Introductory Course: Fourier Analysis and its many uses

Solutions - Exercises 16.1-16.5, 16.9, 16.12, 16.13, 16.15, 16.16, 16.17, 16.20, 16.21, 16.23 from "A First Look at Fourier Analysis" by T.W. Körner

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Exercise 16.1. i) Take $P(t) = \frac{f^{(n)}(0)}{n!}\,t^n + \frac{f^{(n-1)}(0)}{(n-1)!}\,t^{n-1} + \cdots + f(0)$.

ii) Note first that $g(0) = g'(0) = \cdots = g^{(n)}(0) = 0$. Since $g(0) = g(t) = 0$ it follows from Rolle's theorem that there exists a $c_1 \in (0,t)$ such that $g'(c_1) = 0$. Observe now that $g'(0) = g'(c_1) = 0$ so Rolle's theorem implies that there exists a $c_2 \in (0, c_1)$ such that $g''(c_2) = 0$. Repeating this procedure $(n+1)$ times we get that there exists $c = c_{n+1} \in (0,t)$ such that $g^{(n+1)}(c) = 0$. This immediately implies $f(t) = P(t) + \frac{f^{(n+1)}(c)}{(n+1)!}\,t^{n+1}$.

Exercise 16.2. A polynomial P of degree at most $(2n+1)$ with prescribed values for $P(0)$, $P'(0)$, ... $P^{(n)}(0)$, $P(1)$, $P'(1)$, ... $P^{(n)}(1)$ can be obtained by taking a linear combination of the polynomials $\frac{1}{k!}x^k(x-1)^{n+1}$, $0 \leq k \leq n$ and $\frac{1}{k!}x^{n+1}(x-1)^k$, $0 \leq k \leq n$.

We prove now that P is unique. Let Q be a polynomial of degree at most $(2n+1)$ such that $P(0) = Q(0), P'(0) = Q'(0), ...P^{(n)}(0) = Q^{(n)}(0), P(1) = Q(1), P'(1) = Q'(1), ... P^{(n)}(1) = Q^{(n)}(1)$. $R = P - Q$ is a polynomial of degree at most $2n+1$ and $R(0) = R'(0) = \cdots = R^{(n)}(0) = R(1) = R'(1) = \cdots = R^{(n)}(1) = 0$. Therefore the polynomials X^{n+1} and $(X-1)^{n+1}$ divide R and since $\deg R \leq 2n+1$ we obtain $R = 0$.

Let P be the unique polynomial of degree at most $(2n+1)$ such that $P(0) = f(0), P'(0) = f'(0), ... P^{(n)}(0) = f^{(n)}(0), P(1) = f(1), P'(1) = f'(1), ... P^{(n)}(1) = f^{(n)}(1)$. For a fixed $y \in (0,1)$ let

$$g(x) = f(x) - P(x) - \frac{f(y) - P(y)}{y^{n+1}(y-1)^{n+1}}\; x^{n+1}(x-1)^{n+1}$$

Let's observe that $g(0) = g'(0) = \cdots = g^{(n)}(0) = 0$ and $g(1) = g'(1) = \cdots = g^{(n)}(1) = 0$. Furthermore $g(0) = g(y) = g(1) = 0$. Therefore, from Rolle's theorem it follows that there exists two points $c_1^1 \in (0,y)$ and $c_2^1 \in (y,1)$ such that $g'(c_1^1) = g'(c_2^1) = 0$. Since $g'(0) = g'(1) = 0$ we can apply Rolle's theorem again and we get three points $c_1^2 \in (0, c_1^1)$, $c_2^2 \in (c_1^1, c_2^1)$ and $c_3^2 \in (c_2^1, 1)$ such that $g''(c_1^2) = g''(c_2^2) = g''(c_3^2) = 0$. Repeating this procedure $(n+1)$ times we get $(n+2)$ points $c_1^{n+1}, c_2^{n+1}, ... c_{n+2}^{n+1} \in (0,1)$ such that $g^{(n+1)}(c_1^{n+1}) = g^{(n+1)}(c_2^{n+1}) = \cdots = g^{(n+1)}(c_{n+2}^{n+1}) = 0$.

Now we can apply Rolle's theorem again and we get $(n+1)$ points

$$c_1^{n+2} \in (c_1^{n+1}, c_2^{n+1}),\; c_2^{n+2} \in (c_2^{n+1}, c_3^{n+1}),\; ...\; c_{n+1}^{n+2} \in (c_{n+1}^{n+1}, c_{n+2}^{n+1})$$

such that $g^{(n+2)}(c_1^{n+2}) = g^{(n+2)}(c_2^{n+2}) = \cdots = g^{(n+2)}(c_{n+1}^{n+2}) = 0$. Repeating this procedure $(n+1)$ times we get that there exists a point $c = c_1^{2n+2} \in (0,1)$ such that $g^{(2n+2)}(c) = 0$.

Since $g^{(2n+2)}(x) = f^{(2n+2)}(x) - \frac{f(y)-P(y)}{y^{n+1}(y-1)^{n+1}}\,(2n+2)\,!$ we immediately get

$$E(y) = f(y) - P(y) = \frac{f^{(2n+2)}(c)}{(2n+2)\,!}\; y^{n+1}(y-1)^{n+1}$$

Exercise 16.3. Taking imaginary parts in the de Moivre formula it is easy to see that

$$U_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} (t^2-1)^k\; t^{n-2k}$$

Taking $\frac{\partial}{\partial\theta}$ in $T_n(\cos\theta) = \cos n\theta$ we get $T_n'(\cos\theta)\;\sin\theta = n(\sin n\theta)$ which implies that $T_n'(x) = nU_{n-1}(x)$.

Exercise 16.4. For any $t \in \mathbb{R}$ with $|t| < 1$ and for any $\theta \in \mathbb{R}$ we have $|te^{i\theta}| < 1$ so

$$\sum_{n=0}^{\infty} t^n e^{in\theta} = \frac{1}{1 - te^{i\theta}}$$

Since $\mathrm{Re}\left[t^n e^{in\theta}\right] = t^n \cos(n\theta)$ and $\mathrm{Re}\left[\frac{1}{1-te^{i\theta}}\right] = \frac{1-t\cos\theta}{1-2t\cos\theta+t^2}$ we get that $\frac{1-t\cos\theta}{1-2t\cos\theta+t^2} = \sum_{n=0}^{\infty} T_n(\cos\theta)\; t^n$ for any $|t| < 1$ and any $\theta \in \mathbb{R}$, which implies

$$\frac{1-tx}{1-2tx+t^2} = \sum_{n=0}^{\infty} T_n(x)\; t^n$$

for any $|t| < 1$ and $|x| \leq 1$.

Exercise 16.5. Let's observe first that for any $r \in (0,1)$ we have $P_r(f,\theta) = (f * P_r)(\theta)$ where $P_r : [-\pi,\pi] \to \mathbb{R}$, $P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} = \frac{1-r^2}{1+r^2-2r\cos t}$. Furthermore $P_r(t) > 0$ for any $t \in [-\pi,\pi]$ and:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,dt = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} r^{|n|} e^{int} dt = 1$$

Let $\varepsilon > 0$. For any $r \in (0,1)$ we have:

$$|P_r(f,\theta) - f(\theta)| = \left|(f * P_r)(\theta) - f(\theta)\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,dt\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(\theta-t) - f(\theta)\right] P_r(t)\,dt\right|$$

$$\leq \frac{1}{2\pi}\int_{|t|<\delta} |f(\theta-t)-f(\theta)|\, P_r(t)\, dt + \frac{1}{2\pi}\int_{|t|\geq\delta} |f(\theta-t)-f(\theta)|\, P_r(t)\, dt$$

Since $f \in \mathrm{C}(\mathbb{T})$ it follows that f is bounded and uniformly continuous. Let $M > 0$ such that $|f(s)| < M$ for any $s \in [-\pi,\pi]$ and let $\delta_\varepsilon > 0$ such that $|t-t'| < \delta_\varepsilon$ implies $|f(t)-f(t')| < \frac{\varepsilon}{2}$.

For any t with $|t| > \delta_\varepsilon$ we have $\frac{1-r^2}{1+r^2-2r\cos t} < \frac{1-r^2}{1+r^2-2r\cos\delta_\varepsilon} < \frac{1-r^2}{1-(\cos\delta_\varepsilon)^2}$ and therefore

$$\int_{|t|\geq\delta_\varepsilon} P_r(t)\, dt \;<\; \frac{1-r^2}{1-(\cos\delta_\varepsilon)^2}$$

Let $r_\varepsilon > 0$ such that $r \in (r_\varepsilon, 1)$ implies $\frac{1-r^2}{1-(\cos\delta_\varepsilon)^2} < \frac{2\pi\varepsilon}{4M}$. For any $r \in (r_\varepsilon, 1)$ we have:

$$\frac{1}{2\pi}\int_{|t|<\delta_\varepsilon} |f(\theta-t)-f(\theta)|\, P_r(t)\, dt < \frac{\varepsilon}{2}$$

and

$$\frac{1}{2\pi}\int_{|t|\geq\delta_\varepsilon} |f(\theta-t)-f(\theta)|\, P_r(t)\, dt < \frac{1}{2\pi}\, 2M\, \frac{2\pi\varepsilon}{4M} = \frac{\varepsilon}{2}$$

We can conclude now that $|P_r(f,\theta)-f(\theta)| < \varepsilon$ for any $r \in (r_\varepsilon, 1)$ and any $\theta \in [-\pi,\pi]$. Therefore $P_r(f,\cdot) \to f$ uniformly as $r \to 1,\ r < 1$.

Exercise 16.9. i) Let $\varepsilon > 0$. Since f has real values it follows from Theorem 8.1 ii) that there exists a real trigonometric polynomial

$$Q(t) = \sum_{n=0}^{N} (a_n \sin nt + b_n \cos nt)$$

such that $\|f-Q\|_\infty < \varepsilon$. Let $Q_1(t) = Q(-t)$. Since $f(t) = f(-t)$ it follows that $\|f-Q_1\|_\infty < \varepsilon$. For $R(t) = \frac{1}{2}(Q(t)+Q_1(t))$ we have $\|f-R\|_\infty < \varepsilon$. Furthermore a straightforward computation shows that:

$$R(t) = \sum_{n=0}^{N} b_n \cos nt$$

ii) Let $g : [-\pi,\pi] \to \mathbb{R}$ defined by $g(t) = F(|\cos t|)$. Let $\varepsilon > 0$. Since $g \in \mathrm{C}(\mathbb{T})$ and is real valued it follows from i) that there exists a real trigonometric polynomial $R(t) = \sum_{n=0}^{N} c_n \cos nt$ such that $\|f-R\|_\infty < \varepsilon$. Let's observe now that

$$R(t) = \sum_{n=0}^{N} c_n \cos nt = \sum_{n=0}^{N} c_n\, T_n(\cos t) = Q(\cos t)$$

where T_n is the n-th Chebyshev polynomial of the first kind and $Q \in \mathbb{R}[X]$.

Therefore $|g(t) - R(t)| = |F(|\cos t|) - Q(\cos t)| < \varepsilon$ for any $t \in [-\pi, \pi]$ which immediately implies $\|F - Q\|_\infty < \varepsilon$.

iii) Since $\int_0^1 g(t)\, t^n\, dt = 0$ for any n it follows that $\int_0^1 g(t)\, P(t)\, dt = 0$ for any polynomial $P \in \mathbb{R}[X]$. Let P_n be a sequence of polynomials such that $P_n \to \overline{g}$ uniformly. Since

$$\lim_{n\to\infty} \int_0^1 g(t) P_n(t) dt = \int_0^1 g(t)\ \overline{g(x)}\ dt$$

it follows that $\int_0^1 g(t)\ \overline{g(x)}\ dt = \int_0^1 |g(t)|^2\ dt = 0$ so $g = 0$.

Exercise 16.12. i) Since f is continuous it follows immediately that $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$ and Theorem 8.2. implies that $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 < \infty$. We get $\lim_{n\to\infty} \widehat{f}(n) = 0$ and $\lim_{n\to-\infty} \widehat{f}(n) = 0$.

ii) We can write $\sin t = \frac{1}{2i}\left(e^{it} - e^{-it}\right)$ and therefore $f_1(t) = \frac{1}{2i}\left(g_1(t)e^{it} - g_1(t)e^{-it}\right)$. This implies $\widehat{f_1}(j) = \frac{1}{2i}\left(\widehat{g_1}(j-1) - \widehat{g_1}(j+1)\right)$ so for any $n \geq 1$

$$S_n(f_1, 0) = \sum_{j=-n}^{n} \widehat{f_1}(j) = \widehat{f_1}(n) + \widehat{f_1}(n+1) - \widehat{f_1}(-n) - \widehat{f_1}(-n-1)$$

Using part i) we get $\lim_{n\to\infty} S_n(f_1, 0) = 0$.

iii) For any $t \neq n\pi$ we have $g_2(t) = \frac{1}{\sin t} f_2(t)$. We must prove that $\frac{1}{\sin t} f_2(t), t \neq n\pi$ can be extended to a continuous function on $\mathbb{T}$. For any $t \in (-\pi, \pi), t \neq 0$

$$\frac{1}{\sin t}\, f_2(t) = \frac{t}{\sin t}\ \frac{f_2(t)}{t} = \frac{t}{\sin t}\ \frac{f_2(t) - f_2(0)}{t - 0}$$

Therefore $\lim_{t\to 0} \frac{1}{\sin t} f_2(t) = f_2'(0)$, so the function $\frac{1}{\sin t} f_2(t)$ can be extended by continuity at $t = 0$. Similarly

$$\lim_{t\to\pi} \frac{1}{\sin t}\, f_2(t) = \lim_{t\to\pi} \frac{1}{\sin(\pi - t)}\, f_2(t) = \lim_{t\to\pi} \frac{t-\pi}{\sin(\pi - t)}\ \frac{f_2(t) - f_2(\pi)}{t - \pi} = -f_2'(\pi)$$

$$\lim_{t\to-\pi} \frac{1}{\sin t}\, f_2(t) = \lim_{t\to-\pi} \frac{-1}{\sin(\pi + t)}\, f_2(t) = \lim_{t\to-\pi} \frac{-(t+\pi)}{\sin(\pi + t)}\ \frac{f_2(t) - f_2(-\pi)}{t + \pi} = -f_2'(\pi)$$

so the function $\frac{1}{\sin t} f_2(t)$ can be extended to a continuous function $g_2 : \mathbb{T} \to \mathbb{C}$. Using ii) we get $\lim_{n\to\infty} S_n(f_2, 0) = 0$.

iv) It is easy to see that f_4 is continuous and differentiable at 0. Furthermore for any $j = 2n, n \in \mathbb{Z}$ we have:

$$\widehat{f_4}(j) = \widehat{f_4}(2n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_3(2t)\ e^{-2int}dt = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} f_3(s)\ e^{-ins}ds = \widehat{f_3}(n)$$

¿From part iii) we get that there exists a function $g \in \mathrm{C}(\mathbb{T})$ such that $f_4(t) = g(t)\sin t$ and from ii) we get that $\widehat{f_4}(2n) = \frac{1}{2i}(\widehat{g}(2n-1) - \widehat{g}(2n+1))$. We can now conclude that

$$S_N(f_3, 0) = \sum_{n=-N}^{N} \widehat{f_3}(n) = \sum_{n=-N}^{N} \widehat{f_4}(2n) = \frac{1}{2i}\left(\widehat{g}(-2n-1) - \widehat{g}(2n+1)\right)$$

Therefore $\lim_{N\to\infty} S_N(f_3, 0) = 0$.

v) Suppose f is continuous and differentiable at $x_0 \in [-\pi, \pi]$. Let $g : [-\pi, \pi] \to \mathbb{C}$, $g(x) = f(x + x_0) - f(x_0)$. Clearly $g(0) = 0$ and g is differentiable at 0. Using part iv) we get $\lim_{N\to\infty} S_N(g, 0) = 0$. But

$$\begin{aligned} S_N(g,0) = \sum_{n=-N}^{n=N} \widehat{g}(n) \ &= \ \widehat{g}(0) + \sum_{n=-N\ n\neq 0}^{n=N} \widehat{g}(n) = -f(x_0) + \sum_{n=-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+x_0)e^{-inx}dx \\ &= -f(x_0) + \sum_{n=-N}^{N} e^{inx_0}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+x_0)e^{-in(x+x_0)}dx = -f(x_0) + \sum_{n=-N}^{N} e^{inx_0}\widehat{f}(n) \\ &= -f(x_0) + S_N(f, x_0) \end{aligned}$$

which implies $\lim_{N\to\infty} S_N(f, x_0) = f(x_0)$.

Exercise 16.13. - Reformulated

i) Prove that for any $\varepsilon > 0$ and any $K > 0$ we can find a continuous function $f \in \mathrm{C}(\mathbb{T})$ and an integer $M > 0$ such that $\|f\|_\infty < \varepsilon$ and $|S_M(f, 0)| > K$.

ii) Prove that the function f from part i) can be chosen to be a trigonometric polynomial.

iii) Prove that for any $\varepsilon > 0$, any $K > 0$ and any positive integer $m > 0$ we can find a trigonometric polynomial P and a positive integer M such that $\|P\|_\infty < \varepsilon$, $\widehat{P}(r) = 0$ for any integer $|r| \leq m$ and $|S_M(P, 0)| > K$.

iv) For any nonzero trigonometric polynomial $P(t) = \sum_{n=-N}^{N} a_n e^{int}$ we denote by $\deg P = \max\{n,\ n \text{ nonnegative integer s.t. } a_n \neq 0 \text{ or } a_{-n} \neq 0\}$. Prove that we can find a sequence of trigonometric polynomials $\{P_n\}_{n\geq 1}$ such that the sequence $m(n) = \deg(P_{n-1})$ is increasing and:

a) $\|P_n\|_\infty < 2^{-n}$

b) $\widehat{P_n}(r) = 0$ if $|r| \leq m(n)$ or $|r| > m(n+1)$

c) $|S_{\deg P_n}(P_n, 0)| \geq 2^n + \sum_{k=1}^{n-1} |S_{\deg P_k}(P_k, 0)|$

v) Prove that $\sum_{n=1}^{\infty} P_n$ is uniformly convergent to some continuous function f and that for any n and for any integer r such that $m(n) + 1 \leq r \leq m(n+1)$ we have $\widehat{f}(r) = \widehat{P_n}(r)$.

vi) Deduce that $|S_{m(n+1)}(f, 0)| \geq 2^n$ for any n and therefore $\{S_N(f, 0)\}$ diverges.

Solution i) Let M large enough so that $\varepsilon B \log M > K$. From Lemma 6.5. there exists a function $g \in \mathrm{C}(\mathbb{T})$ such that $\|g\|_\infty < 1$ and $|S_M(g, 0)| \geq B \log M > \frac{K}{\varepsilon}$. Let $f = \varepsilon g$. We have $\|f\|_\infty < \varepsilon$ and $|S_M(f, 0)| = \varepsilon |S_M(g, 0)| > K$.

ii) Using part i) we can find a function $f \in \mathrm{C}(\mathbb{T})$ and a positive integer M such that $\|f\|_\infty < \frac{\varepsilon}{2}$ and $|S_M(f, 0)| > (K+1)$. Since the trigonometric polynomials are dense in $\mathrm{C}(\mathbb{T})$ we can find a trigonometric polynomial P such that $\|P - f\|_\infty < \min\{\frac{1}{2M+1}, \frac{\varepsilon}{2}\}$. Clearly $\|P\|_\infty < \varepsilon$. Furthermore:

$$|S_M(f, 0) - S_M(P, 0)| = |S_M(f - P, 0)| \leq \sum_{n=-M}^{M} \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f - P)(t) e^{-int} dt \right| < 1$$

which shows that $|S_M(P, 0)| > K$.

iii) For any nonzero trigonometric polynomial $P(t) = \sum_{n=-N}^{N} a_n e^{int}$ we denote by $\deg P = \max\{\text{n nonnegative integer s.t. } a_n \neq 0 \text{ or } a_{-n} \neq 0\}$. Let Q be a trigonometric polynomial such that $\|Q\|_\infty < \varepsilon$ and $|S_{\deg Q}(Q, 0)| > K$. Let $P(t) = e^{(\deg Q + m + 1)it} Q(t)$. Clearly $\|P\|_\infty < \varepsilon$ and $\widehat{P}(r) = 0$ for any integer $r \leq m$. Let $M = 2 \deg P + m + 1$. Then $S_M(P, 0) = S_{\deg Q}(Q, 0)$ so $|S_M(P, 0)| > K$.

iv) We will repeatedly use iii) to get the sequence of trigonometric polynomials $\{P_n\}$. Let P_1 be a trigonometric polynomial such that $\|P_1\|_\infty \leq 2^{-1}$ and $|S_{\deg P_1}(P_1, 0)| \geq 2$. Let $m(2) = \deg P_2$. Let now P_2 be a trigonometric polynomial such that $\|P_2\|_\infty \leq 2^{-2}$, $|S_{\deg P_2}(P_2, 0)| \geq 2^2 + |S_{\deg P_1}(P_1, 0)|$ and $\widehat{P_2}(r) = 0$ for all integers r, $-m(2) \leq r \leq m(2)$. Let $m(3) = \deg P_2$. Repeating this procedure we get a sequence of trigonometric polynomials $\{P_n\}$ and an increasing sequence of integers $\{m(n)\}$, $m(n) = \deg P_{n-1}$ for any $n \geq 2$, such that $\|P_n\|_\infty \leq 2^{-n}$, $|S_{\deg P_n}(P_n, 0)| \geq 2^n + \sum_{k=1}^{n-1} |S_{\deg P_k}(P_k, 0)|$ and $\widehat{P_n}(r) = 0$ for any r, $|r| \leq m(n)$ or $|r| > m(n+1)$.

v) Since $\|P_n\|_\infty < 2^{-n}$ and the series $\sum_{n=1}^{\infty} 2^{-n}$ is convergent it follows that $\sum_{n=1}^{\infty} P_n$ converges uniformly to a continuous function f. For any integer r, $m(n) + 1 \leq |r| \leq$

$m(n+1)$

$$\widehat{f}(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{irt}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty} P_n(t)\right)e^{irt}dt$$
$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} P_n(t)e^{irt}dt = \sum_{k=1}^{\infty}\widehat{P_k}(r) = \widehat{P_n}(r)$$

Therefore for any integer n we have that

$$S_{m(n+1)}(f,0) = \sum_{k=-m(n+1)}^{m(n+1)} \widehat{f}(k) = \sum_{k=-m(2)}^{m(2)} \widehat{f}(k) + \sum_{l=2}^{n}\left[\sum_{m(l)+1\le|k|\le m(l+1)} \widehat{f}(k)\right]$$
$$= \sum_{k=-m(2)}^{m(2)} \widehat{P_1}(k) + \sum_{l=2}^{n}\left[\sum_{m(l)+1\le|k|\le m(l+1)} \widehat{P_l}(k)\right] = S_{m(2)}(P_1,0) + \sum_{l=2}^{n} S_{m(l+1)}(P_l,0)$$
$$= S_{\deg P_n}(P_n,0) + \sum_{l=1}^{n-1} S_{\deg P_l}(P_l,0)$$

Since $|S_{\deg P_n}(P_n,0)| \ge 2^n + \sum_{k=1}^{n-1}|S_{\deg P_k}(P_k,0)|$ we get $|S_{m(n+1)}(f,0)| \ge 2^n$ for any positive integer n which shows that $\overline{\lim}_{N\to\infty}|S_N(f,0)| = \infty$, so $S_N(f,0) \nrightarrow f(0)$.

The function f constructed before has complex values. Since for any positive integer N we have $S_N(f,0) = S_N(\mathrm{Re}(f),0) + iS_N(\mathrm{Im}(f),0)$ it follows that at least one of $\overline{\lim}_{N\to\infty}|S_N(\mathrm{Re}(f),0)|$ and $\overline{\lim}_{N\to\infty}|S_N(\mathrm{Im}(f),0)|$ is ∞ so at least one of $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ is a real valued continuous function for which the Fourier series diverges at 0.

Exercise 16.15. Let's observe that $\langle \log_{10} n\rangle \in [0,1/2]$ if and only if there exists a $p \in \mathbb{N}$ such that $10^p \le n \le 10^{p+1/2}$. Let

$$a_N = \frac{\mathrm{card}\{1 \le n \le N,\ \langle \log_{10} n\rangle \in [0,1/2]\}}{N}$$

Let $S(k) = 10^k - 1$ and $T(k) = 3\cdot 10^k$. Since $10^{1/2} < 4$ we clearly have

$$\limsup_{k\to\infty} a_{S(k)} < 4/10$$

Also

$$\liminf_{k\to\infty} a_{T(k)} > \lim_{k\to\infty}\frac{1}{3\cdot 10^k}(3\cdot 10^k - 10^k) = 2/3$$

which shows that $\{a_N\}$ does not have a limit as $N \to \infty$. In particular $\langle \log_{10} n \rangle$ are not equidistributed in $[0,1]$.

For $\varepsilon > 0$, $x \in [0,1]$ and $n \in \mathbb{N}$ $n > 0$, $|\langle \log_{10} n \rangle - x| < \varepsilon$ if and only if there exists a $p \in \mathbb{N}$ such that $n \in [10^{p+x-\varepsilon}, 10^{p+x+\varepsilon}]$. Since $\lim_{p\to\infty}(10^{p+x+\varepsilon} - 10^{p+x-\varepsilon}) = \infty$ we can pick a $p \in \mathbb{N}$ large enough such that the set $M = [10^{p+x-\varepsilon}, 10^{p+x+\varepsilon}] \bigcap \mathbb{N}$ is not empty. Any $n \in M$ will satisfy $|\langle \log_{10} n \rangle - x| < \varepsilon$.

Exercise 16.16. i) This is a different proof for the 'Riemann-Lebesgue' lemma. Obviously if f is a trigonometric polynomial then $\widehat{f}(n) = 0$ if $|n|$ is large enough.

Let $\varepsilon > 0$ and $f \in \mathrm{C}(\mathbb{T})$. Let $P(t) = \sum_{k=-N}^{N} a_k e^{ikt}$ be a trigonometric polynomial such that $\|f - P\|_\infty < \varepsilon$. Then for any $n \in \mathbb{Z}$ such that $|n| > N$, we have

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(s) - P(s))\, e^{ins} ds = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(s) - P(s))\, e^{ins} ds + \frac{1}{2\pi}\int_{-\pi}^{\pi} P(s)\, e^{ins} ds$$

so

$$|\widehat{f}(n)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|(f(s) - P(s))\, e^{ins}\right| ds \ < \ \varepsilon$$

which shows that $\lim_{n\to\infty} \widehat{f}(n) = 0$ and $\lim_{n\to-\infty} \widehat{f}(n) = 0$.

Let $f(t) = 1$. Then

$$\begin{aligned}
\int_0^{2\pi} |\sin nt|\, dt &= \frac{1}{n}\int_0^{2n\pi} |\sin u|\, du = \frac{1}{n}\sum_{k=0}^{n-1}\int_{2k\pi}^{2k\pi+2\pi} |\sin u|\, du = \frac{1}{n}\, 4n = 4 \\
&= \frac{4}{2\pi}\int_0^{2\pi} 1\, dt
\end{aligned}$$

Let now $f(t) = e^{imt}, m \in \mathbb{Z}, m \neq 0$ and $n > m$. Then, integrating by parts twice, we get:

$$\begin{aligned}
I_{m,n} &= \int_0^{2\pi} e^{imt}|\sin nt|\, dt = \sum_{k=0}^{2n-1}(-1)^k \int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} e^{imx}\sin nx\, dx \\
&= \sum_{k=0}^{2n-1}(-1)^k \int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} \left(\frac{e^{imx}}{im}\right)' \sin nx\, dx = \frac{n}{im}\sum_{k=0}^{2n-1}(-1)^{k+1}\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} e^{imx}\cos nx\, dx \\
&= \frac{n}{(im)^2}\sum_{k=0}^{2n-1}(-1)^{k+1}\left[\left(e^{imx}\cos nx\right)\Big|_{x=\frac{k\pi}{n}}^{x=\frac{(k+1)\pi}{n}} + n\int_0^{2\pi} e^{imx}\sin nx\, dx\right] \\
&= \frac{n}{(im)^2}\sum_{k=0}^{2n-1}\left(e^{im\frac{(k+1)\pi}{n}} + e^{im\frac{k\pi}{n}}\right) + \frac{n^2}{(im)^2}\sum_{k=0}^{2n-1}(-1)^{k+1}\int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} e^{imx}\sin nx\, dx
\end{aligned}$$

$$= \frac{n}{(im)^2}(e^{im\frac{\pi}{n}} + 1)\sum_{k=0}^{2n-1}\left(e^{\frac{im\pi}{n}}\right)^k + \frac{n^2}{m^2}I_{m,n} = \frac{n^2}{m^2}I_{m,n}$$

Therefore $(1-\frac{n^2}{m^2})I_{m,n} = 0$ so $I_{m,n} = 0$. Since $\int_0^{2\pi} e^{imx}dx = 0$ for any $m \neq 0$ we can conclude now that

$$\lim_{n\to\infty}\int_0^{2\pi} P(t)|\sin nt|\,dt = \frac{4}{2\pi}\int_0^{2\pi} P(t)\,dt$$

for any trigonometric polynomial $P(t) = \sum_{n=-N}^{N} a_n e^{int}$.

We can now finish the proof. Let $f : \mathbb{T} \to \mathbb{R}$ continuous and let $\varepsilon > 0$. Let $P(t) = \sum_{n=-N}^{N} a_n e^{int}$ be a trigonometric polynomial such that $\|f - P\|_\infty < \frac{\varepsilon}{6\pi}$. Let n_0 large enough such that for any $n \geq n_0$ we have

$$\left|\int_0^{2\pi} P(t)|\sin nt|\,dt - \frac{4}{2\pi}\int_0^{2\pi} P(t)\,dt\right| < \frac{\varepsilon}{3}$$

Then for any $n \geq n_0$ we have

$$\left|\int_0^{2\pi} f(t)\,|\sin nt|\,dt - \frac{4}{2\pi}\int_0^{2\pi} f(t)\,dt\right| < \int_0^{2\pi} |f(t)-P(t)|\,|\sin nt|\,dt$$
$$+\left|\int_0^{2\pi} P(t)\,|\sin nt|\,dt - \frac{4}{2\pi}\int_0^{2\pi} P(t)\,dt\right| + \frac{4}{2\pi}\int_0^{2\pi} |f(t)-P(t)|\,dt < \varepsilon$$

and therefore

$$\lim_{n\to\infty}\int_0^{2\pi} f(t)\,|\sin nt|\,dt = \frac{4}{2\pi}\int_0^{2\pi} f(t)\,dt$$

Exercise 16.17. A simple computation shows that $\int_\alpha^\beta e^{2\pi it}dt = 0$ if and only if $(\alpha - \beta) \in \mathbb{Z}$. Without loss of generality we can assume that $R = [0,a] \times [0,b]$. For any integer j, $1 \leq j \leq k$ we have $R(j) = [x_j, x_j + a_j] \times [y_j, y_j + b_j]$. Furthermore:

$$\int\int_{R_j} e^{2\pi i(x+y)}\,dx\,dy = \left(\int_{x_j}^{x_j+a_j} e^{2\pi ix}\,dx\right)\left(\int_{y_j}^{y_j+b_j} e^{2\pi iy}\,dy\right) = 0$$

since at least one of a_j and b_j is an integer. Therefore

$$\left(\int_0^a e^{2\pi ix}\,dx\right)\left(\int_0^b e^{2\pi iy}\,dy\right) = \int\int_R e^{2\pi i(x+y)}\,dx\,dy = \sum_{j=1}^{k}\int\int_{R_j} e^{2\pi i(x+y)}\,dx\,dy = 0$$

which shows that at least one of a and b is an integer.

Exercise 16.20. i) This is the Cauchy Schwarz inequality for $a = (a_{-N}, a_{-N+1}, ...a_{N-1}, a_N)$ and $b = (b_{-N}, b_{-N+1}, ...b_{N-1}, b_N)$ in $\mathbb{C}^{2N+1}$.

ii) For any $N \geq 0$

$$\sum_{j=-N}^{N} |a_j b_j| \leq \left(\sum_{j=-N}^{N} |a_j|^2\right)^{1/2} \left(\sum_{j=-N}^{N} |b_j|^2\right)^{1/2} \leq \left(\sum_{j=-\infty}^{\infty} |a_j|^2\right)^{1/2} \left(\sum_{j=-\infty}^{\infty} |b_j|^2\right)^{1/2}$$

which implies that $\{\sum_{j=-N}^{N} |a_j b_j|\}_{N\in\mathbb{N}}$ converges and

$$\sum_{j=-\infty}^{\infty} |a_j b_j| \ \leq \ \left(\sum_{j=-\infty}^{\infty} |a_j|^2\right)^{1/2} \left(\sum_{j=-\infty}^{\infty} |b_j|^2\right)^{1/2}$$

iii) For any $f \in \mathrm{C}(\mathbb{T})$ and any $j \in \mathbb{Z}$ we get, using integration by parts:

$$\widehat{f'}(j) \ = \ \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(t)\, e^{-ijt}\, dt \ = \ ij\, \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-ijt}\, dt \ = \ ij\, \widehat{f}(j)$$

Since by Plancherel theorem (Theorem 8.2. (i)) we have:

$$\sum_{j=-\infty}^{\infty} |\widehat{f'}(j)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(t) dt$$

we immediately get

$$\sum_{j=-\infty}^{\infty} j^2 |\widehat{f}(j)|^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |f'(t)|^2 dt$$

iv) Using ii) we get

$$\begin{aligned}\sum_{j=-\infty}^{\infty} |\widehat{f}(j)| \ &= \ |\widehat{f}(0)| + \sum_{|j|\geq 1} |\widehat{f}(j)|\, j\, \frac{1}{j} \ \leq \ |\widehat{f}(0)| + \left(\sum_{|j|\geq 1} |\widehat{f}(j)|^2\, j^2\right)^{1/2} \left(\sum_{|j|\geq 1} \frac{1}{j^2}\right)^{1/2} \\ &\leq |\widehat{f}(0)| + \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(t)|^2\, dt\right)^{1/2} \left(\frac{\pi^2}{3}\right)^{1/2}\end{aligned}$$

which shows that $\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|$ converges. Therefore $\sum_{j=-\infty}^{\infty} \widehat{f}(j)\, e^{ijt}$ converges uniformly to a continuous function g. It remains to show that $g = f$. But $\widehat{g}(j) = \widehat{f}(j)$ for any $j \in \mathbb{Z}$. Therefore, from Theorem 7.4 we get $g = f$.

Exercise 16.21. (Wirtinger's inequality) i) Since by hypothesis $\frac{1}{2\pi}\int_{\mathbb{T}} u(t)\,dt = 0$, we have $\widehat{u}(0) = 0$. From Plancherel's theorem and part iii) of the previous problem we have:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(u(t))^2\,dt = \sum_{j=-\infty}^{\infty}|\widehat{u}(j)|^2 = \sum_{|j|\geq 1}|\widehat{u}(j)|^2 \leq \sum_{|j|\geq 1} j^2|\widehat{u}(j)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(u'(t))^2 dt$$

Note that in the previous relation we have equality if and only if $\widehat{u}(j) = 0$ for any $|j| \geq 2$. Therefore $u(t) = \widehat{u}(1)e^{it} + \widehat{u}(-1)e^{-it}$. Since u is real valued we must have $u(t) = a\cos(t) + b\sin(t),\ a, b \in \mathbb{R}$. If $C^2 = a^2 + b^2 \neq 0$ we have

$$u(t) = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos t + \frac{b}{\sqrt{a^2+b^2}}\sin t\right) = C\cos(t+\varphi)$$

where $\varphi \in [0, 2\pi)$ is chosen such that $\sin\varphi = -\frac{b}{\sqrt{a^2+b^2}}$ and $\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}$.

ii) Let $v \in \mathrm{C}^1([0, \frac{\pi}{2}])$ be a real valued function with $v(0) = 0$ and $v'(\frac{\pi}{2}) = 0$. Let $f : [-\pi, \pi] \to \mathbb{R}$ defined by:

$$f(t) = \begin{cases} v(t) & \text{if } \ t \in \left[0, \frac{\pi}{2}\right) \\ v(\pi - t) & \text{if } \ t \in \left[\frac{\pi}{2}, \pi\right) \\ -v(-t) & \text{if } \ t \in \left[-\frac{\pi}{2}, 0\right) \\ -v(\pi + t) & \text{if } \ t \in \left[-\pi, -\frac{\pi}{2}\right) \end{cases}$$

It is easy to see that $f \in \mathrm{C}(\mathbb{T})$. For any $x_0 \in \mathbb{T}$ let's denote by $f'_-(x_0)$ the left-hand derivative of f at x_0 and by $f'_+(x_0)$ the right-hand derivative of f at x_0. Simple computations show that $f'_+(0) = f'_-(0) = v'(0)$, $f'_+(\frac{\pi}{2}) = f'_-(\frac{\pi}{2}) = 0$, $f'_-(\pi) = f'_+(-\pi) = -v'(0)$ and $f'_+(-\frac{\pi}{2}) = f'_-(-\frac{\pi}{2}) = 0$. Therefore $f \in \mathrm{C}^1(\mathbb{T})$. From part i) we get

$$\int_{-\pi}^{\pi}(f(t))^2\,dt \leq \int_{-\pi}^{\pi}(f'(t))^2\,dt$$

Since $\int_{-\pi}^{\pi}(f(t))^2 dt = 4\int_0^{\frac{\pi}{2}}(v(t))^2 dt$ and $\int_{-\pi}^{\pi}(f'(t))^2 dt = 4\int_0^{\frac{\pi}{2}}(v'(t))^2 dt$ we get

$$\int_0^{\frac{\pi}{2}}(v(t))^2\,dt \leq \int_0^{\frac{\pi}{2}}(v'(t))^2\,dt$$

for any $v \in \mathrm{C}^1(\mathbb{T})$.

The previous inequality becomes equality if and only if there exist $C \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$ such that $f(t) = C\cos(t + \varphi)$. When $C \neq 0$, since $f(0) = f'(\frac{\pi}{2}) = 0$, it follows that $\cos\varphi = 0$ and $\sin(\frac{\pi}{2} + \varphi) = 0$. Therefore $\varphi = \frac{\pi}{2}$ or $\varphi = \frac{3\pi}{2}$. In both cases we can conclude that $f(t) = C_1 \sin t$, so $v(t) = C_1 \sin t$ for a constant $C_1 \in \mathbb{R}$.

iii) Let $w \in \mathrm{C}^1([0, \frac{\pi}{2}])$ with $w(0) = 0$. Let $M > 0$ such that $|w(t)| < M$ and $|w'(t)| < M$ for any $t \in [0, \frac{\pi}{2}]$.

Let $\varepsilon > 0$, $\varepsilon < 7M^2$ and let $t_0 = \frac{\pi}{2} - \frac{\varepsilon}{7M^2}$. Define $v : [0, \frac{\pi}{2}] \to \mathbb{R}$ by

$$v(t) = \begin{cases} w(t) & \text{if} \quad t \in [0, t_0] \\ w(t_0) + \frac{1}{2}\, w'(t_0)\left(\frac{\pi}{2} - t_0\right) - \frac{1}{2}\, w'(t_0)\left(\frac{\pi}{2} - t\right)^2 \left(\frac{\pi}{2} - t_0\right)^{-1} & \text{if} \quad t \in (t_0, \frac{\pi}{2}] \end{cases}$$

It is easy to see that $v \in C^1([0, \frac{\pi}{2}]), v(0) = w(0) = 0$ and $v'(\frac{\pi}{2}) = 0$. Therefore, from part ii) we get:

$$\int_0^{\frac{\pi}{2}} (v(t))^2\, dt \le \int_0^{\frac{\pi}{2}} (v'(t))^2\, dt \qquad (1)$$

It is also easy to see that $\|v\|_\infty < 2M$ and $\|v'\|_\infty < M$. Therefore

$$\left| \int_0^{\frac{\pi}{2}} (v(t))^2\, dt - \int_0^{\frac{\pi}{2}} (w(t))^2\, dt \right| = \left| \int_{t_0}^{\frac{\pi}{2}} (v(t))^2\, dt - \int_{t_0}^{\frac{\pi}{2}} (w(t))^2\, dt \right|$$
$$\le \int_{t_0}^{\frac{\pi}{2}} (v(t))^2\, dt + \int_{t_0}^{\frac{\pi}{2}} (w(t))^2\, dt \ \le\ 5M^2 \frac{\varepsilon}{7M^2} = \frac{5\,\varepsilon}{7} \qquad (2)$$

and

$$\left| \int_0^{\frac{\pi}{2}} (v'(t))^2\, dt - \int_0^{\frac{\pi}{2}} (w'(t))^2\, dt \right| = \left| \int_{t_0}^{\frac{\pi}{2}} (v'(t))^2\, dt - \int_{t_0}^{\frac{\pi}{2}} (w'(t))^2\, dt \right|$$
$$\le \int_{t_0}^{\frac{\pi}{2}} (v'(t))^2\, dt + \int_{t_0}^{\frac{\pi}{2}} (w'(t))^2\, dt \ <\ 2M^2 \frac{\varepsilon}{7M^2} = \frac{2\,\varepsilon}{7} \qquad (3)$$

Using (1), (2) and (3) we obtain

$$\int_0^{\frac{\pi}{2}} (w(t))^2\, dt \le \int_0^{\frac{\pi}{2}} (w'(t))^2\, dt + \varepsilon$$

Since ε is arbitrarily small we can conclude that

$$\int_0^{\frac{\pi}{2}} (w(t))^2\, dt \le \int_0^{\frac{\pi}{2}} (w'(t))^2\, dt$$

Exercise 16.23. - (The Gibbs Phenomenon) i) Let $\lambda = \frac{1}{2\pi}(f_+(0) - f_-(0))$ and $g : [-\pi, \pi] \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} f(t) - \lambda F(t) & \text{if } t \neq 0 \\ \frac{1}{2}\left(f_+(0) + f_-(0)\right) & \text{if } t = 0 \end{cases}$$

It is easy to see that $g_-(0) = g_+(0) = \frac{1}{2}(f_+(0) + f_-(0))$ and therefore $g \in \mathrm{C}(\mathbb{T})$. Furthermore, g is continuously differentiable on $\mathbb{T}\backslash\{0\}$. We clearly have $f = g + \lambda F$.

ii) For $r \in \mathbb{Z}, r \neq 0$ we have

$$\begin{aligned}\widehat{F}(r) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\, e^{-irt}\, dt = \frac{1}{2\pi}\int_{-\pi}^{0}(-\pi - t)\, e^{-irt}\, dt + \frac{1}{2\pi}\int_{0}^{\pi}(\pi - t)\, e^{-irt}\, dt \\ &= \frac{1}{2ir}\int_{-\pi}^{0}(e^{-irt})'\, dt - \frac{1}{2ir}\int_{0}^{\pi}(e^{-irt})'\, dt + \frac{1}{2\pi ir}\int_{-\pi}^{\pi} t(e^{-irt})'\, dt \\ &= \frac{1}{2ir}\left(-2(\cos r\pi) + 2\right) + \frac{1}{2\pi ir}2\pi\cos r\pi = \frac{1}{ir}\end{aligned}$$

Since obviously $\widehat{F}(0) = 0$ we have

$$S_n(F, t) = \sum_{r=-n}^{n} \widehat{F}(r)\, e^{irt} = \sum_{r=1}^{n}\left(-\frac{1}{ir}\, e^{-irt} + \frac{1}{ir}\, e^{irt}\right) = 2\sum_{r=1}^{n}\frac{\sin rt}{r}$$

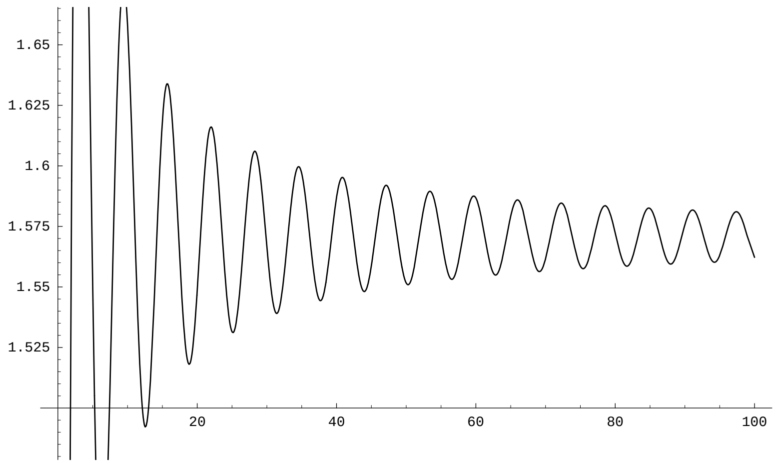
iii) For any $\tau > 0$ we have that $\mathcal{P}_n = \{r\frac{\tau}{n},\ \ 1 \le r \le n\}$ is a partition of $[0, \tau]$ in n intervals of length $\frac{\tau}{n}$. Therefore

$$S_n(F, \frac{\tau}{n}) = 2\,\frac{\tau}{n}\sum_{r=1}^{n}\frac{1}{\frac{r\tau}{n}}\,\sin\frac{r\tau}{n}$$

is the Riemann sum associated to the function $\varphi : [0, \tau] \to \mathbb{R},\ \ \varphi(x) = 2\,\frac{\sin x}{x}$, the partition $\mathcal{P}_n$ of $[0, \tau]$ and the set of intermediate points obtained by taking the right-hand endpoint from each interval $[\frac{r-1}{n}, \frac{r}{n}]$. Since the function φ is Riemann integrable on $[0, \tau]$ we get

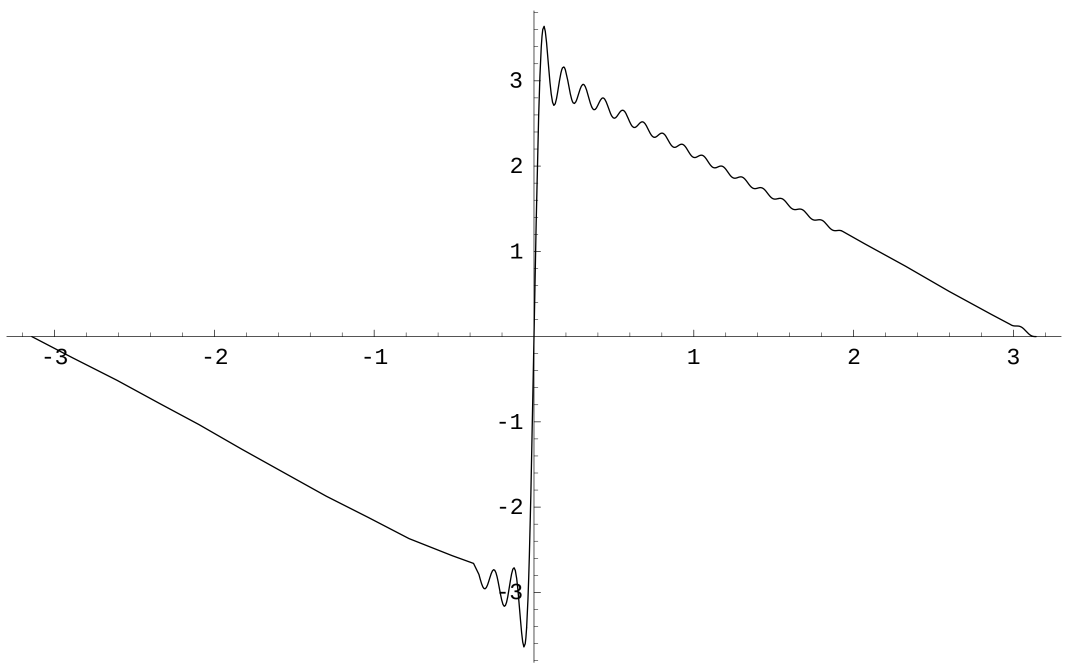
$$\lim_{n\to\infty} S_n(F, \frac{\tau}{n}) = 2\int_0^{\tau}\frac{\sin x}{x}dx$$

iv) The graph of the function $G(\tau) = \int_0^{\tau}\frac{\sin x}{x}\, dx$ is



This suggest that $\lim_{\tau\to\infty}\int_0^\tau \frac{\sin x}{x}dx$ exists. (Actually $\lim_{\tau\to\infty}\int_0^\tau \frac{\sin x}{x}dx = \frac{\pi}{2}$)

v) When τ is small and n is large we have that $t = \tau/n$ is small. The graph of $S_{50}(F,t)$ is

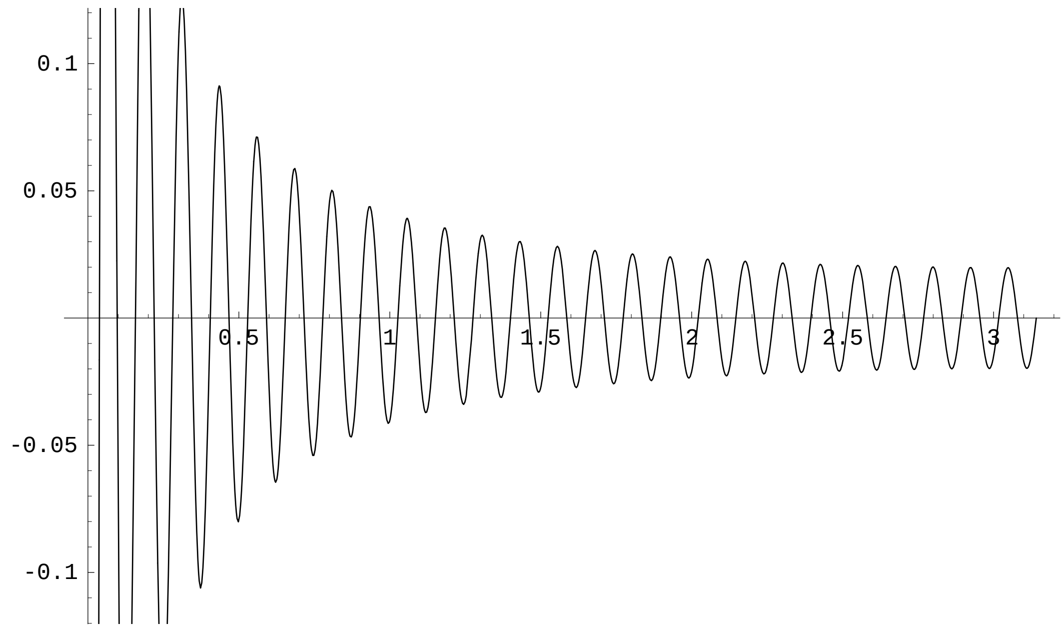


We can observe the two "bumps" near the origin which give a point of maximum to the right of 0 and a point of minimum to the left of 0.

Let's also observe that the graph of the function $h : (0, \infty) \to \mathbb{R}$ defined by

$$h(t) = S_n(F, t) - F(t)$$

is (for $n = 50$)



We can see that this graph is similar to the graph of the function G from part iv).