cohomology, we still have an isomorphism with the space of *cusp* forms and its conjugate, whence the name *Eichler-Shimura isomorphism*. For general Shimura varieties which are not compact, it is not obvious how to replace the cohomology theory — natural candidates are the analytic L^2-*cohomology*, or algebro-geometric *intersection cohomology*; which turns out to be equivalent by the solution of *Zucker conjecture* — but we will not go into that issue in this course. Instead, we will analyze the *compact* Shimura varities where the Matsushima formula works nicely; the absence of boundary simplifies the algebraic geometry as well.

6. Notes on the Literature

The spectral decomposition of $L^2(\Gamma\backslash G)$ is discussed in [GGPS], Chapter 1, §2. See also [Bu], 2.3.

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Harvard University, Department of Mathematics, 1 Oxford Street, Cambridge, MA 02138, USA

E-mail address: yoshida@math.harvard.edu