WEEK 1: AUTOMORPHIC FORMS AND REPRESENTATIONS

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Abstract. Brief introduction to cyclotomic theory over $\mathbb{Q}$ using adeles. Discussion of the definitions of modular forms and automorphic forms. Introducing the adelic automorphic forms via strong approximation theorem. Discussion of the connected components of Shimura varieties (modular curves). Smooth/admissible representations of locally finite groups. Definition and admissibility of (cuspidal) automorphic representations. Tensor product decomposition and the newforms for cuspidal automorphic representations. Central character and the character of classical newform.

Lecture 1 (Sep. 15, 2008)

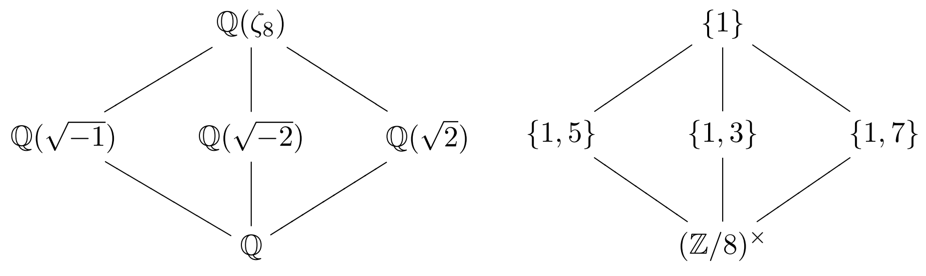
1. Cyclotomic Theory and Adeles

1.1. **Cyclotomic theory.** Let us start by admiring a beautiful theorem.

Theorem 1.1. *For an odd prime p:*

(i) $\exists x, y \in \mathbb{Z},\ p = x^2 + y^2 \iff p \equiv 1 \pmod 4$.
(ii) $\exists x, y \in \mathbb{Z},\ p = x^2 + 2y^2 \iff p \equiv 1, 3 \pmod 8$.
(iii) $\exists x, y \in \mathbb{Z},\ p = x^2 - 2y^2 \iff p \equiv 1, 7 \pmod 8$.

Compare this theorem in elementary number theory with the Galois theory of corresponding quadratic fields, contained in $\mathbb{Q}(\zeta_8)$:



This beautiful correspondence is realized through the *irreducibility of cyclotomic polynomials*, namely the following isomorphism for $N \geq 1$:

$$\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \ni (\zeta_N \mapsto \zeta_N^a) \longmapsto a \bmod N \in (\mathbb{Z}/N)^\times,$$

where the (arithmetic) *Frobenius element* Fr_p is mapped to $p \bmod N$ for all p not dividing N (the *decomposition law*). As we have $M \mid N \Longrightarrow \mathbb{Q}(\zeta_M) \subset \mathbb{Q}(\zeta_N)$, we can vary N and pass to the limit to describe the Galois group of $\mathbb{Q}(\boldsymbol{\mu}_\infty) := \bigcup_N \mathbb{Q}(\zeta_N)$. Then:

$$\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_\infty)/\mathbb{Q}) \ni (\zeta_N \mapsto \zeta_N^{a_N}) \longmapsto (a_N)_N \in \widehat{\mathbb{Z}}^\times,$$

where the *profinite completion* $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$ is a ring defined by:

$$\widehat{\mathbb{Z}} := \varprojlim_{N \geq 1} \mathbb{Z}/N = \Big\{ (a_N)_N \in \prod_{N \geq 1} \mathbb{Z}/N \;\Big|\; \text{if } M \mid N \text{ then } a_N \bmod M = a_M \Big\}.$$

Note the canonical direct product decomposition as rings (the *Chinese remainder theorem*):

$$\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p, \quad \mathbb{Z}_p := \varprojlim_m \mathbb{Z}/p^m \quad (\textit{the ring of } p\textit{-adic integers}).$$

1.2. **Adeles.** Now consider the *ring of finite adeles* $\mathbb{A}^\infty := \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Naturally $\mathbb{Q}, \widehat{\mathbb{Z}} \subset \mathbb{A}^\infty$, and also $\mathbb{Q}^\times, \widehat{\mathbb{Z}}^\times \subset (\mathbb{A}^\infty)^\times$. We have $\mathbb{Q}^\times \widehat{\mathbb{Z}}^\times = (\mathbb{A}^\infty)^\times$ (the *uniqueness of prime decomposition*) and $\mathbb{Q}^\times \cap \widehat{\mathbb{Z}}^\times = \{\pm 1\}$. To separate $\pm 1 \in \mathbb{Q}^\times$, we use $\mathbb{R}^\times/\mathbb{R}^\times_{>0} \cong \{\pm 1\}$ to make an isomorphism:

$$\mathbb{Q}^\times \times \widehat{\mathbb{Z}}^\times \ni (x, u) \overset{\cong}{\longmapsto} \big(xu,\ x \bmod \mathbb{R}^\times_{>0}\big) \in (\mathbb{A}^\infty)^\times \times \big(\mathbb{R}^\times/\mathbb{R}^\times_{>0}\big).$$

Now define the *ring of adeles* $\mathbb{A} := \mathbb{A}^\infty \times \mathbb{R}$ as the direct product of $\mathbb{Q}$-algebras. Then:

$$\mathbb{Q}^\times \times \widehat{\mathbb{Z}}^\times \overset{\cong}{\longrightarrow} \mathbb{A}^\times/\mathbb{R}^\times_{>0}, \text{or}$$
$$\widehat{\mathbb{Z}}^\times \overset{\cong}{\longrightarrow} \mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}^\times_{>0}.$$

In other words, we have a direct product decomposition (as groups): $\mathbb{A}^\times = \mathbb{Q}^\times \widehat{\mathbb{Z}}^\times \mathbb{R}^\times_{>0}$. Here the distinction of left/right quotients is insignificant as the groups are abelian, but $\mathbb{Q}^\times$ is a discrete subgroup of $\mathbb{A}^\times$, as opposed to $\mathbb{R}^\times_{>0}$ which is a connected component of $\mathbb{A}^\times$.

Now define the *p-adic field* by $\mathbb{Q}_p := \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, which turns out to be the fraction field of $\mathbb{Z}_p$, and tensoring $\mathbb{Q}$ on $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ gives the injection $\mathbb{A}^\infty \to \prod_p \mathbb{Q}_p$. So we consider $\mathbb{A}^\infty$ as a subring of $\prod_p \mathbb{Q}_p$, and writing $(\mathbb{A}^\infty)^\times$ in the similar way (the *restricted product*):

$$\mathbb{A}^\infty = \Big\{ (x_p)_p \in \prod_p \mathbb{Q}_p \;\Big|\; x_p \in \mathbb{Z}_p \text{ for almost all } p \Big\}, \text{ and}$$
$$(\mathbb{A}^\infty)^\times = \Big\{ (x_p)_p \in \prod_p \mathbb{Q}_p^\times \;\Big|\; x_p \in \mathbb{Z}_p^\times \text{ for almost all } p \Big\},$$

where *almost all* p means "except for finitely many p". Thus we have a commutative diagram with the inclusion of $\mathbb{Q}_p^\times$ into $(\mathbb{A}^\infty)^\times$ as a direct factor:

$$\begin{array}{ccc} \mathrm{Art}_{\mathbb{Q}} : \mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}^\times_{>0} & \overset{\cong}{\longrightarrow} & \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_\infty)/\mathbb{Q}) = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{\mathrm{ab}} \\ \uparrow & & \downarrow \\ \mathbb{Q}_p^\times \ni p^{-1} & \longrightarrow & \mathrm{Fr}_p \in \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_\infty^p)/\mathbb{Q}) \end{array}$$

where $\mathbb{Q}(\boldsymbol{\mu}_\infty^p) := \bigcup_{(p,N)=1} \mathbb{Q}(\zeta_N)$, and the equality in the upper right is the consequence of the *Kronecker-Weber theorem* $\mathbb{Q}(\boldsymbol{\mu}_\infty) = \mathbb{Q}^{\mathrm{ab}}$. (This commutativity describes the decomposition law as the compatibility of the global Artin map with the unramified local class field theory at p.) We get a bijection between the finite order characters of the Galois group and the Dirichlet characters – the first instance of Langlands correspondence for $GL_1/\mathbb{Q}$:

$$\{\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times\} \overset{1:1}{\longleftrightarrow} \{\omega : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times \mid \omega|_{\mathbb{R}^\times_{>0}} = 1\}$$
$$\rho \circ \mathrm{Art}_{\mathbb{Q}} = \omega.$$

2. Modular Forms

Modular forms are "highly symmetric functions on upper half plane" (see Barry Mazur's comment in the BBC program on Fermat's Last Theorem, on YouTube). The reason why there are such functions is that the upper half plane $\mathcal{H}^+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ itself has a big symmetry, namely a left action of a Lie group

$$GL_2^+(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R},\ ad - bc > 0 \right\},$$

where the action is defined by

$$GL_2^+(\mathbb{R}) \times \mathcal{H}^+ \ni (g,z) \longmapsto gz \in \mathcal{H}^+, \quad gz := \frac{az+b}{cz+d} \ \text{ for } \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Exercise 2.1. Check that this is an action: $gz \in \mathcal{H}^+$, $\ (gg')z = g(g'z)$.

Let $\mathcal{O}$ be the space of all holomorphic functions on $\mathcal{H}^+$. It has the right action of $GL_2^+(\mathbb{R})$ coming from the left action on $\mathcal{H}^+$:

$$\mathcal{O} \times GL_2^+(\mathbb{R}) \ni (f,g) \longmapsto f \cdot g \in \mathcal{O}, \quad (f \cdot g)(z) := f(gz) \ \ (\forall z \in \mathcal{H}^+).$$

Now we will define some *twisted actions*, by introducing the *automorphic factor*:

$$\mu : GL_2^+(\mathbb{R}) \times \mathcal{H}^+ \longrightarrow \mathbb{C}, \quad \mu(g,z) := cz + d \ \text{ for } \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is not quite an "action", but something close to it.

Exercise 2.2. Check that this μ satisfies a "*cocycle condition*" $\mu(gg', z) = \mu(g', z)\mu(g, g'z)$.

Now for each $k \in \mathbb{Z}_{\geq 0}$, define a new action by:

$$\mathcal{O} \times GL_2^+(\mathbb{R}) \ni (f,g) \longmapsto f \cdot [g]_k \in \mathcal{O}, \quad (f \cdot [g]_k)(z) := \mu(g,z)^{-k} f(gz) \ \ (\forall z \in \mathcal{H}^+).$$

This is indeed a right action because, using the cocycle condition:

$$\begin{aligned}(f \cdot [g]) \cdot [g'](z) &= \mu(g', z)^{-k}(f \cdot [g])(g'z) \\ &= \mu(g', z)^{-k}\mu(g, g'z)^{-k} f(gg'z) = (f \cdot [gg'])(z).\end{aligned}$$

Modular forms are the holomorphic functions $f \in \mathcal{O}$ that are invariant under this twisted action of some discrete subgroup of $GL_2^+(\mathbb{R})$. Let Γ be a finite index subgroup of:

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\} \subset GL_2^+(\mathbb{R}).$$

We will need some growth conditions near the cusps. Observe that $SL_2(\mathbb{Z})$ has a following subgroup, which is isomorphic to the additive group $\mathbb{Z}$:

$$N := \left\{ n_b := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{Z} \right\} \subset SL_2(\mathbb{Z})$$

As $\mu(n,z) = 1$ for $n \in N$, we have $f \cdot [n]_w = f \cdot n$. If $\Gamma \cap N = \langle n_b \rangle$ and $f = f \cdot [\gamma]_k$ for all $\gamma \in \Gamma$, then $f = f \cdot [n_b]_k = f \cdot n_b$, i.e.

$$f(z) = (f \cdot n_b)(z) = f(z + b),$$

in which case there is a holomorphic function F on $\mathbb{C}\backslash\{0\}$ satisfying

$$f(z) = F(\exp(2\pi i z/b)).$$

We say that f is *holomorphic* (resp. *vanishes*) *at* ∞ if F extends to a holomorphic function around 0 (resp. holomorphic function which vanishes at 0).

Definition 2.3. A holomorphic function $f \in \mathcal{O}$ is a *modular form* (resp. *cusp form*) of *weight* k with respect to Γ, if:

(i) $f \cdot [\gamma]_k = f \quad (\forall \gamma \in \Gamma)$,
(ii) $f \cdot [\gamma]_k = f$ is holomorphic (resp. vanishes) at ∞ for all $\gamma \in SL_2(\mathbb{Z})$.

Example 2.4. For each $k \in \mathbb{Z}_{\geq 0}$, the *Eisenstein series of weight* k:

$$E_k(z) = \sum_{(m,n) \in \mathbb{Z}\times\mathbb{Z}\backslash\{(0,0)\}} \frac{1}{(mz+n)^k}$$

is a modular form of weight k with respect to $\Gamma = SL_2(\mathbb{Z})$. It is non-zero only when k is even, and convergent only for $k > 2$. The *Ramanujan* Δ *function* $\Delta := (60E_4)^3 - 27(140E_6)^2$ is a cusp form of weight 12 with respect to $\Gamma = SL_2(\mathbb{Z})$.

3. Automorphic Forms

The term *automorphic forms* is used almost synonymously with *modular forms*, but it is a slightly different way of looking at the functions – automorphic forms are more like functions on the group itself. Note that the action of $GL_2^+(\mathbb{R})$ on $\mathcal{H}^+$ is *transitive*, because

$$\mathcal{H}^+ \ni z = x + iy = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot i \quad (\forall z \in \mathcal{H}^+),$$

and the *stabilizer* of $i \in \mathcal{H}^+$ is given by:

$$U_\infty := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}) \,\middle|\, \frac{ai+b}{ci+d} = i \right\} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \,\middle|\, a,b \in \mathbb{R} \right\},$$

which is isomorphic to the group $\mathbb{C}^\times$ by:

$$U_\infty \ni u = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \longmapsto \mu(u,i) = a + bi \in \mathbb{C}^\times.$$

Then, we have:

$$\mathcal{H}^+ \xrightarrow{\cong} GL_2^+(\mathbb{R})/U_\infty \quad \text{by} \quad gi \longmapsto g \cdot U_\infty$$

as left $GL_2^+(\mathbb{R})$-sets. Now pull back $f \in \mathcal{O}$ to $\tilde{f} : GL_2^+(\mathbb{R}) \longrightarrow \mathcal{H}^+ \xrightarrow{f} \mathbb{C}$, i.e.:

$$\tilde{f} : GL_2^+(\mathbb{R}) \to \mathbb{C}, \quad \tilde{f}(g) := f(gi) \quad (\forall g \in GL_2^+(\mathbb{R})).$$

Then, using the cocycle condition, we compute:

$$\begin{aligned} f \cdot [\gamma]_k = f &\iff \mu(\gamma,z)^{-k} f(\gamma z) = f(z) \quad (\forall z \in \mathcal{H}^+) \\ &\iff \mu(\gamma,gi)^{-k} f(\gamma gi) = f(gi) \quad (\forall g \in GL_2^+(\mathbb{R})) \\ &\iff \mu(\gamma g,i)^{-k} \tilde{f}(\gamma g) = \mu(g,i)^{-k} \tilde{f}(g) \quad (\forall g \in GL_2^+(\mathbb{R})). \end{aligned}$$

Therefore, if we set

$$\phi : GL_2^+(\mathbb{R}) \to \mathbb{C}, \quad \phi(g) := \mu(g,i)^{-k} \tilde{f}(g) = \mu(g,i)^{-k} f(gi),$$

then we can write simply as follows:

$$f \cdot [\gamma]_k = f \iff \phi(\gamma g) = \phi(g).$$

Also, as f was a function on $\mathcal{H}^+$, we have:

$$\begin{aligned} \tilde{f}(gu) = \tilde{f}(g) \quad (\forall u \in U_\infty) &\iff \phi(gu) = \mu(gu,i)^{-k} \tilde{f}(g) \quad (\forall u \in U_\infty) \\ &\iff \phi(gu) = \mu(u,i)^{-k} \mu(g,ui)^{-k} \tilde{f}(g) \quad (\forall u \in U_\infty) \\ &\iff \phi(gu) = \mu(u,i)^{-k} \phi(g) \quad (\forall u \in U_\infty) \quad (\text{use } ui = i). \end{aligned}$$

Therefore, if we define the character χ_k of U_∞ by:

$$\chi_k : U_\infty \to \mathbb{C}^\times, \quad \chi_k(u) := \mu(u,i)^{-k} \in \mathbb{C}^\times,$$

then we have $\phi(gu) = \chi_k(u)\phi(g)$ for all $u \in U_\infty$.

Summarizing, we have the correspondence:

$$\{ \ f : \mathcal{H}^+ \to \mathbb{C} \mid f \cdot [\gamma]_k = f \ \} \stackrel{1:1}{\longleftrightarrow} \{ \ \phi : \Gamma \backslash GL_2^+(\mathbb{R}) \to \mathbb{C} \mid \phi(gu) = \chi_k(u)\phi(g) \ \}.$$

In other words, by giving up the right U_∞-invariance of f, the transformation property of f with respect to the left Γ-action is translated into the transformation property with respect to the right U_∞-action of the Γ-invariant functions. This is the point of view which is more amenable to adelic formulations.

Before getting to the adelic version of automorphic forms, we interpret above functions ϕ back to objects over $\mathcal{H}^+$, namely the sections of certain holomorphic line bundles over $\mathcal{H}^+$. Consider the graph of ϕ:

$$(\mathrm{id}, \phi) : GL_2^+(\mathbb{R}) \ni g \longmapsto (g, \phi(g)) \in GL_2^+(\mathbb{R}) \times \mathbb{C},$$

and give a right U_∞-action on $GL_2^+(\mathbb{R}) \times \mathbb{C}$ by:

$$(g,x) \cdot u := (gu, \chi_k(u)x) \quad (\forall (g,x) \in GL_2^+(\mathbb{R}) \times \mathbb{C}, \quad \forall u \in U_\infty).$$

Then the map (id, ϕ) is visibly equivariant under the right U_∞-action of domain and target, hence descends to the quotient spaces. Define:

$$\mathcal{V}_k := (GL_2^+(\mathbb{R}) \times \mathbb{C})/U_\infty,$$

which naturally becomes a holomorphic line bundle over $\mathcal{H}^+ = GL_2^+(\mathbb{R})/U_\infty$ by the projection map

$$\pi : \mathcal{V}_k \ni (g, x) \bmod U_\infty \longmapsto gi \in \mathcal{H}^+.$$

Therefore, denoting again by ϕ the map between the quotient spaces induced by (id, ϕ):

$$\phi : \mathcal{H}^+ \longrightarrow \mathcal{V}_k, \quad \pi \circ \phi = \mathrm{id},$$

i.e. we can think of ϕ as a section of the line bundle $\mathcal{V}_k$.

Exercise 3.1. (i) Check that this $\mathcal{V}_k$ is indeed a holomorphic line bundle over $\mathcal{H}^+$.
(ii) Check that $\mathcal{V}_k \cong \mathcal{V}_1^{\otimes k}$. (The $\mathcal{V}_1$ is often denoted by ω in the literature.)

In the next section, in order to receive a left $GL_2(\mathbb{Q})$-action (which has elements with negative determinant), we will consider

$$\mathcal{H} := \mathbb{C} \backslash \mathbb{R} = \mathcal{H}^+ \amalg \mathcal{H}^-, \quad \mathcal{H}^- := \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\},$$

which has the left action of $GL_2(\mathbb{R})$ by linear fractional transformations, which extends the action of $GL_2^+(\mathbb{R})$ on $\mathcal{H}$. We have $\mathcal{H} \cong GL_2(\mathbb{R})/U_\infty$ by the same argument as before. Note that $GL_2^+(\mathbb{R})$ acts on each of $\mathcal{H}^+$ and $\mathcal{H}^-$. The same definition of the line bundles:

$$\mathcal{V}_k := (GL_2(\mathbb{R}) \times \mathbb{C})/U_\infty \longrightarrow \mathcal{H},$$

where U_∞ acts on $\mathbb{C}$ by χ_k, works over $\mathcal{H}$ and restricts to the previously constructed $\mathcal{V}_k$ on $\mathcal{H}^+$. The sections of $\mathcal{V}_k$ over $\mathcal{H}$ can be thought of as functions on $GL_2(\mathbb{R})$ with the same transformation property with respect to the right U_∞-action:

$$\phi : GL_2(\mathbb{R}) \to \mathbb{C}, \quad \phi(gu) = \chi_k(u)\phi(g) \quad (\forall u \in U_\infty).$$

4. Adelic Automorphic Forms

In order to give adelic formulations to the automorphic forms, we start with the adelic description of $\Gamma\backslash\mathcal{H}^+$. For this, we will enlarge the space from $\mathcal{H}^+$ to $GL_2(\mathbb{A}^\infty) \times \mathcal{H}$, in order to interpret the left Γ-invariance of functions on $\mathcal{H}^+$ as a $GL_2(\mathbb{Q})$-invariance of functions on $GL_2(\mathbb{A}^\infty) \times \mathcal{H}$.

Proposition 4.1. (use strong approximation theorem) *Let $U \subset GL_2(\widehat{\mathbb{Z}})$ be a finite index subgroup such that $\det U = \widehat{\mathbb{Z}}^\times$, and set $\Gamma := SL_2(\mathbb{Z}) \cap U$. Then we have:*

$$\Gamma\backslash GL_2^+(\mathbb{R}) \xrightarrow{\cong} GL_2(\mathbb{Q})\backslash\Big(\big(GL_2(\mathbb{A}^\infty)/U\big) \times GL_2(\mathbb{R})\Big),$$

$$\Gamma\backslash\mathcal{H}^+ \xrightarrow{\cong} GL_2(\mathbb{Q})\backslash\Big(\big(GL_2(\mathbb{A}^\infty)/U\big) \times \mathcal{H}\Big) =: X_U.$$

The second isomorphism (of open Riemann surfaces) is obtained from the first by taking the quotient under right U_∞-action. We can write in a slightly simpler way as follows:

$$\Gamma\backslash GL_2^+(\mathbb{R}) \xrightarrow{\cong} GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/U, \quad \Gamma\backslash\mathcal{H}^+ \xrightarrow{\cong} GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/(U\cdot U_\infty) =: X_U.$$

We will treat the necessary definitions and proofs in the next section, and will prove a generalized statement in Proposition 6.1 for arbitrary finite index subgroup $U \subset GL_2(\widehat{\mathbb{Z}})$, where the LHS is in general a disjoint union of finitely many $\Gamma_i\backslash\mathcal{H}^+$ for finite index subgroups $\Gamma_i \subset SL_2(\mathbb{Z})$, indexed by the finite set $\widehat{\mathbb{Z}}^\times/\det U$.

We denote the RHS of the second isomorphism, for arbitrary U, by X_U (the *Shimura variety*, or *modular curve* for U). Now $\Gamma\backslash\mathcal{H}^+$ is an open Riemann surface with a compactification $\overline{\Gamma\backslash\mathcal{H}^+}$ whose holomorphic structure at the *cusps* (missing points) are essentially described when we discussed the behaviour of modular forms at ∞. Therefore X_U is compactified into $\overline{X}_U$, which is in general a disjoint union of such spaces. The holomorphic line bundles $\mathcal{V}_k$ on $\mathcal{H}^+$ are equivariant under the left action of Γ, hence descends to X_U, and extends naturally to $\overline{\mathcal{V}}_k$ over $\overline{X}_U$. Looking at this on the RHS, we arrive at the following definition of adelic automorphic forms.

Exercise 4.2. Check that the line bundle $\mathcal{V}_k$ over $\mathcal{H}^+$ descends to X_U and extends to $\overline{X}_U$.

Definition 4.3. For a finite index subgroup $U \subset GL_2(\widehat{\mathbb{Z}})$ and $k \in \mathbb{Z}_{\geq 0}$, an *automorphic form* of *level* U and *weight* k on $\mathcal{H}$ is a section ϕ of the holomorphic line bundle $\mathcal{V}_k$ over $GL_2(\mathbb{A}^\infty) \times \mathcal{H}$ (defined as the pull-back from $\mathcal{H}$), such that:

(i) $\phi(\gamma g, \gamma z) = \phi(g, z)$ $(\forall\gamma \in GL_2(\mathbb{Q}))$,
(ii) $\phi(gu, z) = \phi(g, z)$ $(\forall u \in U)$.
(iii) ϕ extends to a section of $\overline{\mathcal{V}}_k$ over $\overline{X}_U$. It is called a *cusp form* if it extends by 0.

By definition of X_U, the condition (i) and (ii) mean that ϕ is a section of $\mathcal{V}_k$ over X_U. We can pull back ϕ further and consider them as functions on $GL_2(\mathbb{A}) = GL_2(\mathbb{A}^\infty) \times GL_2(\mathbb{R})$ (or $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$, which is what you find in many of the literature).

By the remark above, for general U an automorphic form ϕ corresponds to a set of finitely many modular forms f_i, each with respect to possibly different Γ_i. This happened to be invisible in the classical theory of modular forms, mainly because all the newforms were modular with respect to $\Gamma_1(N)$, and for this we could take $U = U_1(N)$, so that $\det U = \widehat{\mathbb{Z}}^\times$ (see Example 6.2 and Theorem 9.2). But the disconnectedness of Shimura varieties will be inherent in general, e.g. the Hilbert modular case for totally real fields with non-trivial ideal class group.

We denote the space of all automorphic forms (resp. cusp forms) of level U and weight k by $\mathcal{A}_k(U)$ (resp. $\mathcal{A}_k^0(U)$). If $U' \subset U$, then naturally $\mathcal{A}_k(U) \subset \mathcal{A}_k(U')$ and $\mathcal{A}_k^0(U) \subset \mathcal{A}_k^0(U')$, hence we can take the direct limit as we make U smaller:

$$\mathcal{A}_k := \varinjlim_U \mathcal{A}_k(U), \quad \mathcal{A}_k^0 := \varinjlim_U \mathcal{A}_k^0(U).$$

Once taking the limit, these spaces receive the left action of the huge group $GL_2(\mathbb{A}^\infty)$ (more on this later), called the *Hecke action* or *Hecke operators*. The *automorphic representations* (resp. *cuspidal automorphic representations*) of $GL_2(\mathbb{A}^\infty)$ are defined to be the irreducible representations of $GL_2(\mathbb{A}^\infty)$ appearing in $\mathcal{A}_k$ (resp. $\mathcal{A}_k^0$).

Note that this "passing to the limit with respect to the levels" is parallel to what we did with the cyclotomic fields when we passed from $\mathbb{Q}(\zeta_N)$ to $\mathbb{Q}(\boldsymbol{\mu}_\infty)$. In fact, we can think of $\mathbb{Q}(\zeta_N)\otimes\mathbb{C}$ as a (0-dimensional) Shimura variety for GL_1, because for $U := \mathrm{Ker}(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times)$, we can write:

$$(\mathbb{Z}/N)^\times \xrightarrow{\cong} \mathbb{Q}^\times\backslash\mathbb{A}^\times/(U\cdot\mathbb{R}^\times_{>0}) = \mathbb{Q}^\times\backslash\big((\mathbb{A}^\infty)^\times/U\big)\times(\mathbb{R}^\times/\mathbb{R}^\times_{>0})\big).$$

Lecture 2 (Sep. 17, 2008)

5. Strong Approximation on GL_n

We will prove the strong approximation, needed for Proposition 4.1. Recall the notation:

$$GL_2(R) := \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\,\middle|\, a,b,c,d\in R,\ ad-bc\in R^\times\right\},$$

for arbitrary ring R, and similar one for GL_n (we have in mind the fact that GL_n is a *group scheme* over $\mathbb{Z}$, hence we can think of the group of R-rational points for any R). Recall $\mathbb{A}^\infty := \mathbb{Q}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$ and $\mathbb{A} := \mathbb{A}^\infty\times\mathbb{R}$. We had the embedding $\mathbb{A}^\infty\to\prod_p\mathbb{Q}_p$ (compatible with the canonical decomposition $\widehat{\mathbb{Z}}\cong\prod_p\mathbb{Z}_p$), under which the natural ring homomorphism $\mathbb{Q}\to\mathbb{A}^\infty\to\prod_p\mathbb{Q}_p$ is a diagonal embedding.

Now for GL_1, i.e. the multiplicative ring, we had the embeddings $\mathbb{Q}^\times,\widehat{\mathbb{Z}}^\times\subset(\mathbb{A}^\infty)^\times$, and for $\mathbb{A}^\times\cong(\mathbb{A}^\infty)^\times\times\mathbb{R}^\times$, we had a direct product decomposition $\mathbb{A}^\times\cong\mathbb{Q}^\times\widehat{\mathbb{Z}}^\times\mathbb{R}^\times_{>0}$. It is good to remind ourselves of how each factor is embedded in terms of $\mathbb{A}^\times = (\mathbb{A}^\infty)^\times\times\mathbb{R}^\times$:

$$\mathbb{Q}^\times\ni\gamma\longmapsto(\gamma,\gamma)\in\mathbb{A}^\times,$$
$$\widehat{\mathbb{Z}}^\times\ni u\longmapsto(u,1)\in\mathbb{A}^\times,$$
$$\mathbb{R}^\times_{>0}\ni g\longmapsto(1,g)\in\mathbb{A}^\times.$$

We write $G = G_1\cdots G_t$ for a group G and its subgroups $G_1,\dots,G_t$ if all $g\in G$ can be written as $g = g_1\cdots g_t$ for some $g_i\in G_i$, not necessarily uniquely. If it is a *direct product*, i.e. the elements of G_i and G_j commute with each other when $i\neq j$ and $G_1\cdots G_{i-1}G_{i+1}\cdots G_t\cap G_i = \{1\}$ for all i, then the expression $g = g_1\cdots g_t$ is unique.

Exercise 5.1. Check the direct product decomposition $\mathbb{A}^\times\cong\mathbb{Q}^\times\widehat{\mathbb{Z}}^\times\mathbb{R}^\times_{>0}$.

Now we want to think of the analogue of this decomposition for GL_n. We have:

$$GL_n(\mathbb{A}) = GL_n(\mathbb{A}^\infty)\times GL_n(\mathbb{R}),$$
$$GL_n(\mathbb{Q}),\ GL_n(\widehat{\mathbb{Z}}) = \prod_p GL_n(\mathbb{Z}_p)\subset GL_n(\mathbb{A}^\infty)\subset\prod_p GL_n(\mathbb{Q}_p).$$

Exercise 5.2. (i) Check that, by the last embedding above, we can think of $GL_n(\mathbb{A}^\infty)$ as the restricted product:

$$GL_n(\mathbb{A}^\infty) = \Big\{(g_p)_p \in \prod_p GL_n(\mathbb{Q}_p) \;\Big|\; g_p \in GL_n(\mathbb{Z}_p) \text{ for almost all } p\Big\}.$$

(ii) Check that the following canonical morphisms are isomorphisms:

$$GL_n(\widehat{\mathbb{Z}}) \cong \varprojlim_N GL_n(\mathbb{Z}/N), \quad GL_n(\mathbb{Z}_p) \cong \varprojlim_m GL_n(\mathbb{Z}/p^m).$$

Hence $GL_n(\widehat{\mathbb{Z}})$ and $GL_n(\mathbb{Z}_p)$ are profinite groups. We will use the term *locally profinite groups* for the groups like $GL_n(\mathbb{A}^\infty)$ and $GL_n(\mathbb{Q}_p)$, whose topology is determined by declaring a profinite subgroup to be an open subgroup, where this profinite group has the profinite topology. This means that as a topological space it is a disjoint union of open cosets, each of which having a profinite (hence compact) topology (think of $\mathbb{Q}_p^\times$ as a disjoint union of cosets by $\mathbb{Z}_p^\times$). Thus we do not need much topological intuition for these groups.

Remark 5.3. A great thing about $GL_n(\mathbb{A}^\infty)$ (resp. $GL_n(\mathbb{Q}_p)$) is that the $GL_n(\widehat{\mathbb{Z}})$ (resp. $GL_n(\mathbb{Z}_p)$) is the unique maximal compact subgroup up to conjugation, hence every *open compact subgroup* can be conjugated into a finite index subgroup of $GL_n(\widehat{\mathbb{Z}})$ (resp. $GL_n(\mathbb{Z}_p)$). (This is not necessarily true for general reductive adelic/p-adic groups.)

Proposition 5.4. *$GL_n(\mathbb{A}^\infty) = GL_n(\mathbb{Q})GL_n(\widehat{\mathbb{Z}})$.*

Proof. We begin with a lemma. Here, for a ring R, an *R-lattice* in $(R\otimes_{\mathbb{Z}}\mathbb{Q})^n$ means a free R-submodule $L \subset (R\otimes_{\mathbb{Z}}\mathbb{Q})^n$ of rank n such that $L\otimes_{\mathbb{Z}}\mathbb{Q} = (R\otimes_{\mathbb{Z}}\mathbb{Q})^n$. Recall that in $\mathbb{A}^\infty$ we have $\widehat{\mathbb{Z}}\cap\mathbb{Q} = \mathbb{Z}$ and $\widehat{\mathbb{Z}}+\mathbb{Q} = \mathbb{A}^\infty$. In fact for any $N \in \mathbb{Z}_{>0}$ we have $N\widehat{\mathbb{Z}}+\mathbb{Q} = \mathbb{A}^\infty$, which is a (weak) approximation that follows from the fact that $\mathbb{Z}$ is a UFD.

Lemma 5.5. *We have a bijection* $\{\mathbb{Z}\text{-lattice } \Lambda \subset \mathbb{Q}^n\} \xleftrightarrow{1:1} \{\widehat{\mathbb{Z}}\text{-lattice } \widehat{\Lambda} \subset (\mathbb{A}^\infty)^n\}$*, defined by* $\Lambda \mapsto \widehat{\Lambda} := \Lambda\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$ *and* $\widehat{\Lambda} \mapsto \Lambda := \widehat{\Lambda}\cap\mathbb{Q}^n$.

Proof. As $\widehat{\mathbb{Z}}$ is $\mathbb{Z}$-torsion free, it is $\mathbb{Z}$-flat, hence $\Lambda \mapsto \widehat{\Lambda}$ is injective. Now for any $\widehat{\Lambda}$, writing the canonical basis of $\widehat{\mathbb{Z}}^n$ as $\mathbb{Q}$-linear combinations of a basis of $\widehat{\Lambda}$ and taking $N \in \mathbb{Z}_{>0}$ which is divisible by the denominators of all coefficients, we see that $(N\widehat{\mathbb{Z}})^n \subset \widehat{\Lambda}$. Using $N\widehat{\mathbb{Z}}+\mathbb{Q} = \mathbb{A}^\infty$, we have $\widehat{\Lambda} = (N\widehat{\mathbb{Z}})^n + (\widehat{\Lambda}\cap\mathbb{Q}^n)$. As $\widehat{\Lambda}\cap\mathbb{Q}^n$ contains $(N\widehat{\mathbb{Z}})^n\cap\mathbb{Q}^n = (N\mathbb{Z})^n$, we get $\widehat{\Lambda} = (\widehat{\Lambda}\cap\mathbb{Q}^n)\otimes\widehat{\mathbb{Z}}$, hence $\Lambda\mapsto\widehat{\Lambda}$ is surjective and inverse to $\widehat{\Lambda}\mapsto\Lambda$. □

As $GL_n(\mathbb{Q})$ acts transitively on the set $\{\mathbb{Z}\text{-lattice } \Lambda \subset \mathbb{Q}^n\}$, and the stabilizer of $\mathbb{Z}^n$ is $GL_n(\mathbb{Z})$, this set is isomorphic to $GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$. Similarly, the set on RHS is isomorphic to $GL_n(\mathbb{A}^\infty)/GL_n(\widehat{\mathbb{Z}})$. The bijection of the lemma is $GL_n(\mathbb{Q})$-equivariant and gives

$$GL_n(\mathbb{Q})/GL_n(\mathbb{Z}) \xrightarrow{\cong} GL_n(\mathbb{A}^\infty)/GL_n(\widehat{\mathbb{Z}}),$$

induced by the canonical inclusion $GL_n(\mathbb{Q}) \subset GL_n(\mathbb{A}^\infty)$. Thus the proposition follows. □

Proposition 5.6. *$GL_n(\mathbb{A}) = GL_n(\mathbb{Q})GL_n(\widehat{\mathbb{Z}})GL_n^+(\mathbb{R})$.*

Proof. At this point it is helpful to recall how these subgroups are embedded into $GL_n(\mathbb{A})$ in terms of $GL_n(\mathbb{A}) = GL_n(\mathbb{A}^\infty) \times GL_n(\mathbb{R})$, which is exactly similar to the GL_1 case:

$$\begin{aligned} GL_n(\mathbb{Q}) \ni \gamma &\longmapsto (\gamma, \gamma) \in GL_n(\mathbb{A}), \\ GL_n(\widehat{\mathbb{Z}}) \ni u &\longmapsto (u, 1) \in GL_n(\mathbb{A}), \\ GL_n^+(\mathbb{R}) \ni g &\longmapsto (1, g) \in GL_n(\mathbb{A}). \end{aligned}$$

Now note that

$$GL_n(\mathbb{R}) = GL_n^+(\mathbb{R}) \amalg e \cdot GL_n^+(\mathbb{R}), \text{ where } e = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \in GL_n(\mathbb{Z}).$$

Hence, for any element $(g^\infty, g_\infty) \in GL_n(\mathbb{A}^\infty) \times GL_n(\mathbb{R}) = GL_n(\mathbb{A})$, write $g^\infty = \gamma u$ with $\gamma \in GL_n(\mathbb{Q})$ and $u \in GL_n(\widehat{\mathbb{Z}})$ by the previous proposition, and compute:

$$\begin{aligned} (g^\infty, g_\infty) &= (\gamma, \gamma) \cdot (u, g) \quad \text{where } g = \gamma^{-1} g_\infty \in GL_n(\mathbb{R}) \\ &= \begin{cases} (\gamma, \gamma) \cdot (u, 1) \cdot (1, g) & (g \in GL_n^+(\mathbb{R})) \\ (e\gamma, e\gamma) \cdot (eu, 1) \cdot (1, eg) & (eg \in GL_n^+(\mathbb{R})) \end{cases} \end{aligned}$$

because $e^2 = 1$. □

This last proposition is sometimes quoted as a consequence of the strong approximation theorem, but it is significantly weaker than that – all we used was a kind of "weak approximation" in $\mathbb{A}^\infty$ when proving the lemma. Now by taking the determinant:

$$\begin{array}{ccc} GL_n(\mathbb{A}) & \xleftarrow{\cong} & GL_n(\mathbb{Q}) \cdot GL_n(\widehat{\mathbb{Z}}) \cdot GL_n^+(\mathbb{R}) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ \mathbb{A}^\times & \xleftarrow{\cong} & \mathbb{Q}^\times \cdot \widehat{\mathbb{Z}}^\times \cdot \mathbb{R}^\times_{>0} \end{array}$$

the expression $g_{\mathbb{A}} = \gamma u g$ gives $\det(g_{\mathbb{A}}) = \det(\gamma)\det(u)\det(g)$. As the bottom row is a *direct product* decomposition, if $\det(g_{\mathbb{A}}) = 1$, then $\det(\gamma) = \det(u) = \det(g) = 1$. Hence we conclude $SL_n(\mathbb{A}) = SL_n(\mathbb{Q}) SL_n(\widehat{\mathbb{Z}}) SL_n(\mathbb{R})$, where we write, for any ring R,

$$SL_2(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in R,\ ad - bc = 1 \right\}, \text{ and similarly for } SL_n(R).$$

We turn to the (real) strong approximation. It boils down to the following lemma:

Lemma 5.7. *For any $N \in \mathbb{Z}_{>0}$, the natural map $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/N)$ is surjective.*

Proof. We give the proof for $n = 2$. We need to find, for any $X \in M_2(\mathbb{Z})$ such that $X \bmod N \in SL_2(\mathbb{Z}/N)$, i.e. $\det X \equiv 1 \pmod N$, an element $Y \in SL_2(\mathbb{Z})$ such that $Y \equiv X \pmod N$. As the theory of elementary divisors shows that we can multiply (possibly different) elements of $SL_2(\mathbb{Z})$ from left and right to X to diagonalize X, it suffices to show when X is diagonal. When $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, then $Y := \begin{pmatrix} x & -(1-xy) \\ 1-xy & y(2-xy) \end{pmatrix}$ will do (note that $xy \equiv 1 \pmod N$). This can be made into an inductive argument for general n (see [Sh], Lemma 1.38). □

Note that the above lemma works only in the "semisimple group" setting, because for GL_1, obviously $\mathbb{Z}^\times$ does not surject onto $(\mathbb{Z}/N)^\times$! Now we can prove the main statement:

Theorem 5.8. *(Strong approximation theorem for SL_n and GL_n)*

(i) *Let $U \subset SL_n(\widehat{\mathbb{Z}})$ be a finite index subgroup. Then $SL_n(\mathbb{A}) = SL_n(\mathbb{Q}) \cdot U \cdot SL_n(\mathbb{R})$.*
(ii) *Let $U \subset GL_n(\widehat{\mathbb{Z}})$ be a finite index subgroup. Then $GL_n(\mathbb{A}) = GL_n(\mathbb{Q}) \cdot U \cdot GL_n^+(\mathbb{R})$, provided that $\det U = \widehat{\mathbb{Z}}^\times$.*

Proof. (i): We can find $N \in \mathbb{Z}_{>0}$ such that $U \supset \mathrm{Ker}(SL_n(\widehat{\mathbb{Z}}) \to SL_n(\mathbb{Z}/N))$. Then the above lemma shows $SL_n(\widehat{\mathbb{Z}}) \subset SL_n(\mathbb{Z}) \cdot U$. Now use $SL_n(\mathbb{A}) = SL_n(\mathbb{Q})SL_n(\widehat{\mathbb{Z}})SL_n(\mathbb{R})$.
(ii): The RHS contains $SL_n(\mathbb{A})$ by applying (i) to $U \cap SL_n(\widehat{\mathbb{Z}})$. By $\det U = \widehat{\mathbb{Z}}^\times$, the det of RHS is $\mathbb{Q}^\times \widehat{\mathbb{Z}}^\times \mathbb{R}^\times_{>0} = \mathbb{A}^\times$. □

Now it is not difficult to deduce Proposition 4.1, although we will generalize it to general finite index subgroup $U \subset GL_n(\widehat{\mathbb{Z}})$ in the next section. Here are some examples of U with $\det U = \widehat{\mathbb{Z}}^\times$ and $\det U \neq \widehat{\mathbb{Z}}^\times$, respectively.

Example 5.9. For $N \in \mathbb{Z}_{>0}$, we define:

$$U_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \equiv 0 \pmod N \right\} \subset GL_2(\widehat{\mathbb{Z}}),$$

$$U_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \equiv 0,\ d \equiv 1 \pmod N \right\} \subset GL_2(\widehat{\mathbb{Z}}).$$

$$U(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\} \subset GL_2(\widehat{\mathbb{Z}}).$$

Then we have $\det(U_0(N)) = \det(U_1(N)) = \widehat{\mathbb{Z}}^\times$, but $\det(U(N)) = \mathrm{Ker}\big(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times\big)$.

Note that $U(N) = \mathrm{Ker}(GL_2(\widehat{\mathbb{Z}}) \to GL_2(\mathbb{Z}/N))$, and any finite index subgroup $U \subset GL_2(\widehat{\mathbb{Z}})$ contains $U(N)$ for some N.

6. Connected Components of Modular Curves

Then, what happens when $\det U \neq \widehat{\mathbb{Z}}^\times$? The failure of strong approximation is by $\widehat{\mathbb{Z}}^\times / \det U$, which is a finite group because U has finite index in $GL_2(\widehat{\mathbb{Z}})$. If we take a set of representatives $\Sigma \subset GL_2(\widehat{\mathbb{Z}})/U$ which maps bijectively onto $\widehat{\mathbb{Z}}^\times / \det U$, then we can generate the whole of $GL_2(\mathbb{A})$ with $GL_2(\mathbb{Q}) \cdot \sigma U \cdot GL_2^+(\mathbb{R})$ for $\sigma \in \Sigma$. Generalizing Proposition 4.1, we will see that the connected components of the Shimura variety X_U are labeled by $\Sigma \cong \widehat{\mathbb{Z}}^\times / \det U$.

Proposition 6.1. *Let $U \subset GL_2(\widehat{\mathbb{Z}})$ be a finite index subgroup, and let $\Sigma \subset GL_2(\widehat{\mathbb{Z}})/U$ be a set which maps bijectively onto $\widehat{\mathbb{Z}}^\times / \det U$ under* det. *For each $\sigma \in \Sigma$, let $\Gamma_\sigma := SL_2(\mathbb{Z}) \cap$*

$\sigma U \sigma^{-1}$ (where we took the intersection inside $GL_2(\widehat{\mathbb{Z}})$). Then we have the isomorphisms of sets (resp. of Riemann surfaces):

$$\begin{aligned}
\coprod_{\sigma\in\Sigma} \Gamma_\sigma \backslash GL_2^+(\mathbb{R}) &\xrightarrow{\cong} GL_2(\mathbb{Q})\backslash\Big(\big(GL_2(\mathbb{A}^\infty)/U\big) \times GL_2(\mathbb{R})\Big), \\
(\sigma,\ g \bmod \Gamma_\sigma) &\longmapsto (\sigma, g) \bmod GL_2(\mathbb{Q}) \\
\coprod_{\sigma\in\Sigma} \Gamma_\sigma \backslash \mathcal{H}^+ &\xrightarrow{\cong} GL_2(\mathbb{Q})\backslash\Big(\big(GL_2(\mathbb{A}^\infty)/U\big) \times \mathcal{H}\Big) = X_U. \\
(\sigma,\ z \bmod \Gamma_\sigma) &\longmapsto (\sigma, z) \bmod GL_2(\mathbb{Q})
\end{aligned}$$

In particular, the set of connected components of X_U is in canonical bijection with $\widehat{\mathbb{Z}}^\times / \det U$ (by $\det$ on the $GL_2(\mathbb{A}^\infty)$-component).

Proof. The second isomorphism is obtained from the first by taking the quotient by right U_∞-action. We discussed before the proposition that $\coprod_{\sigma\in\Sigma} GL_2^+(\mathbb{R})$ surjects on to the RHS of the first isomorphism. When are (σ, g) and (σ', g') equivalent $\bmod\, GL_2(\mathbb{Q})$? We need $\sigma = \sigma'$ in order to land on the same element of $\widehat{\mathbb{Z}}^\times / \det U$ under det. Just to make sure: the det of the map $\coprod_{\sigma\in\Sigma} GL_2^+(\mathbb{R}) \longrightarrow$ RHS is the inclusion:

$$\mathbb{R}^\times_{>0} \longrightarrow \mathbb{Q}^\times \backslash \Big(\big((\mathbb{A}^\infty)^\times / \det U\big) \times \mathbb{R}^\times\Big) = \mathbb{Q}^\times\backslash \mathbb{A}^\times / \det U = \big(\widehat{\mathbb{Z}}^\times / \det U\big) \times \mathbb{R}^\times_{>0}.$$

Now (σ, g) and (σ, g') are equivalent $\bmod\, GL_2(\mathbb{Q})$ if:

$$(\gamma\sigma U, \gamma g) = (\sigma U, g') \quad (\gamma \in GL_2(\mathbb{Q})),$$

where we chose $\sigma \in GL_2(\widehat{\mathbb{Z}})$ representing $\sigma \in GL_2(\widehat{\mathbb{Z}})/U$. This identity means that:

$$\begin{aligned}
&\gamma \in \sigma U \sigma^{-1} \quad \text{as an element of } GL_2(\mathbb{A}^\infty), \\
&\gamma = g' g^{-1} \quad \text{as an element of } GL_2(\mathbb{R}).
\end{aligned}$$

These two conditions are equivalent to say that $\gamma \in SL_2(\mathbb{Z}) \cap \sigma U \sigma^{-1}$, because the first condition says $\gamma \in GL_2(\mathbb{Q}) \cap GL_2(\widehat{\mathbb{Z}}) = GL_2(\mathbb{Z})$ and the second says $\gamma \in GL_2^+(\mathbb{R})$, hence $\gamma \in SL_2(\mathbb{Z})$. □

Example 6.2. For $N \in \mathbb{Z}_{>0}$, we define:

$$\begin{aligned}
\Gamma_0(N) &:= SL_2(\mathbb{Z}) \cap U_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \equiv 0 \pmod N \right\} \subset SL_2(\mathbb{Z}), \\
\Gamma_1(N) &:= SL_2(\mathbb{Z}) \cap U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \equiv 0,\ d \equiv 1 \pmod N \right\} \subset SL_2(\mathbb{Z}), \\
\Gamma(N) &:= SL_2(\mathbb{Z}) \cap U(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\} \subset SL_2(\mathbb{Z}).
\end{aligned}$$

Then we have $X_{U_0(N)} = \Gamma_0(N)\backslash\mathcal{H}^+$ and $X_{U_1(N)} = \Gamma_1(N)\backslash\mathcal{H}^+$, but $X_{U(N)} = \coprod_\sigma \Gamma(N)_\sigma \backslash \mathcal{H}^+$. For the last one as $U(N) = \mathrm{Ker}\big(GL_2(\widehat{\mathbb{Z}}) \to GL_2(\mathbb{Z}/N)\big)$ is a normal subgroup of $GL_2(\widehat{\mathbb{Z}})$, the conjugates $\Gamma(N)_\sigma$ are all equal to $\Gamma(N)$. Thus:

$$X_{U(N)} = \coprod_{\sigma \in (\mathbb{Z}/N)^\times} \Gamma(N)\backslash\mathcal{H}^+.$$

The compactifications of these Shimura varieties (modular curves) are traditionally denoted by $X_0(N), X_1(N)$ and $X(N)$.

It is not a coincidence that we found $(\mathbb{Z}/N)^\times$ as the group of components of modular curves and the Galois group of cyclotomic fields. The fact that the connected components (or 0-th cohomology group) of Shimura varieties describe the class field theory of the base field (or the field of definition of X_U) was one of the initial motivations for studying Shimura varieties, in the context of constructing class fields (the 12th problem of Hilbert). A general statement can be found in [De].

Remark 6.3. Note that the correspondence $U \longmapsto \Gamma := SL_2(\mathbb{Z}) \cap U$ is neither surjective nor injective. The subgroups Γ that are obtained from U in this way are called the *congruence subgroups* of $SL_2(\mathbb{Z})$. Check that if we define $U_1'(N)$ by $c \equiv 0$ and $a \equiv 1 \pmod N$, it gives the same $\Gamma = \Gamma_1(N)$ as $U_1(N)$.

Some authors prefer to use $\Gamma = GL_2(\mathbb{Q}) \cap U$. The translation of our language into this one is easy. The strong approximation works *a fortiori* with $GL_2^+(\mathbb{R})$ replaced by $GL_2(\mathbb{R})$, except that after taking the determinant we have a redundancy of $\mathbb{R}^\times/\mathbb{R}^\times_{>0} \cong \{\pm 1\}$, hence we can take the "failure set " $\Sigma^\pm \subset GL_2(\widehat{\mathbb{Z}})/U$ so that it maps bijectively onto $\{\pm 1\}\backslash\widehat{\mathbb{Z}}^\times/\det U$ under det. Then the same proof shows that

$$\coprod_{\sigma \in \Sigma^\pm} \Gamma_\sigma\backslash GL_2(\mathbb{R}) \xrightarrow{\cong} GL_2(\mathbb{Q})\backslash\Big(\big(GL_2(\mathbb{A}^\infty)/U\big) \times GL_2(\mathbb{R})\Big)$$

$$\coprod_{\sigma \in \Sigma^\pm} \Gamma_\sigma\backslash\mathcal{H} \xrightarrow{\cong} GL_2(\mathbb{Q})\backslash\Big(\big(GL_2(\mathbb{A}^\infty)/U\big) \times \mathcal{H}\Big) = X_U,$$

where $\Gamma_\sigma := GL_2(\mathbb{Q}) \cap \sigma U \sigma^{-1}$. Note that $\Gamma_\sigma = GL_2(\mathbb{Z}) \cap \sigma U \sigma^{-1}$, because $\sigma U \sigma^{-1} \subset GL_2(\widehat{\mathbb{Z}})$. Note that the RHS of these isomorphisms are the same as the ones in Proposition 6.1, namely $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/U$ and X_U respectively. Therefore the LHS is just a different grouping of the connected components, depending on whether $-1 \in \det U$ or not. Denote our previous Γ_σ by $\Gamma_\sigma^+ := SL_2(\mathbb{Z}) \cap U$ to distinguish from Γ_σ. Then:

$$-1 \in \det U \iff \Sigma = \Sigma^\pm \iff (\Gamma_\sigma : \Gamma_\sigma^+) = 2 \iff \Gamma_\sigma\backslash\mathcal{H} \cong \Gamma_\sigma^+\backslash\mathcal{H}^+$$

$$-1 \notin \det U \iff \Sigma \neq \Sigma^\pm \iff \Gamma_\sigma = \Gamma_\sigma^+ \iff \Gamma_\sigma\backslash\mathcal{H} = (\Gamma_\sigma^+\backslash\mathcal{H}^+) \amalg (\Gamma_\sigma^+\backslash\mathcal{H}^-),$$

In the second case, we can take $\Sigma = \Sigma^\pm \amalg e\Sigma^\pm$ for $e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and as $\Gamma_{e\sigma}^+ = e\Gamma_\sigma^+ e^{-1}$, the isomorphism $e : \mathcal{H}^- \ni z \mapsto -z \in \mathcal{H}^+$ induces $\Gamma_\sigma^+\backslash\mathcal{H}^- \xrightarrow{\cong} \Gamma_{e\sigma}^+\backslash\mathcal{H}^+$, which recovers the expression in Proposition 6.1.

One good thing about the convention $\Gamma = GL_2(\mathbb{Q}) \cap U$ is that we can consider arbitrary *open compact* subgroup $U \subset GL_2(\mathbb{A}^\infty)$, and do not have to fix a specific maximal compact subgroup $GL_2(\widehat{\mathbb{Z}}) \subset GL_2(\mathbb{A}^\infty)$. In the GL_2-context this is not important because of Remark 5.3, but for general groups it is more canonical to be able to say "for any open compact subgroup U". (For that matter, we could have said $\Gamma = GL_2^+(\mathbb{Q}) \cap U$ for $GL_2^+(\mathbb{Q}) := GL_2(\mathbb{Q}) \cap GL_2^+(\mathbb{R})$...)

7. Automorphic Representations

Now we get back to the Definition 4.3. The $\mathbb{C}$-vector space $\mathcal{A}_k(U)$ of *automorphic forms* of level U and weight k was the space of sections of a holomorphic line bundle $\overline{\mathcal{V}}_k$ on $\overline{X}_U$. The first non-trivial property of $\mathcal{A}_k(U)$ is that they are finite dimensional.

Theorem 7.1. *Let $k \in \mathbb{Z}_{\geq 0}$ and let $U \subset GL_2(\widehat{\mathbb{Z}})$ be a finite index subgroup. Then $\mathcal{A}_k(U)$ is a finite dimensional $\mathbb{C}$-vector space.*

Proof. It is a direct consequence of the finiteness of the space of sections of holomorphic line bundle over compact Riemann surfaces. Later we will see that X_U are (naturally) algebraic curves over $\mathbb{C}$, hence the finiteness is shown algebraically. □

Now we haste to add remarks — although we know that all compact Riemann surfaces are algebraic curves, this needs hard analysis. And for higher dimensional Shimura varieties, it is essential to know that they are algebraic varieties, and then we can resort to the finiteness of sections of coherent sheaves over proper varieties to prove Theorem 7.1. But the finiteness generalizes to much greater generality, including the cases where the automorphic forms do not have geometric definition but only real-analytic definition (like Maass forms on GL_2, or automorphic forms on GL_n for $n > 2$). Once we have the "correct" real-analytic definition of automorphic forms, the finiteness follows from the *regularity theorems for elliptic differential operators*. I hope to come back to the real-analytic definition of automorphic forms at some point. See [Bo] for the real-analytic discussion of our GL_2 case, in the guise of $SL_2(\mathbb{R})$...

Now the whole point of thinking adelically is to consider automorphic forms of all levels (with a fixed weight) simultaneously. At the end of §4, we defined the *infinite dimensional* spaces

$$\mathcal{A}_k := \varinjlim_U \mathcal{A}_k(U), \quad \mathcal{A}_k^0 := \varinjlim_U \mathcal{A}_k^0(U).$$

(The $\mathcal{A}_k^0(U)$ denoted the space of cusp forms.) These direct limits are unions, because $\mathcal{A}_k(U) \subset \mathcal{A}_k(U')$ and $\mathcal{A}_k^0(U) \subset \mathcal{A}_k^0(U')$ for $U' \subset U$. How does $GL_2(\mathbb{A}^\infty)$ act on these infinite dimensional spaces?

The left action of $GL_2(\mathbb{A}^\infty)$ on $\mathcal{A}_k$ is the natural one coming from the *right* action of $GL_2(\mathbb{A}^\infty)$ on the first component of $GL_2(\mathbb{A}^\infty) \times \mathcal{H}$. (Remember that we are reserving the *left* action on the space in order to take the quotient by $GL_2(\mathbb{Q})$!) Let us spell this out: take $\phi \in \mathcal{A}_k(U)$, i.e. $\phi(hu, z) = \phi(h, z)$ for all $u \in U$ and $h \in GL_2(\mathbb{A}^\infty)$. The left action of $g \in GL_2(\mathbb{A}^\infty)$ on ϕ is given by:

$$(g\phi)(h, z) = \phi(hg, z) \quad (\forall h \in GL_2(\mathbb{A}^\infty)).$$

This $\phi \cdot g$ is an automorphic form of level gUg^{-1}, because for all $u \in U$,

$$(g\phi)(h \cdot gug^{-1}, z) = \phi(hgu, z) = \phi(hg, z) = (g\phi)(h, z).$$

Thus we have an isomorphism of finite dimensional spaces:

$$g : \mathcal{A}_k(U) \ni \phi \longmapsto g\phi \in \mathcal{A}_k(gUg^{-1}),$$

and this action piles up into the action on the infinite dimensional space $\mathcal{A}_k$. On the other hand, for each U, the space $\mathcal{A}_k(U)$ is recovered from the whole space $\mathcal{A}_k$ as the automorphic forms that are invariant under the right action of U, i.e. the *U-invariant subspace* $\mathcal{A}_k^U$ of $\mathcal{A}_k$. (Everything is parallel for the space of cusp forms $\mathcal{A}_k^0$.) What you have learned about the classical *Hecke operators* sending one modular form to another were all fragments of this single action of $GL_2(\mathbb{A}^\infty)$ on $\mathcal{A}_k$. We propose to understand this big space $\mathcal{A}_k$ systematically as representations of $GL_2(\mathbb{A}^\infty)$.

Before defining the automorphic representations (which are simply the representations of $GL_2(\mathbb{A}^\infty)$ that appear in $\mathcal{A}_k$), we prepare a small amount of terminology concerning the representations of *locally profinite groups* (see §5).

Let G be a locally profinite group, and let $\pi : G \to \mathrm{Aut}(V)$, or (π, V), be a representation of G on a $\mathbb{C}$-vector space V. (The whole theory is equivalent if we work with vector spaces over any algebraically closed field isomorphic to $\mathbb{C}$, for example $\overline{\mathbb{Q}}_\ell$, which we will do later. Note that what follows basically treats V as a discrete space.) The following are the properties that are abstracted from the representation of $GL_2(\mathbb{A}^\infty)$ on $\mathcal{A}_k$, i.e. (i) it is the union of $\mathcal{A}_k(U) = \mathcal{A}_k^U$, and (ii) each $\mathcal{A}_k(U)$ are finite-dimensional. For each open compact subgroup $U \subset G$, denote the *U-invariant subspace* of V by:

$$V^U := \{v \in V \mid uv = v \ (\forall u \in U)\}.$$

We say that V is a *smooth* representation of G, if all $v \in V$ belongs to V^U for some U, i.e. $V = \bigcup_U V^U$. We will work in the category $\mathrm{Rep}(G)$ of all smooth representations of G, which is an abelian category and we can talk about *irreducible* representations, *subquotients*, etc. A smooth representation V is called *admissible* if V^U is finite dimensional for all open compact subgroup U. All subquotients of admissible representation are admissible. As we have $V^U \subset V^{U'}$ for $U' \subset U$, to check the admissibility it is enough to look at V^U for small enough U. Also, as each element $g \in G$ gives the isomorphism:

$$g : V^U \ni v \overset{\cong}{\longmapsto} gv \in V^{gUg^{-1}},$$

it is enough to look at a conjugate subgroup.

In this language, we can rephrase Theorem 7.1 as follows:

Corollary 7.2. *Let $k \in \mathbb{Z}_{\geq 0}$. Then $\mathcal{A}_k$ and $\mathcal{A}_k^0$ are admissible representations of $GL_2(\mathbb{A}^\infty)$.*

Definition 7.3. An irreducible smooth representation π^∞ of $GL_2(\mathbb{A}^\infty)$ is called an *automorphic representation* (resp. *cuspidal automorphic representation*) of *weight k* if it is isomorphic to a subquotient of $\mathcal{A}_k$ (resp. $\mathcal{A}_k^0$).

Lecture 3 (Sep. 19, 2008)

8. Restricted Tensor Products

One of the most important features of the adelic formulation of the theory is that the theory decomposes into *local*, i.e. *p-adic* theories for each prime p, as we saw a faint shadow in §1, or as in class field theory over a general number field. This decomposition was not clearly seen from the modular forms themselves, although it asserted itself in the *Euler product* decomposition of the L-functions associated to modular forms. For that matter, even *automorphic* forms do not decompose into local functions — you need to look at the automorphic *representations*.

Generalizing the situation for GL_1 (the idele groups), any irreducible admissible representation of $GL_n(\mathbb{A}^\infty)$ (in particular the automorphic representations) is written as a *restricted tensor product* of irreducible admissible representations of $GL_n(\mathbb{Q}_p)$. We will give the definition of restricted tensor products here.

We would like to construct a representation of $GL_n(\mathbb{A}^\infty)$ from a family of representations $(\pi_p, V_p) \in \mathrm{Rep}(GL_n(\mathbb{Q}_p))$ for each p. This seems quite formidable, considering the fact that each representation is usually infinite dimensional. But the admissibility helps – any irreducible representation V is generated (as a representation) by any of its non-zero vector, hence in particular from a non-zero U-invariant subspace V^U, and for an irreducible *admissible* representation, each of the spaces V^U is finite dimensional, and the whole V is generated by V^U. So if we choose $U_p \subset GL_n(\mathbb{Z}_p)$ for each p such that $V_p^{U_p} \neq 0$, we may try to generate a representation from $\otimes_p V_p^{U_p}$ — but still there are infinitely many primes, so in order to make sense of this we would like to have $\dim_{\mathbb{C}} V_p^{U_p} = 1$ for almost all p.

And this is what happens in our case. Let us go back to admissible representations of $GL_n(\mathbb{A}^\infty)$ to see the heurisics. Any open compact subgroup of $GL_n(\mathbb{A}^\infty)$ is conjugate to a finite index subgroup $U \subset GL_n(\widehat{\mathbb{Z}})$, and the projection map $U \to GL_n(\mathbb{Z}_p)$ is surjective for almost all p. Therefore, by shrinking U into a smaller subgroup of finite index, it will have the form

$$U = \prod_p U_p, \quad U_p \subset GL_n(\mathbb{Z}_p) : \text{finite index}, \quad U_p = GL_n(\mathbb{Z}_p) \text{ for almost all } p. \tag{8.0.1}$$

Therefore, we can take this type of U as a typical finite index subgroup in $GL_n(\mathbb{A}^\infty)$. Having this observation in mind, the following fact will come in handy for us:

Proposition 8.1. *Let $(\pi_p, V_p) \in \mathrm{Rep}(GL_n(\mathbb{Q}_p))$ be an irreducible smooth representation of $GL_n(\mathbb{Q}_p)$ such that $V_p^{GL_n(\mathbb{Z}_p)} \neq 0$. Then $\dim_{\mathbb{C}} V_p^{GL_n(\mathbb{Z}_p)} = 1$. (We call these representations* unramified *representations.)*

Proof. At the moment we can treat this proposition as a black box. We will come back to this topic later. It is the consequence of a so-called *Satake isomorphism* for $GL_n(\mathbb{Q}_p)$. If V_p is irreducible, then $V_p^{GL_n(\mathbb{Z}_p)}$ has to be an irreducible module over the *unramified Hecke* *algebra* (a double coset algebra in the sense of [Sh], §3.1):

$$\mathbb{C}\big[GL_n(\mathbb{Z}_p)\backslash GL_n(\mathbb{Q}_p)/GL_n(\mathbb{Z}_p)\big] \xrightarrow{\cong} \mathbb{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]^{S_n},$$

which turns out to be commutative (the RHS is the ring of symmetric Laurent polynomials in n variables). Being an irreducible module over a commutative ring, the $V_p^{GL_n(\mathbb{Z}_p)}$ has to be 1-dimensional. □

Now get back to our original setting. We have irreducible $(\pi_p, V_p) \in \mathrm{Rep}(GL_n(\mathbb{Q}_p))$ for each p, such that $V_p^{GL_n(\mathbb{Z}_p)} \neq 0$ for almost all p, namely for all p outside the finite set of primes S. then for each such p we can choose a non-zero vector $w_p \in V_p^{GL_n(\mathbb{Z}_p)}$, which are unique up to constant by the previous proposition. Then form the following space:

$$V := \Big\langle \bigotimes_p v_p \;\Big|\; v_p = w_p \text{ for almost all } p \Big\rangle \subset \bigotimes_p V_p.$$

Then this space is a stable under the action of $GL_2(\mathbb{A}^\infty)$. Why? For any $g = (g_p)_p \in GL_n(\mathbb{A}^\infty)$, there is a finite set of primes $S' \supset S$ such that $g_p \in GL_n(\mathbb{Z}_p)$ for all $p \notin S'$. Then:

$$g \cdot \bigotimes_p v_p = \bigotimes_{p\in S'} g_p v_p \otimes \bigotimes_{p \notin S'} w_p \in V.$$

It is a smooth representation, because if each v_p is fixed by $U_p \subset GL_n(\mathbb{Z}_p)$, where we take $U_p = GL_n(\mathbb{Z}_p)$ for all $p \notin S$, then $\bigotimes_p v_p$ is fixed by $U = \prod_p U_p$, which has finite index in $GL_n(\mathbb{A}^\infty)$. Moreover, for $U \subset GL_n(\widehat{\mathbb{Z}})$ of the form (8.0.1) with $U_p = GL_n(\mathbb{Z}_p)$ for $p \notin S$, we have:

$$V^U = \bigotimes_{p\in S} V_p^{U_p} \otimes \Big\langle \bigotimes_{p\notin S} w_p \Big\rangle \cong \bigotimes_{p \in S} V_p^{U_p}, \tag{8.0.2}$$

which is finite dimensional, hence it is an admissible representation. Moreover, if we start with a different choice of $\{w_p'\}_{p\notin S}$, then we will arrive at an isomorphic representation of $GL_2(\mathbb{A}^\infty)$, where the isomorphism is given by

$$\bigotimes_{p\in S'} v_p \otimes \bigotimes_{p\notin S'} w_p \longmapsto \bigotimes_{p\in S'} v_p \otimes \bigotimes_{p\notin S'} w_p'$$

for finite set of primes $S' \supset S$. Therefore we obtained a well-defined admissible representation π^∞ of $GL_n(\mathbb{A}^\infty)$, which we call the *restricted tensor product* of $\{\pi_p\}_p$. We usually write $\pi^\infty = \bigotimes_p \pi_p$ by abuse of notation.

A (presumably straightforward) algebraic argument using Hecke algebras shows the following theorem, which is can be formulated for arbitrary restricted product of locally profinite groups having a property analogous to Proposition 8.1 (which does hold for fairly general p-adic reductive groups):

Theorem 8.2. (Decompostion Theorem [Fl]) *For a collection of irreducible* $(\pi_p, V_p) \in \mathrm{Rep}(GL_n(\mathbb{Q}_p))$ *for each* p*, such that* $V_p^{GL_n(\mathbb{Z}_p)} \neq 0$ *for almost all* p*, their restrict tensor product* $\pi^\infty = \bigotimes_p \pi_p$ *is an* irreducible *admissible representation of* $GL_n(\mathbb{A}^\infty)$*. Conversely,* *every irreducible admissible representation of $GL_n(\mathbb{A}^\infty)$ is isomorphic to $\bigotimes_p \pi_p$ for some collection $\{\pi_p\}_p$ as above, where each π_p is uniquely determined up to isomorphism.*

When $\pi^\infty \cong \bigotimes_p \pi_p$, we call π_p the *local components* of π^∞.

Corollary 8.3. *Every automorphic representation π^∞ of $GL_2(\mathbb{A}^\infty)$ is decomposed into a restricted tensor product of irreducible admissible representations of $GL_2(\mathbb{Q}_p)$.*

9. Cuspidal Automorphic Representations and Newforms

Now we will restrict our attention to the *cuspidal* automorphic representations, which are the main object of study for this course. The cuspidal automorphic representations have a very useful realization called *Whittaker models* on certain space of functions on $GL_2(\mathbb{A})$. It follows that its local components also have similar realizations, and using them the following is proved:

Theorem 9.1. *Let π^∞ be a cuspidal automorphic representation of $GL_2(\mathbb{A}^\infty)$.*

(i) (Multiplicity One) *There is a unique subrepresentation V of $\mathcal{A}_k^0$ realizing π^∞.*
(ii) (Local conductor) *Let $(\pi_p, V_p) \in \mathrm{Rep}(GL_2(\mathbb{Q}_p))$ be a local component of π^∞. There exists $n \in \mathbb{Z}_{\geq 0}$ such that $V_p^{U_1(p^n)} \neq 0$, where:*

$$U_1(p^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \equiv 0,\ d \equiv 1 \pmod{p^n} \right\} \subset GL_2(\mathbb{Z}_p).$$

Moreover, if we take the minimal one among such n (called the conductor *of π_p), then* $\dim_{\mathbb{C}} V_p^{U_1(p^n)} = 1$.

Note that (ii) generalizes the statement of Proposition 8.1 (the unramified case), because $U_1(1) = GL_2(\mathbb{Z}_p)$. These theorems are generalized to (the suitably defined) cuspidal automorphic representations of GL_n over arbitrary number field, but fails for general reductive groups. Now, admitting Theorem 9.1, we conclude:

Theorem 9.2. (Newform) *Let π^∞ be a cuspidal automorphic representation of $GL_2(\mathbb{A}^\infty)$. There exists $N \in \mathbb{Z}_{>0}$ with $V^{U_1(N)} \neq 0$, and for minimal such N, we have* $\dim_{\mathbb{C}} V^{U_1(N)} = 1$. *For the unique $V \subset \mathcal{A}_k^0$ realizing π^∞, we call the cusp form $\phi \in \mathcal{A}_k^0(U_1(N))$ which generates $V^{U_1(N)}$ (well-defined up to multiples by $\mathbb{C}^\times$) the* newform *associated to π^∞.*

Proof. Realize π^∞ uniquely as $V \subset \mathcal{A}_k^0$. Let $\pi^\infty = \otimes_p \pi_p$ by Corollary 8.3, and let n_p be the conductor of π_p, by Theorem 9.1(ii). Then $n_p = 1$ for almost all p, and set $N := \prod_p p^{n_p}$. Then:

$$U_1(N) = \prod_p U_1(p^{n_p}), \quad V^{U_1(N)} \cong \bigotimes_{p|N} V_p^{U_1(p^{n_p})}$$

by (8.0.2), which has dimension 1 by Theorem 9.1(ii). □

Recall that the cusp form $\phi \in \mathcal{A}_k^0(U_1(N))$ is a certain section of $\overline{\mathcal{V}}_k$ over the compactification of:

$$X_{U_1(N)} = GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/U_1(N)U_\infty \cong \Gamma_1(N)\backslash\mathcal{H}^+.$$

Rewinding our definitions, we recover the newforms as classical cusp forms with respect to $\Gamma_1(N)$ by the definition:

$$f(gi) := \mu(g,i)^k \phi(1,g) \quad (\forall g \in GL_2^+(\mathbb{R})),$$

where we consider ϕ as functions on $GL_2(\mathbb{A}) = GL_2(\mathbb{A}^\infty) \times GL_2(\mathbb{R})$.

Lecture 4 (Sep. 22, 2008)

Let us see the correspondence between the *central character* of cuspidal automorphic representation π^∞ and the *(Dirichlet) character* of the associated classical newform f. Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a character. We have short exact sequence:

$$1 \longrightarrow \Gamma_1(N) \longrightarrow \Gamma_0(N) \longrightarrow (\mathbb{Z}/N)^\times \longrightarrow 1,$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \bmod N$$

and consider χ as a character of $\Gamma_0(N)$ by $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N)^\times$. For a modular form f with respect to $\Gamma_1(N)$, we say that f has *character* χ if:

$$f \cdot [\gamma]_k = \chi(\gamma) f \quad (\forall \gamma \in \Gamma_0(N)).$$

Now the corresponding $\phi \in \mathcal{A}_0(U_1(N))$ satisfies, for all $\gamma \in \Gamma_0(N)$:

$$\begin{aligned}\phi(1,\gamma g) &= \mu(\gamma g,i)^{-k} f(\gamma g i) = \mu(g,i)^{-k}\mu(\gamma,gi)^{-k} f(\gamma g i) \\ &= \mu(g,i)^{-k}(f\cdot[\gamma]_k)(gi) = \chi(\gamma)\mu(g,i)^{-k} f(gi) = \chi(\gamma)\phi(1,g).\end{aligned}$$

We have a similar exact sequence:

$$1 \longrightarrow U_1(N) \longrightarrow U_0(N) \longrightarrow (\mathbb{Z}/N)^\times \longrightarrow 1,$$

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \bmod N$$

but if we consider $\widehat{\mathbb{Z}}^\times \subset GL_2(\widehat{\mathbb{Z}})$ as the diagonal matrices (the center), then $U_0(N) = \widehat{\mathbb{Z}}^\times U_1(N)$, hence we can describe the action of $(\mathbb{Z}/N)^\times$ in terms of the action of the center $\widehat{\mathbb{Z}}^\times$. As ϕ is contained in the irreducible representation π^∞, the center $(\mathbb{A}^\infty)^\times \subset GL_2(\mathbb{A}^\infty)$ should act by a character, say $\omega^\infty : (\mathbb{A}^\infty)^\times \to \mathbb{C}^\times$, i.e.:

$$\phi(u,g) = \omega^\infty(u)\phi(1,g) \quad (\forall u \in (\mathbb{A}^\infty)^\times),$$

where we wrote u for the diagonal matrix $u \cdot I_2 \in GL_2(\widehat{\mathbb{Z}})$. Getting back to our situation, let $\gamma \in \Gamma_0(N)$ with $\chi(\gamma) = \chi(d \bmod N)$. If we choose $\widetilde{d} \in \widehat{\mathbb{Z}}^\times$ which lifts $d \bmod N$ under $\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times$, then we have $\gamma\widetilde{d}^{-1} \in U_1(N)$ inside $GL_2(\widehat{\mathbb{Z}})$, hence:

$$\phi(1,\gamma g) = \phi(\gamma^{-1}, g) = \phi(\widetilde{d}^{-1}, g) = \omega^\infty(\widetilde{d}^{-1})\phi(1,g).$$

Then, comparing above computations, we conclude:

$$\chi(d \bmod N)\phi(1,g) = \omega^\infty(\widetilde{d}^{-1}), \text{ i.e. } \quad \omega^\infty|_{\widehat{\mathbb{Z}}^\times} = \chi^{-1} : \widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times \to \mathbb{C}^\times.$$

Actually, as considered as functions on $GL_2(\mathbb{A})$, the group U_∞ acts on automorphic forms in $\mathcal{A}_k$ through χ_k:

$$\phi(h, gu) = \chi_k(u)\phi(h,g),$$

and U_∞ contains the center $\mathbb{R}^\times \cdot I_2 \subset GL_2(\mathbb{R})$, which acts through:

$$\chi_k|_{\mathbb{R}^\times} : \mathbb{R}^\times \ni a \longmapsto a^{-k} \in \mathbb{C}^\times.$$

Therefore we can think of the *central character* of an automorphic representation π^∞ as the character ω of $\mathbb{A}^\times = (\mathbb{A}^\infty)^\times \times \mathbb{R}^\times$, which factors through $\mathbb{Q}^\times$ because automorphic forms are left $GL_2(\mathbb{Q})$-invariant. As $\omega|_{\mathbb{R}^\times_{>0}}(a) = a^{-k}$, this ω is an *algebraic Hecke character* of $\mathbb{A}^\times$ of *weight* k; it is a (cuspidal) automorphic representation of $GL_1(\mathbb{A})$ (in this convention, the *absolute value* has weight -1). Using $\mathbb{Q}^\times\backslash\mathbb{A}^\times \cong \widehat{\mathbb{Z}}^\times \times \mathbb{R}^\times_{>0}$, we have a bijection:

$$\begin{array}{ccc} \{\text{ Dirichlet character } \chi \} & \overset{1:1}{\longleftrightarrow} & \{ \text{ algebraic Hecke character } \omega \text{ of weight } k \} \\ \chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times & \longleftrightarrow & \omega \text{ such that } \omega|_{\widehat{\mathbb{Z}}^\times} = \chi^{-1}, \ \ \omega|_{\mathbb{R}^\times_{>0}}(a) = a^{-k} \end{array}$$

This is set up so that for p prime to N, we have $\chi(p \bmod N) = \omega(p)$ where p is considered as an element of $\mathbb{Q}_p^\times$. We summarize the above argument:

Proposition 9.3. *Let π^∞ be a cuspidal automorphic representation of $GL_2(\mathbb{A}^\infty)$, and ω be its central character. If f is the classical cusp form corresponding to the newform $\phi \in \mathcal{A}^0_k(U_1(N))$ associated to π^∞, then the character χ of f is the Dirichlet character corresponding to ω.*

This correspondence is sensitive to some choices, e.g. the change of choice from $U_1(N)$ to $U_1'(N)$ as in Remark 6.3 for the adelic subgroup corresponding to $\Gamma_1(N)$ will change the sign.

10. Notes on the Literature

The finiteness of the space of automorphic forms for general semisimple group is proved in [HC]; see [Bo] for the treatment of $SL_2(\mathbb{R})$. The article of Casselman [Ca] includes in §3 the dictionary between classical and adelic description of newforms, and also the the local conductors in §1. See also [Bu], [Ge]. The whole adelic framework goes back to [GGPS] and [JL], which both discuss the tensor product decomposition. The book [JL] proves the multiplicity one theorem.

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