Topological Groups
Part III, Spring 2008

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March 8, 2008

Small print This is just a first draft for the course. The content of the course will be what I say, not what these notes say. Experience shows that skeleton notes (at least when I write them) are very error prone so use these notes with care. I should **very much** appreciate being told of any corrections or possible improvements and might even part with a small reward to the first finder of particular errors. This document is written in LaTeX2e and available in tex, dvi, ps and pdf form from my home page `http://www.dpmms.cam.ac.uk/~twk/`. My e-mail address is `twk@dpmms.cam.ac.uk`.

In the middle of the 20th century it was realised that classical Fourier Analysis could be extended to locally compact Hausdorff Abelian groups. The object of this course (which may not be completely achieved) is to show how this is done. (Specifically we wish to get as far as the first two chapters of the book of Rudin [6].) The main topics will thus be topological groups in general, Haar measure, Fourier Analysis on locally compact Hausdorff Abelian groups, Pontryagin duality and the principal structure theorem.

Although we will not need deep results, we will use elementary functional analysis, measure theory and the elementary theory of commutative Banach algebras. (If you know two out of three you should have no problems, if only one out of three then the course is probably a bridge too far.) Preliminary reading is not expected but the book by Deitmar [1] is a good introduction.

Contents

1 Prelude

We start with the following observations[1].

Lemma 1.1. *Consider* $\mathbb{R}$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ *and* $\mathbb{Z}$ *with their usual (Euclidean) metrics. Then* $\mathbb{R}$, $\mathbb{T}$ *and* $\mathbb{Z}$ *are Abelian groups under addition and* S^1 *is a group under multiplication.*

(i) The continuous homomorphisms $\chi : \mathbb{R} \to S^1$ *are precisely the maps* $\chi_a(t) = \exp(iat)$ $[t \in \mathbb{R}]$ *with* $a \in \mathbb{R}$.

(ii) The continuous homomorphisms $\chi : \mathbb{T} \to S^1$ *are precisely the maps* $\chi_a(t) = \exp(2\pi iat)$ $[t \in \mathbb{T}]$ *with* $a \in \mathbb{Z}$.

(iii) The continuous homomorphisms $\chi : \mathbb{Z} \to S^1$ *are precisely the maps* $\chi_a(t) = \exp(2\pi iat)$ $[t \in \mathbb{Z}]$ *with* $a \in \mathbb{T}$.

Exercise 1.2. *We use the notation of Lemma 1.1.*

(i) Show that the non-zero Borel measures μ *on* $\mathbb{R}$ *such that*

$$\int_{\mathbb{R}} f(x-y)\, d\mu(x) = \int_{\mathbb{R}} f(x)\, d\mu(x)$$

for all continuous functions $f : \mathbb{R} \to \mathbb{R}$ *of compact support and all* $y \in \mathbb{R}$ *are precisely the non-zero multiples of* m *the Lebesgue measure on* $\mathbb{R}$. *In other*

[1] I shall assume a lot of notation and results, not because I assume that my audience will know it all but because I assume they will know much of it. If something seems strange *just ask*.

words

$$\int_{\mathbb{R}} f(x)\,d\mu(x) = A\int_{\mathbb{R}} f(x)\,dx$$

for all continuous functions $f:\mathbb{R}\to\mathbb{C}$ *of compact support and some constant* $A\neq 0$.

(ii) Show that the unique measure μ *on* $\mathbb{T}$ *such that*

$$\int_{\mathbb{T}} f(x-y)\,d\mu(x) = \int_{\mathbb{T}} f(x)\,d\mu(x)$$

for all continuous functions $f:\mathbb{T}\to\mathbb{R}$ *and all* $y\in\mathbb{T}$ *and such that*

$$\int_{\mathbb{T}} 1\,d\mu(x) = 1$$

is $(2\pi)^{-1}m$ *where* m *is the Lebesgue measure on* $\mathbb{T}$. *In other words*

$$\int_{\mathbb{T}} f(x)\,d\mu(x) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\,dx$$

for all continuous functions $f:\mathbb{T}\to\mathbb{C}$.

(iii) Show that the unique measure μ *on* $\mathbb{Z}$ *such that*

$$\int_{\mathbb{Z}} f(x-y)\,d\mu(x) = \int_{\mathbb{Z}} f(x)\,d\mu(x)$$

for all continuous functions $f:\mathbb{Z}\to\mathbb{C}$ *of compact support and all* $y\in\mathbb{Z}$ *and such that*

$$\int_{\mathbb{Z}} I_{\{0\}}(x)\,d\mu(x) = 1$$

(where $I_{\{0\}}(0)=1$ *and* $I_{\{0\}}(x)=0$ *when* $x\neq 0$*) is the counting measure. In other words*

$$\int_{\mathbb{Z}} f(x)\,d\mu(x) = \sum_{x\in\mathbb{Z}} f(x)$$

for all continuous functions $f:\mathbb{Z}\to\mathbb{C}$.

Exercise 1.3. *We continue with the notation and ideas of Lemma 1.1 and Exercise 1.2. We write* $\chi_a(t)=\langle t,a\rangle$. *We let* $\mu_{\mathbb{R}}=(2\pi)^{-1/2}m$ *where* m *is Lebesgue measure on* $\mathbb{R}$, $\mu_{\mathbb{T}}=(2\pi)^{-1}m$ *where* m *is Lebesgue measure on* $\mathbb{T}$ *and* $\mu_{\mathbb{Z}}$ *be the counting measure on* $\mathbb{Z}$. *Let* (G,H) *be one of the pairs* $(\mathbb{R},\mathbb{R})$, $(\mathbb{T},\mathbb{Z})$ *or* $(\mathbb{Z},\mathbb{T})$. *Then*

(i) If $f\in L^1(G,\mu_G)$, *then*

$$\hat{f}(a) = \int_G f(t)\langle -t,a\rangle\,d\mu_G(t)$$

is well defined for all $a \in H$.

(ii) If $f,\ g \in L^1(G, \mu_G)$, then the convolution

$$f * g(s) = \int_G f(s-t)g(t)d\mu_G(t)$$

is defined μ_G almost everywhere and $f * g \in L^1(G, \mu_G)$.

(iii) If $f,\ g \in L^1(G, \mu_G)$, then

$$\widehat{f * g}(a) = \hat{f}(a)\hat{g}(a)$$

for all $a \in H$.

(Part (ii) uses Fubini's theorem to establish the existence of $f * g$. It is not necessary to know the results of Exercise 1.3 in the generality given but you ought to know some version of these results.)

Theorem 1.4. *We continue with the notation and ideas of Lemma 1.1 and Exercises 1.2 and 1.3. If $f \in L^1(G, \mu_G)$ is sufficiently well behaved, then*

$$f(x) = \int_H \hat{f}(a)\langle x, a\rangle \, d\mu_H(a).$$

Exercise 1.5. *Show that Theorem 1.4 is true for $(G, H) = (\mathbb{Z}, \mathbb{T})$ without any extra conditions on f (beyond that $f \in L^1(G, \mu_G)$).*

As it stands Theorem 1.4 is more of an aspiration than a theorem. It is only useful if the 'sufficiently well behaved functions' form a 'sufficiently large class'. For example, if $(G, H) = (\mathbb{T}, \mathbb{Z})$ any twice continuously differentiable function is sufficiently well behaved and if $(G, H) = (\mathbb{R}, \mathbb{R})$, then any twice differentiable function f with $x^{-2}f(x)$, $x^{-2}f'(x)$, $x^{-2}f''(x) \to 0$ as $|x| \to \infty$ is sufficiently well behaved. (We can do much better but it is well known that, in both cases, the result is not true for all continuous f.)

The object of this course is to illustrate the group theoretic content of classical Fourier analysis by extending the ideas and results of this section to the general context of locally compact groups.

Exercise 1.6. *(i) Go through the results of this section with $\mathbb{R}^2$, $\mathbb{T}^2$ and $\mathbb{Z}^2$ in place of $\mathbb{R}$, $\mathbb{T}$ and $\mathbb{Z}$ making the appropriate changes.*

(ii) Go through the results of this section with the finite cyclic group C_n.

2 Topological groups

The following method of marrying algebraic and topological structures will be routine for many readers and should strike the rest as natural.

Definition 2.1. *We say that $(G, \times, \tau)$ is a topological group if $(G, \times)$ is a group and (G, τ) a topological space such that, writing $M(x, y) = x \times y$ and $Jx = x^{-1}$ the multiplication map $M : G^2 \to G$ and the inversion map $J : G \to G$ are continuous.*

Exercise 2.2. *Show that $(G, \times, \tau)$ is a topological group if $(G, \times)$ is a group and (G, τ) a topological space such that the map $K : G^2 \to G$ given by $K(x, y) = xy^{-1}$ is continuous.*

The systems $\mathbb{R}$, S^1, $\mathbb{T}$ and $\mathbb{Z}$ discussed in Section 1 are readily seen to be examples of topological groups. Any group G becomes a topological group when equipped with the discrete or the indiscrete topology. (Note that this shows that the mere fact that something is a topological group tells us little unless we know more about the topology.)

Exercise 2.3. *Verify the statements just made.*

There is a natural definition of isomorphism.

Definition 2.4. *If $(G, \times_G, \tau_G)$ and $(H, \times_H, \tau_H)$ are topological groups we say that $\theta : G \to H$ is an isomorphism if it is a group isomorphism and a topological homeomorphism.*

Exercise 2.5. *Show that $\mathbb{T}$ and S^1 are isomorphic as topological groups. Find all the isomorphisms.*

The next result gives us a source of interesting non-commutative topological groups.

Lemma 2.6. *Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. The multiplicative group $GL_n(\mathbb{F})$ of invertible $n \times n$ matrices with the topology induced by the usual norm is a topological group.*

Exercise 2.7. *Prove the following generalisation of Lemma 2.6. Let U be a Banach space. If we give the multiplicative group $GL(U)$ of continuous linear maps $T : U \to U$ with continuous inverses the topology induced by the operator norm then we obtain a topological group.*

It is, perhaps, worth noting that we can not replace joint continuity of multiplication by left and right continuity (this contrasts with the Banach algebra case).

Exercise 2.8. *Let τ be the usual topology on $\mathbb{R}$ and let τ_1 be the set of $U \in \tau$ such that there exists a $K > 0$ such that*

$$U \supseteq \mathbb{R} \setminus (-K, K)$$

together with the empty set.

Then τ_1 is a topology on $\mathbb{R}$ such that the map $x \mapsto -x$ and the map $x \mapsto x + a$ are continuous (as maps $(\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_1)$) for all $a \in \mathbb{R}$ but the map $(x, y) \mapsto x + y$ is not continuous (as a map $(\mathbb{R}^2, \tau_1^2) \to (\mathbb{R}, \tau_1)$).

Similarly, joint continuity of multiplication does not imply continuity of the inverse.

Exercise 2.9. *Let τ be the usual topology on $\mathbb{R}$ and let τ_2 be the set of $U \in \tau$ such that there exists a K such that*

$$U \supseteq (K, \infty)$$

together with the empty set.

Then τ_2 is a topology on $\mathbb{R}$ such that the map $(x, y) \mapsto x+y$ is continuous (as a map $(\mathbb{R}^2, \tau_1^2) \to (\mathbb{R}, \tau_1)$) but the map $x \to -x$ is not continuous (as a map $(\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_1)$).

Returning to more central matters we make the following simple but basic observations.

Lemma 2.10. *Let $(G, \times, \tau)$ be a topological group. Then*

(i) $xU = \{xu : u \in U\}$ is open if and only if U is.

(ii) V is a neighbourhood of x if and only if $x^{-1}V$ is a neighbourhood of e.

(Here and elsewhere we will use e to denote the unit of a multiplicative group and 0 to denote the unit of an additive one.)

Lemma 2.11. *Suppose that $(G, \times)$ is a group and $\mathcal{N}_e$ is a collection of sets N with $e \in N \subseteq G$. Then there exists a topology τ on G having $\mathcal{N}_e$ as a neighbourhood basis of e and making $(G, \times, \tau)$ into a topological group if and only if*

(1) If N, $M \in \mathcal{N}_e$, then there exists a $P \in \mathcal{N}_e$ with $P \subseteq N \cap M$.

(2) If $N \in \mathcal{N}_e$, then there exists an $M \in \mathcal{N}_e$ with $M^2 \subseteq N$.

(3) If $N \in \mathcal{N}_e$, then there exists an $M \in \mathcal{N}_e$ with $M \subseteq N^{-1}$.

(4) If $N \in \mathcal{N}_e$ and $a \in G$, then there exists an $M \in \mathcal{N}_e$ with $M \subseteq aNa^{-1}$.

Exercise 2.12. *Suppose that* $(G,\times)$ *is a group and* (G,τ) *a topological space. Then* $(G,\times,\tau)$ *is a topological group if and only if, writing* $\mathcal{N}_a$ *for the set of open neighbourhoods of* a*, we have*

(1) Let $a \in G$*. Then* $N \in \mathcal{N}_e$ *if and only if* $aN \in \mathcal{N}_a$*.*
(2) If $N \in \mathcal{N}_e$*, then there exists an* $M \in \mathcal{N}_e$ *with* $M^2 \subseteq N$*.*
(3) If $N \in \mathcal{N}_e$*, then there exists an* $M \in \mathcal{N}_e$ *with* $M \subseteq N^{-1}$*.*
(4) If $N \in \mathcal{N}_e$ *and* $a \in G$*, then there exists an* $M \in \mathcal{N}_e$ *with* $M \subseteq aNa^{-1}$*.*

From time to time it is useful to have neighbourhood bases with further properties.

Lemma 2.13. *If* $(G,\times,\tau)$ *is a topological group we can find a neighbourhood basis* $\mathcal{N}_e$ *for* e *consisting of open sets* N *with* $N^{-1} = N$*.*

The following example shows that we can not choose M in condition (4) of Lemma 2.11 and Exercise 2.12 independently of a.

Example 2.14. *Given any* $\epsilon > 0$ *and any* $K > 0$ *we can find* $A,\ B \in GL_2(\mathbb{R})$ *such that* $\|I - B\| < \epsilon$ *but* $\|I - ABA^{-1}\| > K$*.*

In some sense a topological group is an object were the the statement 'the neighbourhood of every point looks the same' is (a) meaningful and (b) true. (There is a more general notion of a uniform topology.)

Exercise 2.15. *Let* $(G,\times,\tau)$ *be a topological group. What ought it to mean to say that* $f : G \to \mathbb{C}$ *is right (or left) uniform continuous. If* $(G,\times,\tau)$ *is a compact topological group, show that any continuous function* $f : G \to \mathbb{C}$ *is right uniformly continuous.*

3 Subgroups and quotients

It is easy to define topological subgroups and quotient groups along the lines given in the next lemma.

Lemma 3.1. *If* $(G,\times,\tau)$ *is a topological group and* H *is a subgroup of* G*, then* H *equipped with the standard subspace topology is a topological group.*

If H *is a normal subgroup of* G *then* G/H *equipped with the standard quotient space topology (formally, the finest topology on* G/H *which makes the map* $G \to G/H$ *given by* $x \mapsto xH$ *continuous) is a topological group.*

However, it is important to realise that, without further conditions quotient topological groups may not behave well.

Exercise 3.2. *Consider* $(\mathbb{R}, +, \tau)$ *with the usual Euclidean topology* τ*. Show that* $\mathbb{Q}$ *is a normal subgroup and* $\mathbb{R}/\mathbb{Q}$ *is uncountable but has the indiscrete topology.*

Let us look a little closer at how a subgroup can sit in a topological group.

Lemma 3.3. *Let* $(G, \times, \tau)$ *be a topological group and* H *a subgroup of* G*.*
(i) The (topological) closure $\bar{H}$ *of* H *is a subgroup.*
(ii) If H *is normal, so is* $\bar{H}$*.*
(iii) If H *contains an open set, then* H *is open.*
(iv) If H *is open, then* H *is closed.*
(v) If H *is closed and of finite index in* G*, then* H *is open.*

Lemma 3.4. *Let* $(G, \times, \tau)$ *be a topological group. The connected component* L *containing* e *is an open normal subgroup.*
Any open subgroup of G *contains* L*.*

Lemma 3.5. *If* $(G, \times, \tau)$ *is a topological group then* $I = \overline{\{e\}}$ *is a closed normal subgroup.*

Lemma 3.6. *Let* $(G, \times, \tau)$ *is a topological group. The following conditions are equivalent.*
(i) $\{e\}$ *is closed.*
(ii) G *is Hausdorff.*
(iii) G *is* T_0*.*

(Part (iii) is just an observation. Note that the spaces in Exercises 2.8 and 2.9 are T_1 but not Hausdorff.)

Lemma 3.7. *Let* $(G, \times, \tau)$ *be a topological group. If* H *is a closed normal subgroup, then* G/H *is Hausdorff.*

Lemma 3.8. *Let* $(G, \times, \tau)$ *be a topological group. Then* $G/\overline{\{e\}}$ *is Hausdorff.*

If G is a topological group, then any continuous function $f : G \to \mathbb{F}$ must be constant on any coset $x + \overline{\{e\}}$ so it is clear that, if we are interested in continuous functions, nothing is lost by confining ourselves to Hausdorff topological groups. Rudin defines a topological group to be what we would call a Hausdorff topological group and Bourbaki defines a compact space to be what we would call a Hausdorff compact space. Unless we explicitly state otherwise all our topological groups will be Hausdorff topological groups and we will only quotient by closed normal subgroups.

4 Products

We shall be interested in two different kinds of products for topological groups.

Definition 4.1. *Suppose A is non-empty and $(G_\alpha, \times_\alpha, \tau_\alpha)$ is a topological group for each $\alpha \in A$. Their* **complete direct product** *is $G = \prod_\alpha G_\alpha$ equipped with the usual product topology τ_G and with multiplication given by $(\mathbf{x} \times_G \mathbf{y})_\alpha = x_\alpha \times_\alpha y_\alpha$.*

Definition 4.2. *Suppose A is non-empty and $(G_\alpha, \times_\alpha)$ is a group for each $\alpha \in A$. Their* **direct product** *is the subgroup of the group $G = \prod_\alpha G_\alpha$ with multiplication given by $(\mathbf{x} \times_G \mathbf{y})_\alpha = x_\alpha \times_\alpha y_\alpha$ consisting of those $\mathbf{x} \in G$ such that $x_\alpha = e$ for all but finitely many $\alpha \in A$*

Our definition of direct product does not involve topology but we shall usually give the direct product the discrete topology.

Exercise 4.3. *(i) Verify that the complete direct product of topological groups is indeed a topological group.*

(ii) Verify that the direct product of topological groups is indeed a group.

(iii) Show that the complete direct product of Abelian topological groups is Abelian.

(iv) Show that the direct product of Abelian groups is Abelian.

(v) Show that the complete direct product of Hausdorff topological groups is Hausdorff.

(vi) Explain why the complete direct product of compact topological groups is compact.

We illustrate these ideas by looking at D^∞ the complete direct product of G_1, G_2, ... where each G_i is a copy of D_2 the additive group of two elements 0 and 1 and D_0^∞ the direct product of H_1, H_2, ..., where each H_i is a copy of C_2 the multiplicative group of two elements -1 and 1, equipped with the discrete topology. (Of course, D_2 and C_2 are isomorphic as groups.)

Lemma 4.4. *D^∞ is homeomorphic as a topological space to the Cantor set. It is compact, Hausdorff, totally disconnected and has no isolated points.*

Lemma 4.5. *(i) The continuous homomorphisms $\chi : D^\infty \to S^1$ are precisely the maps $\chi_{\mathbf{a}}(\mathbf{t}) = \prod_{j=1}^\infty a_j^{t_j}$ $[\mathbf{t} \in D^\infty]$ with $\mathbf{a} \in D_0^\infty$.*

(ii) The continuous homomorphisms $\chi : D_0^\infty \to S^1$ are precisely the maps $\chi_{\mathbf{t}}(\mathbf{a}) = \prod_{j=1}^\infty a_j^{t_j}$ $[\mathbf{a} \in D_0^\infty]$ with $\mathbf{t} \in D^\infty$.

Exercise 4.6. *There is a unique Borel measure* μ *on* D^∞ *such that*

$$\int_{D^\infty} f(\mathbf{x}-\mathbf{y})\,d\mu(\mathbf{x}) = \int_{D^\infty} f(\mathbf{x})\,d\mu(\mathbf{x})$$

for all continuous functions $f : D^\infty \to \mathbb{R}$ *and all* $\mathbf{y} \in D^\infty$ *and* $\int_{D^\infty} 1\,d\mu(\mathbf{x}) = 1$.

If we write $K_m = \{\mathbf{x} \in D^\infty : x_1 = x_2 = \cdots = x_m = 1\}$, *then* $\mu(K_m) = 2^{-m}$.

Exercise 4.7. *We continue with the notation of this section. We write* $\langle \mathbf{t}, \mathbf{a}\rangle = \prod_{j=1}^\infty a_j^{t_j}$ *when* $\mathbf{t} \in D^\infty$ *with* $\mathbf{a} \in D_0^\infty$. *We let* μ *be the measure of Exercise 4.6 and* μ_0 *be the counting measure on* D_0^∞. *Then*

(i) If $f \in L^1(D^\infty, \mu)$, *then*

$$\hat{f}(\mathbf{a}) = \int_{D^\infty} f(\mathbf{t})\langle -\mathbf{t}, \mathbf{a}\rangle\,d\mu(\mathbf{t})$$

is well defined for all $\mathbf{a} \in D_0^\infty$.

(ii) If $f,\ g \in L^1(D^\infty, \mu)$, *then the convolution*

$$f * g(\mathbf{s}) = \int_{D^\infty} f(\mathbf{s}-\mathbf{t})g(\mathbf{t})d\mu(\mathbf{t})$$

is defined μ *almost everywhere and* $f * g \in L^1(D^\infty, \mu)$.

(iii) If $f,\ g \in L^1(D^\infty, \mu)$, *then*

$$\widehat{f * g}(\mathbf{a}) = \hat{f}(\mathbf{a})\hat{g}(\mathbf{a})$$

for all $\mathbf{a} \in D_0^\infty$.

Theorem 4.8. (Inversion theorem for D^∞) *We continue with the notation of this section. If* $f : D^\infty \to \mathbb{C}$ *has the form*

$$f = \sum_{m=1}^{M} a_m \mathbb{I}_{x_m + K_{n(m)}}$$

(that is, f *is a step function constant on cosets of subgroups of the type* K_m *described in Exercise 4.6), then*

$$f(\mathbf{x}) = \int_{D_0^\infty} \hat{f}(\mathbf{a})\langle \mathbf{x}, \mathbf{a}\rangle\,d\mu_0(\mathbf{a}).$$

Theorem 4.8 asserts that an inversion theorem is true for functions of a certain well behaved type. At first sight the reader might feel that the class involved is too small to be useful but the next result shows that this is not the case.

Theorem 4.9. (A Plancherel theorem for D^∞) *If $f \in L^2(D^\infty, \mu)$, then $\hat{f} \in L^2(D_0^\infty, \mu_0)$ and*

$$\int_{D_0^\infty} |\hat{f}(\mathbf{a})|^2 d\mu_0(\mathbf{a}) = \int_{D^\infty} |f(\mathbf{x})|^2 d\mu(\mathbf{x}).$$

Exercise 4.10. (Parseval's theorem for D^∞.) *If $f,\ g \in L^2(D^\infty, \mu)$, then $fg^* \in L^1(D^\infty, \mu)$, $\hat{f}(\hat{g})^* \in L^1(D_0^\infty, \mu_0)$ and*

$$\int_{D_0^\infty} (\hat{f})(\mathbf{a})(\hat{g})(\mathbf{a})^* d\mu_0(\mathbf{a}) = \int_{D^\infty} f(\mathbf{x})g(\mathbf{x})^* d\mu(\mathbf{x}).$$

In many ways D^∞ is the simplest non-trivial Hausdorff compact topological group and, if a result is hard for $\mathbb{T}$ it may well be easier to prove or understand for D^∞.

Exercise 4.11. *If $f \in L^1(D_0^\infty, \mu_0) \cap L^2(D_0^\infty, \mu_0)$, then $\hat{f} \in L^2(D^\infty, \mu)$ and*

$$\int_{D^\infty} |\hat{f}(\mathbf{a})|^2 d\mu(\mathbf{a}) = \int_{D_0^\infty} |f(\mathbf{x})|^2 d\mu_0(\mathbf{x}).$$

(This is not very deep. Essentially we repeat the the easy proof of exercise 1.5 but remember that $L^2(D_0^\infty, \mu_0)$ is not a subset of $L^1(D_0^\infty, \mu_0)$.)

The reader is probably familiar with the classical version of the next step.

Theorem 4.12. *There exists a linear isometry $\mathcal{F} : L^2(D_0^\infty, \mu_0) \to L^2(D^\infty, \mu)$ such that $\mathcal{F}(f) = \hat{f}$ whenever $f \in L^1(D_0^\infty, \mu_0) \cap L^2(D_0^\infty, \mu_0)$.*

Exercise 4.13. (Parseval's theorem for D_0^∞.) *If $f,\ g \in L^2(D_0^\infty, \mu_0)$, then $fg^* \in L^1(D_0^\infty, \mu_0)$, $\mathcal{F}f\mathcal{F}g^* \in L^1(D^\infty, \mu)$ and*

$$\int_{D^\infty} (\mathcal{F}f)(\mathbf{a})(\mathcal{F}g)(\mathbf{a})^* d\mu(\mathbf{a}) = \int_{D_0^\infty} f(\mathbf{x})g(\mathbf{x})^* d\mu_0(\mathbf{x}).$$

5 Metrisability

The following easy result is well known.

Exercise 5.1. *If (X, d) is a metric space and τ is the topology derived from the metric, then (X, τ) is Hausdorff and given any $x \in X$ we can find open neighbourhoods N_j of x such that $\bigcap_{j=1}^{\infty} N_j = \{x\}$.*

The converse is false.

Example 5.2. *We work in $\mathbb{R}$. Let us set*

$$\mathcal{N}_x = \{(x-\delta, x+\delta) \cap \mathbb{Q} \,:\, \delta > 0\} \qquad \text{if } x \in \mathbb{Q},$$
$$\mathcal{N}_x = \{(x-\delta, x+\delta) \,:\, \delta > 0\} \qquad \text{otherwise.}$$

Then there is a unique topology τ on $\mathbb{R}$ with the $\mathcal{N}_x$ as neighbourhood bases. The topology τ is Hausdorff and given any $x \in \mathbb{R}$ we can find open neighbourhoods N_j of x such that $\bigcap_{j=1}^{\infty} N_j = \{x\}$. However there is no metric on $\mathbb{R}$ which will induce the topology.

The homogeneity imposed by the group structure means that the necessary condition introduced in Exercise 5.1 is actually sufficient. Indeed, we have an even stronger result.

Theorem 5.3. *Let $(G, \times, \tau)$ be a topological group. If we can find a base of neighbourhoods of N_j of e such that $\bigcap_{j=1}^{\infty} N_j = \{e\}$, then there exists a metric d on G which induces τ. Moreover, we can take d left invariant, that is to say $d(gx, gy) = d(x, y)$ for all x, y, $g \in G$.*

The method of proof is illustrated by the following exercise.

Exercise 5.4. *We work on $\mathbb{R}$.*

(i) Suppose that $|x| > 2^{-k}$ for some integer $k \geq 0$. Show, by induction on N, or otherwise, that if

$$x = \sum_{j=1}^{N} x_j \text{ with } |x_j| \leq 4^{-n(j)} \text{ for some integer } n(j) \geq 0 \ [1 \leq j \leq N],$$

then $\sum_{j=1}^{N} 2^{-n(j)} \geq 2^{-k}$.

(ii) Show that

$$d(x, y) = \inf\{\sum_{j=1}^{N} 2^{-n(j)} : x - y = \sum_{j=1}^{N} x_j \text{ with } |x_j| \leq 4^{-n(j)} \text{ for some integer } n(j) \geq 0 \ [1 \leq j \leq N]\}$$

defines a metric on $\mathbb{R}$ *such that* $d(x+a, y+a) = d(x,y)$ *for all* x, y, $a \in \mathbb{R}$.

(iii) Show that we can find a constant $K > 1$ *such that*

$$K|x-y|^{1/2} \geq d(x,y) \geq K^{-1}|x-y|^{1/2}$$

for all x, $y \in \mathbb{R}$.

(iv) Sketch the graph of the function $x \mapsto d(0,x)$.

We make the following remarks (recall Example 2.14).

Exercise 5.5. *(i) There exist topological groups with a topology induced by a left invariant metric where the topology is not induced by a left and right invariant metric. (Briefly, a metrisable topological group may not have a metric which is both left and right invariant.)*

(ii) The metric of Theorem 5.3 is not unique (unless G *is the trivial group* $\{e\}$*).*

Exercise 5.6. *Let* A *be uncountable and, for each* $\alpha \in A$ *let* G_α *be a copy of* D_2 *(the additive group of two elements* 0 *and* 1*) equipped with the discrete topology. Then the complete direct product of the* G_α *is a non-metrisable compact Abelian group.*

In the rest of this section we introduce a simple but useful definition.

Definition 5.7. *(i) A topological space* (X, τ) *is locally compact if every point of* X *has a compact neighbourhood.*

(ii) A topological space (X, τ) *is* σ*-compact if it is the countable union of compact sets.*

Lemma 5.8. *Suppose that* (X, τ) *is locally compact and Hausdorff. If we consider* $X \cup \{\infty\}$ *with a topology* τ_∞ *in which the open sets are the open sets of* τ *together with the sets* $\{\infty\} \cup (X \setminus K)$ *with* K *compact in* τ *then* $(X \cup \{\infty\}, \tau_\infty)$ *(the 'one point compactification' of* X*) is a compact Hausdorff space.*

Combined with Urysohn's lemma which we quote without proof as Theorem 5.9 this gives us a plentiful supply of continuous functions.

Theorem 5.9. (Urysohn's lemma.) *If* (X, τ) *is a compact Hausdorff space and* E_1 *and* E_2 *are disjoint closed sets we can find a continuous function* $f : X \to \mathbb{R}$ *such that* $1 \geq f(x) \geq 0$ *for all* $x \in X$, $f(e) = 1$ *for all* $e \in E_1$ *and* $f(e) = 0$ *for all* $e \in E_0$.

Lemma 5.10. *Every point in a locally compact space has a basis of compact neighbourhoods.*

Since the discrete topology is locally compact, locally compact groups can be very big indeed. However for many purposes we can restrict ourselves to σ-compact topological groups.

Lemma 5.11. *Let $(G, \times, \tau)$ be a locally compact Hausdorff topological group. If K_j is a compact subset of G for $j \geq 1$, then there exists an open (so closed) subgroup H of G which is σ-compact and such that $H \supseteq \bigcup_{j=1}^{\infty} K_j$.*

We may also be able to restrict ourselves to metrisable locally compact groups.

Lemma 5.12. *Let $(G, \times, \tau)$ be a σ-compact Hausdorff topological group. If N_j is an open neighbourhood of e for $j \geq 1$, then we can find a closed normal subgroup H of G with $H \subseteq \bigcap_{j=1}^{\infty} N_j$ and G/H metrisable and σ-compact.*

6 The Haar integral

The natural (certainly, a natural) requirement for a topology for which we wish to develop a theory of continuous functions and integrals is that it should be Hausdorff and locally compact.

Definition 6.1. *Let $(G, \times, \tau)$ be a Hausdorff locally compact group. Write $C_{00}(G)$ for the collection of continuous functions $f : G \to \mathbb{R}$ with compact support and $C_{00}^{+}(G)$ for the set of $f \in C_{00}(G)$ such that $f(x) \geq 0$ for all $x \in G$. If $f \in C_{00}(G)$ and $y \in G$ we write $f_y(x) = f(y^{-1}x)$ for all $x \in G$. A non-zero linear map $I : C_{00}(G) \to \mathbb{R}$ such that $If \geq 0$ when $f \in C_{00}^{+}(G)$ and $If_y = If$ for all $y \in G$ and $f \in C_{00}(G)$ is called a left invariant Haar integral.*

Measure theory gives us much more powerful weapons than those developed in the next exercise but for the moment we do not need them.

Exercise 6.2. *Let $(G, \times, \tau)$ be a Hausdorff locally compact group and let I, $J : C_{00}(G) \to \mathbb{R}$ be non-zero linear maps such that $If \geq 0$ and $Jf \geq 0$ when $f \in C_{00}^{+}(G)$.*

(i) Given any compact set K we can find a constant $\gamma(K)$ such that $If \leq \gamma(K)\|f\|_{\infty}$ for all $f \in C_{00}^{+}(G)$ with support in K.

(ii)[2] *Suppose that K is a compact set in G and H is a compact neighbourhood of e. If $F : G \times G \to \mathbb{R}$ is a continuous function with support in*

[2] In the course I proved a slightly harder result, but this is all we need for (iii) and (iv).

$K \times K$, then given $\epsilon > 0$ we can find $u_j,\ v_j \in C_{00}(G)$ with support in KH such that

$$|F(x,y) - \sum_{j=1}^{N} u_j(x)v_j(y)| \leq \epsilon$$

for all $x,\ y \in G$.

(iii) If $F : G \times G \to \mathbb{R}$ is a continuous function with compact support and $y \in G$, then (using dummy variable notation) $x \mapsto J_y F(x,y)$ is a continuous function of compact support. Thus $I_x J_y F(xy)$ exists.

(iv) $F : G \times G \to \mathbb{R}$ is a continuous function with compact support, then

$$I_x J_y F(x,y) = J_y I_x F(x,y)$$

(v) If I is a Haar measure $g \in C_{00}^{+}(G)$ and $Ig = 0$ then $g = 0$.

Theorem 6.3. *If a left invariant Haar integral exists it is unique up to multiplication by a strictly positive constant.*

Here are some examples. (Note that if G is Abelian a left invariant Haar integral must be right invariant.)

Exercise 6.4. *(i) Show that the set G of all matrices of the form*

$$\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$$

with x and y real and $x > 0$ with the usual norm gives rise to a locally compact metrisable Abelian group.

By informal but reasonably coherent arguments show that a Haar integral must be a multiple of

$$\int_G f\, d\mu = \int_{-\infty}^{\infty} \int_0^{\infty} f \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \frac{1}{x^2}\, dx\, dy.$$

Verify that this is indeed a Haar integral.

(ii) Show that the set G of all matrices of the form

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$$

with x and y real and $x > 0$ gives rise to a locally compact metrisable non-Abelian group.

By informal but reasonably coherent arguments show that the left invariant Haar integral must be a multiple of

$$\int_G f\, d\mu = \int_{-\infty}^{\infty} \int_0^{\infty} f \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \frac{1}{x^2}\, dx\, dy.$$

Verify that this is indeed a left invariant Haar integral.

By informal but reasonably coherent arguments show that the right invariant Haar integral must be a multiple of

$$\int_G f\,d\mu = \int_{-\infty}^{\infty}\int_0^{\infty} f\begin{pmatrix} x & y \\ 0 & 1\end{pmatrix}\frac{1}{x}\,dx\,dy.$$

Verify that this is indeed a right invariant Haar integral.

Once the ideas of Exercise 6.4 have been understood it is fairly easy to extend them to more complicated situations.

Exercise 6.5. *(i) Show that $GL(\mathbb{R}^n)$ is a locally compact group which is non-Abelian if $n \geq 2$.*

If we identify $GL(\mathbb{R}^n)$ as a subset of $\mathbb{R}^{n^2}$ in the usual manner and let m be Lebesgue measure on $\mathbb{R}^{n^2}$ show by informal but reasonably coherent arguments that a left invariant Haar integral must be a multiple of

$$\int_G f(A)\,d\mu = \int_{\mathbb{R}^{n^2}} f(A)|\det A|^{-n} dm$$

where we define $f(A) = 0$ when $A \notin GL(\mathbb{R}^n)$. Verify that this is both a left and a right invariant Haar integral.

(ii) Consider the set $GA(\mathbb{R}^n)$ of invertible affine transformations $\mathbf{x} \to \mathbf{t} + A\mathbf{x}$ with $\mathbf{t} \in \mathbb{R}^n$ and $A \in GL(\mathbb{R}^n)$.

If we identify $GA(\mathbb{R}^n)$ as a subset of $\mathbb{R}^{n+n^2}$ in the usual manner and let m be Lebesgue measure on $\mathbb{R}^{n+n^2}$ show by informal but reasonably coherent arguments that the left invariant Haar integral must be a multiple of

$$\int_G f(A)\,d\mu_L = \int_{\mathbb{R}^{n+n^2}} f(\mathbf{t}, A)|\det A|^{-n} dm$$

and the right invariant Haar integral must be a multiple of

$$\int_G f(A)\,d\mu_R = \int_{\mathbb{R}^{n+n^2}} f(A)|\det A|^{-n-1} dm$$

(with natural conventions). Verify that we have indeed left invariant and right invariant Haar integrals.

The following remarks range from the obvious to the fairly obvious.

Lemma 6.6. *(i) Let $(G, \times, \tau)$ be a Hausdorff locally compact group. If G has a left invariant Haar integral it has a right invariant Haar integral.*

(ii) Suppose I is a left invariant Haar integral for G. If $t \in G$ there exists a $\Delta(t) > 0$ such that

$$I_x f(xt) = \Delta(t) I_x f(x)$$

for all $f \in C_{00}(G)$.

(iii) The function $\Delta : G \to (0, \infty)$ depends only on G (and not on the particular choice of I).

The function Δ is called the modular function. (Although we discuss it briefly, we do not use it.)

The statement that $\Delta = 1$ is equivalent to the statement that left invariant Haar integrals are right invariant and vice versa. A group with this property is called unimodular.

Exercise 6.7. *Let $(G, \times, \tau)$ be a Hausdorff locally compact group with a left invariant Haar measure.*

(i) Give $(0, \infty)$ its standard multiplicative structure and topology. The map $\Delta : G \to (0, \infty)$ is a continuous homomorphism.

(ii) If G is Abelian or compact or discrete, then it is unimodular.

(iii) If I is a left invariant Haar integral for G, then

$$I_x(f(x^{-1})\Delta(x^{-1})) = I_x f(x).$$

Exercise 6.8. *Calculate the modular function for the groups of Exercises 6.4 and 6.5.*

7 Existence of the Haar integral

The main business of this section is the proof of the following theorem.

Theorem 7.1. *Any metrisable σ-compact topological group $(G, \times, \tau)$ has a left invariant Haar integral.*

We need a preliminary result.

Lemma 7.2. *(i) Any σ-compact metric space has a countable dense subset.*

(ii) If G is a σ-compact metric group then we can find a countable collection of compactly supported, everywhere non-negative, continuous functions $g_i : X \to \mathbb{R}$ with the following property. If $\epsilon > 0$, H is a compact neighbourhood of e and $g : X \to \mathbb{R}$ is a non-negative continuous function vanishing outside a compact set K we can find $i \geq 1$ such that g_i vanishes outside KH and

$$|g(x) - g_i(x)| < \epsilon$$

for all $x \in X$

From now on until the completion of the proof of Theorem 7.1, $(G, \times, \tau)$ will be a σ-compact topological group with topology derived from a metric d. If $u : G \to \mathbb{R}$ we write $(T_y u)(x) = u(xy)$ for $x, y \in G$.

Although the discovery of the Haar integral appears to have been unexpected, much of the construction follows very natural lines. If f and ϕ are members of $C_{00}^+(G)$ and ϕ is non zero we define

$$(f;\phi) = \inf\{\sum_{j=1}^{n} c_j \, : \, \sum_{j=1}^{n} c_j \phi(y_j x) \geq f(x) \text{ for all } x \in G, \ c_j \geq 0, y_j \in G, in \geq 1\}.$$

The following results are routine.

Lemma 7.3. *Let f, f_1, f_2 be members of $C_{00}^+(G)$ and let ϕ, ψ be non zero members of $C_{00}^+(G)$. Let $y \in G$ and $\lambda > 0$. Then*
(i) $(f;\phi)$ is well defined.
(ii) $(f_1 + f_2;\phi) \leq (f_1;\phi) + (f_2;\phi)$.
(iii) $(\lambda f;\phi) = \lambda(f;\phi)$.
(iv) If $f_1(x) \leq f_2(x)$ for all $x \in G$ then $(f_1;\phi) \leq (f_2;\phi)$.
(v) $(T_y f;\phi) = (f;\phi)$.
(vi) $(f;\psi) \leq (f;\phi)(\phi;\psi)$.
(vii) $(f;\phi) \geq \|f\|_\infty/\|\phi\|_\infty$.

We need to normalise the 'upper approximation' $(f;\phi)$. To do this, fix, once and for all, f_0 as a particular non-zero element of $C_{00}^+(G)$ and set

$$I_\phi f = \frac{(f;\phi)}{(f_0;\phi)}$$

Exercise 7.4. *Interpret the results of Lemma 7.3 in terms of I_ϕ and I_ψ. In particular note that*

$$\frac{1}{(f_0, f)} \leq I_\phi f \leq (f; f_0)$$

whenever f is non-zero.

As might be hoped the 'quality' of the 'upper approximation' is improved by taking ϕ of small support. (Note, however, that the approximation is not uniform.)

Lemma 7.5. *Given f_1, $f_2 \in C_{00}^+(G)$ and $\epsilon > 0$, we can find a neighbourhood V of e such that if $\phi \in C_{00}^+(G)$ is non-zero and* $\operatorname{supp} \phi \subseteq V$ *then*

$$I_\phi f_1 + I_\phi f_2 \leq I_\phi(f_1 + f_2) + \epsilon.$$

A sequential compactness argument now gives Theorem 7.1 (the existence of a left invariant Haar integral on a σ-compact metrisable group).

Using Lemmas 5.12 and 5.11 we obtain first Lemma 7.6 and then the general Theorem 7.7

Lemma 7.6. *Any σ-compact Hausdorff topological group $(G, \times, \tau)$ has a left invariant Haar integral.*

Theorem 7.7. *Any locally compact Hausdorff topological group $(G, \times, \tau)$ has a left invariant Haar integral.*

Note: The standard argument for obtaining the Haar integral on a general Hausdorff locally compact group from Lemma 7.5 uses Tychonoff's theorem and so the axiom of choice (see [5]). Our treatment uses countable choice. Cartan produced a proof which avoids any use of the axiom of choice (see [5] again).

The reader probably does not need to be told how remarkable and useful Theorem 7.7 is. The absence of such a result makes possible the following phenomenon.

Example 7.8. *If G is the group freely generated by two generators then we can find a function $f : G \to \mathbb{R}$ such that $1 \geq f(x) \geq 0$ for all $x \in G$, together with y_1, y_2, y_3, $y_4 \in G$ such that*

$$f(y_1x) + f(y_2x) - f(y_3x) - f(y_4x) \leq -1$$

for all $x \in G$.

8 The space $L^1(G)$

Once we have a Haar integral I on a locally compact Hausdorff topological group we can develop the standard theory of integration. The quickest way is simply to define $L^1(G)$ as the completion of the space $C_{00}(G)$ normed by

$$\|f\|_1 = I(|f|).$$

This would cover all the integration theory that we need but does not connect the integral with the underlying space. The natural path is to define

$$\lambda(K) = \inf\{If \ : \ f \in C_{00}(G), \ f(x) \geq 1 \text{ for } x \in K\}$$

whenever K is a compact set and show that λ satisfies the appropriate consistency conditions which allow it to be extended to a measure μ on all Borel sets such that

$$If = \int_G f \, d\mu$$

for all $f \in C_{00}(G)$. (For details, see for example, [3] Chapter X.)

An important consequence of Lemma 5.11 is that, although the measure μ itself may not be a σ-finite measure, the pathologies (failure of Fubini's theorem etc) associated with non-σ-finite measures cannot occur. We shall not give a specific proof of the following result.

Lemma 8.1. *Let I be a left invariant Haar integral on a Hausdorff locally compact topological group $(G, \times, \tau)$. Then $C_{00}(G)$ is dense in $(L^1(G), \| \|)$.*

If $f \in L^1(G, m)$ then, setting $f_y(x) = f(yx)$, we have $f_y \in L^1(G, m)$ and

$$\int_G f\,dm = \int_G f_y\,dm.$$

Exercise 8.2. *If $f \in L^1(G, m)$, then the map $y \mapsto f_y$ from G to $L^1(G)$ is continuous.*

Exercise 8.3. *$\int_G 1\,dm < \infty$ if and only if G is compact.*

Unless specifically stated we normalise Haar measure on compact groups to give $\int_G 1\,dm = 1$ and on discrete groups so that $m(\{e\}) = 1$.

Lemma 8.4. *If f, $g \in L^1(G, m)$ then*

$$f * g(x) = \int_G f(xy)g(y^{-1})\,dm(y)$$

is well defined m almost everywhere and $f * g \in L^1(G, m)$ with $\|f * g\|_1 \leq \|f\|_1\|g\|_1$.

From now on we only deal with Hausdorff locally compact **Abelian** groups (although some of the results carry over to the non-Abelian case).

Theorem 8.5. *If $(G, +, \tau)$ is a locally compact Hausdorff Abelian group with Haar measure m. then the Banach space $L^1(G, m)$ equipped with convolution $*$ as multiplication is a commutative Banach algebra.*

Further, the map $f \mapsto f^*$ is an involution. (That is to say $f^{**} = f$, $(f + g)^* = f^* + g^*$, $(\lambda f)^* = \lambda^* f^*$ and $(f * g)^* = f^* * g^*$.)

Exercise 8.6. *If $(G, +, \tau)$ is a locally compact Hausdorff Abelian group the map $f \mapsto \tilde{f}$ with $\tilde{f}(x) = f^*(-x)$ is an involution on $L^1(G, m)$ equipped with convolution.*

Exercise 8.7. *Convolution is commutative if and only if G is commutative.*

9 Characters

Recall that a character θ on a commutative Banach algebra B is non-zero linear functional $\theta : B \to \mathbb{C}$ such that $\theta(ab) = \theta(a)\theta(b)$ for all a, $b \in B$. We give the collection $\mathcal{M}$ the weak-* topology in which θ_1 has neighbourhood basis formed by sets of the form

$$\{\theta \,:\, |\theta(a_j) - \theta_1(a_j)| < \epsilon_j \ [1 \leq j \leq n]\}.$$

We define $\hat{a}(\theta) = \theta(a)$. The Gelfand transform $a \mapsto \hat{a}$ is a continuous algebra homomorphism from B to $C(\mathcal{M})$ with $\|\hat{a}\|_\infty = \rho(a)$.

How does this fit with the Banach algebra of Theorem 8.5 and the classical Fourier analysis of section 1? Following the pattern of section 1 we make the following definition.

Definition 9.1. *If G is a Hausdorff locally compact Abelian group we say that a continuous group homomorphism $\chi : G \to S^1$ is a character of the group. We write $\hat{G}$ for the set of such characters and $\langle x, \chi \rangle = \chi(x)$ for all $x \in G$ and $\chi \in \hat{G}$.*

Theorem 9.2. *If G is a Hausdorff locally compact Abelian group with Haar measure m then if we fix χ and write*

$$(\chi)(f) = \int_G \langle -x, \chi \rangle f(x)\, dm(x),$$

the map $L^1 \to \mathbb{C}$ is a non-zero multiplicative linear functional. Every multiplicative linear functional arises in this way and distinct χ give rise to distinct multiplicative linear functionals.

Thus if we give $\hat{G}$ the topology in which a character χ_1 has neighbourhood basis formed by sets of the form

$$\{\chi \,:\, |\chi(f_j) - \chi_1(f_j)| < \epsilon_j \ [1 \leq j \leq n]\}$$

(with $\epsilon_j > 0$, $f_j \in L^1(G)$, $1 \leq j \leq n$, $n \geq 1$) we can identify $\hat{G}$ with $\mathcal{M}$ and obtain

$$\hat{f}(\chi) = \int_G \langle -x, \chi \rangle f(x)\, dm(x)$$

satisfactorily uniting the Gelfand and the Fourier transform. Note that we have extended the classical slogan 'Fourier transformation is a way to convert convolution into multiplication' by showing that it is the only (reasonable) way.

The nature of the Gelfand topology on $\hat{G}$ is illuminated by the following lemma.

Lemma 9.3. *If G is a Hausdorff locally compact Abelian group then*

(i) The map $G \times \hat{G} \to \mathbb{C}$ given by $(x, \chi) \mapsto \langle x, \chi \rangle$ is continuous.

$(ii)_a$ Sets of the form

$$\{\chi \in \hat{G} : |\langle x, \chi\rangle - \langle x, \chi_1\rangle| < \epsilon \text{ for all } x \in K\}$$

with K compact in G and $\epsilon > 0$ are open in $\hat{G}$.

$(ii)_b$ Sets of the form

$$\{x \in G : |\langle x, \chi\rangle - \langle x_1, \chi\rangle| < \epsilon \text{ for all } \chi \in K\}$$

with K compact in $\hat{G}$ and $\epsilon > 0$ are open in G.

$(iii)_a$ Sets of the form

$$\{\chi \in \hat{G} : |\langle x, \chi\rangle - \langle x, \chi_1\rangle| < \epsilon \text{ for all } x \in K\}$$

with K compact in G and $\epsilon > 0$ form neighbourhood bases at each $\chi_1 \in \hat{G}$.

(iv) $\hat{G}$ is a locally compact Hausdorff Abelian group.

Note the absence (for the time being) of any part $(\text{iii})_b$.

Exercise 9.4. *(i) If G is compact, $\hat{G}$ is discrete.*

(ii)If G is discrete, $\hat{G}$ is compact.

10 Fourier transforms of measures

Although we want an inversion theorem stated in terms of Haar measures alone, the standard treatments (see [6] which we follow closely and [2]) require us to look at more general measures. This is not surprising since $L^1(G)$ is not weak-* closed in the appropriate space of measures.

Definition 10.1. *If X is a locally compact space then $M(X)$ is the set of measures μ on the Borel sets of X such that $\|\mu\|$ is finite and μ is* regular *that is to say:-*

Given any $\epsilon > 0$ we can find a compact set K such that $|\mu|(X \setminus K) < \epsilon$.

(If we give the group $\mathbb{T}$ the discrete topology and define $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if $\mathbb{T} \setminus E$ is countable then μ is not regular.)

Exercise 10.2. *Consider the following statements about a measure μ on a locally compact Hausdorff group G.*

(i) μ is σ-finite.

(ii) G is σ-compact.

(iii) μ is regular.

(iv) μ is finite.

What relations, if any, hold between these concepts. Show that if μ is regular its support lies inside a σ-compact normal open subgroup of G.

Lemma 10.3. *Let $(G, +, \tau)$ be a locally compact Hausdorff Abelian group.*

(i) If E is a Borel set in G then

$$\tilde{E} = \{(x, y) \in G^2 \,:\, x + y \in E\}$$

is Borel in G^2.

(ii) If μ_1, $\mu_2 \in M(G)$ then writing

$$\mu_1 * \mu_2(E) = \mu_1 \times \mu_2(\tilde{E})$$

gives us $\mu_1 * \mu_2 \in M(G)$.

Exercise 10.4. *If $(G, +, \tau)$ is a locally compact Hausdorff Abelian group then $(M(G), +, *, \| \|)$ is a commutative Banach algebra with unit.*

If $\mu \in M(G)$ we can define the Fourier transform

$$\hat{\mu}(\chi) = \int_G \langle -x, \chi \rangle d\mu(x).$$

Exercise 10.5. *(i) If $\mu \in M(G)$ then $\hat{\mu}$ is a bounded uniformly continuous function $G \to \mathbb{R}$.*

(ii) If $\mu_1\, \mu_2 \in M(G)$ then $\widehat{\mu_1 * \mu_2}(\chi) = \hat{\mu}_1(\chi)\hat{\mu}_2(\chi)$.

We can now prove our first uniqueness theorem.

Theorem 10.6. *Let $(G, +, \tau)$ be a locally compact Hausdorff Abelian group. If $\mu \in M(\hat{G})$ and*

$$\int_{\hat{G}} \langle x, \chi \rangle \, \mu(\chi) = 0$$

for all $x \in G$, then $\mu = 0$.

11 Discussion of the inversion theorem

We want a theorem of the form

$$\int_{\hat{G}} \langle \chi, x \rangle \hat{f}(\chi) \, dm_{\hat{G}}(\chi) \stackrel{?}{=} Af(x), \qquad \bigstar$$

but we know from the classical case that such a result does not hold without restriction.

Example 11.1. *(i) Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1$ for $|x| \leq 1$, $f(x) = 0$ otherwise. Then $f \in L^1$ but $\hat{f} \notin L^1$.*

(ii) There exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ of compact support such that $\hat{g} \notin L^1$.

It is trivial that if ★ is to hold we must have $f(x) = \hat{\mu}(x)$ for some $\mu \in M(\hat{G})$. Remarkably the converse holds.

Theorem A. *If $f \in L^1(G)$ and $f = \hat{\mu}$ for some $\mu \in M(\hat{G})$ then*

$$\int_{\hat{G}} \langle \chi, x \rangle \hat{f}(\chi) \, dm_{\hat{G}}\chi = Af(x)$$

m_G almost everywhere for some A independent of f.

How can recognise that $f = \hat{\mu}$? In general, we can not but, remarkably, we can characterise $\hat{\mu}$ when μ is a positive measure in $M(G)$. (Since every member μ of $M(\hat{G})$ an be written $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ with $\mu_j \in M^+(\hat{G})$ this also gives a handle on general elements of $M(\hat{G})$.

One again we start with a near triviality.

Lemma 11.2. *If $\mu \in M^+(\hat{G})$ then $\hat{\mu}$ is continuous and*

$$\sum_{1 \leq j,k \leq n} c_j c_k^* \hat{\mu}(x_j - x_k) \geq 0$$

for all $x_j \in \mathbb{C}$ $c_j \in \mathbb{C}$ $n \geq 1$.

Remarkably the converse holds (this was first observed by Herglotz for $G = \mathbb{Z}$ and then by Bochner for the deeper case $G = \mathbb{R}$.)

Definition 11.3. *A function $\phi : G \to \mathbb{C}$ is called positive definite if*

$$\sum_{1 \leq j,k \leq n} c_j c_k^* \phi(x_j - x_k) \geq 0$$

for all $x_j \in G$ $c_j \in \mathbb{C}$ $n \geq 1$.

Theorem B. (Bochner's theorem.) *A continuous function $f : G \to \mathbb{C}$ is positive definite if and only if $f = \hat{\mu}$ for some $\mu \in M^+(\hat{G})$.*

Although we introduced Theorems A and B in alphabetical order we shall first prove B and then A.

As an indication of their utility we make the following observation.

Lemma 11.4. *(i) If $f, g \in L^2(G)$ then $f * g$ is a well defined continuous function.*

(ii) If $f \in L^2(G)$ and we set $\tilde{f}(x) = f(-x)^*$, then $f * \tilde{f}$ is positive definite.

12 The inversion theorem

Before proving Bochner's theorem we need some simple results on positive definite functions.

Lemma 12.1. *Suppose that $(G, +, \tau)$ is a locally compact Abelian Hausdorff group and $\phi : G \to \mathbb{C}$ is positive definite. Then $\phi(0)$ is real and positive and*
(i) $\phi(-x) = \phi^*(x)$,
(ii) $|\phi(x)| \leq \phi(0)$
(iii) $|\phi(x) - \phi(y)|^2 \leq 2\phi(0)\Re(\phi(0) - \phi(x - y))$
for all x, $y \in G$.

In particular, if ϕ is continuous at 0, then ϕ is uniformly continuous.

Lemma 12.2. *Suppose that $(G, +, \tau)$ is a locally compact Abelian Hausdorff group and $\phi : G \to \mathbb{C}$ is positive definite and continuous. Then*

$$\int_G \int_G f(x)f^*(y)\phi(x-y)\, dm_G(x)\, dm_G(y) \geq 0$$

for all $f \in L^1(G)$.

We now prove Bochner's theorem.

Theorem 12.3. *Suppose that $(G, +, \tau)$ is a locally compact Abelian Hausdorff group. Then a continuous function $\phi : G \to \mathbb{C}$ is positive definite if and only if $\phi = \hat{\mu}$ for some $\mu \in M^+(\hat{G})$.*

Now we can prove our main inversion theorem.

Theorem 12.4. *Suppose that $(G, +, \tau)$ is a locally compact Abelian Hausdorff group. Let m_G be a Haar measure on G.*

Then there exists a Haar measure $m_{\hat{G}}$ on $\hat{G}$ such that, if $f \in L^1(G)$ and $f = \hat{\mu}$ for some $\mu \in M(\hat{G})$, then $\hat{f} \in L^1(\hat{G})$ and

$$\int_{\hat{G}} \langle \chi, x \rangle \hat{f}(\chi)\, dm_{\hat{G}}(\chi) = f(x)$$

m_G almost everywhere.

Henceforward we shall assume that our Haar measures on G and $\hat{G}$ are chosen so that the formula of Theorem 12.4 holds.

Exercise 12.5. *Recall Exercise 9.4. Check that our new convention for Haar measure is consistent with our previous conventions for choosing Haar measures for discrete and compact groups at least if the groups are not both compact and discrete.*

What happens if our group is both compact and discrete?

Theorem 12.6. (Plancherel.) *Suppose that* $(G, +, \tau)$ *is a locally compact Hausdorff group. If* $f \in L^1(G) \cap L^2(G)$*, then* $\hat{f} \in L^2(\hat{G})$ *and*

$$\int_{\hat{G}} |\hat{f}(\chi)|^2 \, dm_{\hat{G}}(\chi) = \int_G |f(x)|^2 \, dm_G(x).$$

Theorem 12.7. *Suppose that* $(G, +, \tau)$ *is a locally compact Hausdorff group. There exists a linear isometry* $\mathcal{F} : L^2(G) \to L^2(G)$ *such that* $\mathcal{F}(f) = \hat{f}$ *whenever* $f \in L^1(G) \cap L^2(G)$*.*

Lemma 12.8. *If* $F, G \in L^2(\hat{G})$ *then we can find a* $k \in L^1 G$ *with* $\hat{f} = F * G$*.*

13 Pontryagin duality

Using the inversion theorem we can increase the symmetry between G and $\hat{G}$. Our first step is to obtain the missing part $(\text{iii})_b$ of Lemma 9.3.

Lemma 13.1. *If* G *is a Hausdorff locally compact Abelian group then sets of the form*

$$\{x \in G \,:\, |\langle x, \chi\rangle - \langle x_1, \chi\rangle| < \epsilon \text{ for all } \chi \in K\}$$

with K *compact in* $\hat{G}$ *and* $\epsilon > 0$ *are open and form neighbourhood bases at each* $x_1 \in G$*.*

Lemma 13.2. *If* $\gamma \in \hat{G}$ *and* K *is a compact neighbourhood of* γ *we can find an* $f \in L^1(G)(G)$ *with* $\hat{f}(\gamma) > 0$*,* $\hat{f}(\chi) \geq 0$ *for all* $\chi \in \hat{G}$ *and* $\hat{f}(\chi) = 0$ *for all* $\chi \notin K$*.*

Theorem 13.3. (Pontryagin's duality theorem.) *If* G *is a Hausdorff locally compact Abelian group, then the map* $\Phi : G \to \hat{\hat{G}}$ *defined by*

$$\Phi(x)(\chi) = \chi(x)$$

for $x \in G$ *and* $\chi \in \hat{G}$ *is an algebraic isomorphism and a topological homeomorphism.*

Thus we can identify G with $\hat{\hat{G}}$ in a natural manner.

Here are some immediate corollaries.

Lemma 13.4. *Let* G *be a Hausdorff locally compact Abelian group.*

(i) If $\mu \in M(G)$ *and* $\hat{\mu}(\chi) = 0$ *for all* $\chi \in \hat{G}$ *then* $\mu = 0$*.*

(ii) If $\mu \in M(G)$ and $\hat{\mu} \in L^1(\hat{G})$ then there exists an $f \in L^1(G)$ such that $\mu = f m_G$ and

$$f(x) = \int_{\hat{G}} \hat{\mu}(\chi)\langle x, \chi\rangle \, dm_{\hat{G}}(\chi).$$

(iii) If G is not discrete then $L^1(G)$ has no unit. Thus $L^1(G) = M(G)$ if and only if G is discrete.

It is worth noting that since the dual of a compact Hausdorff Abelian group G is a discrete group $\hat{G}$ and the dual of $\hat{G}$ is G all the topological information about G is encoded as algebraic information about $\hat{G}$.

14 Structure theorems

We conclude by deriving a substantial amount of information about the structure of a general locally compact Hausdorff Abelian group.

Our first results are not unexpected.

Definition 14.1. *Let G be a Hausdorff locally compact Abelian group and let H be a closed subgroup of G. We write*

$$H^\perp = \{\chi \in \hat{G} \,:\, \langle x, \chi\rangle = 1 \text{ for all } x \in H\}$$

and call $H^\perp$ the annihilator of H.

Lemma 14.2. *Let G be a Hausdorff locally compact Abelian group and let H be a closed subgroup of G.*
(i) $H^\perp$ is a closed subgroup of $\hat{G}$.
(ii) $H^{\perp\perp} = H$.
(iii) $H^\perp$ is isomorphic as a topological group with $(G/H)\hat{}$.
(iv) $\hat{G}/H^\perp$ is isomorphic as a topological group with $\hat{H}$.

Theorem 14.3. *Let G be a Hausdorff locally compact Abelian group and let H be a closed subgroup of G. Then any character of H can be extended to a character of G.*

The next results are also expected. Let us write $G \oplus H$ for the direct sum of two Abelian topological groups G and H.

Lemma 14.4. *Let G_j be a Hausdorff locally compact Abelian group for each $1 \leq j \leq n$ Then*

$$(G_1 \oplus G_2 \oplus G_3 \cdots \oplus G_n)\hat{} = \hat{G}_1 \oplus \hat{G}_2 \oplus \hat{G}_3 \cdots \oplus \hat{G}_n.$$

The next exercise echos Lemma 4.5.

Exercise 14.5. *The dual of the complete direct sum of a collection G_α $[\alpha \in A]$ of compact Hausdorff Abelian groups is the direct sum of the dual groups $\hat{G}_\alpha$ $[\alpha \in A]$.*

We can now state the structure theorem we wish to prove.

Theorem 14.6. (Principal structure theorem.) *If G is a Hausdorff locally compact Abelian group then we can find an open (so closed) subgroup H such that $H = W \oplus \mathbb{R}^n$ with W a compact group.*

The lemmas that follow are directed towards the proof of the principal structure theorem.

Lemma 14.7. *Suppose that G is a locally compact Hausdorff topological group containing a dense cyclic group. If G is not compact then G is (topological group isomorphic to) $\mathbb{Z}$.*

Lemma 14.8. *Suppose that G is a locally compact Hausdorff topological group generated by a compact neighbourhood V of 0. Then G contains a closed subgroup (topological group isomorphic to) $\mathbb{Z}^n$ such that $G/\mathbb{Z}^n$ is compact and $V \cap \mathbb{Z}^n = \{0\}$.*

The next two lemmas concern groups like D^∞.

Lemma 14.9. *Suppose E is a compact open set in a locally compact Hausdorff topological group G.*

(i) There exists a neighbourhood W of 0 with $W = -W$ and $E + W = E$.

(ii) If $0 \in E$ then E contains a compact open subgroup of G.

(iii) E is the finite union of open cosets in G.

Recall that a totally disconnected topological space is one in which the connected components are singletons.

Lemma 14.10. *IF G is a totally disconnected locally compact Hausdorff topological group then every neighbourhood of 0 contains a compact open subgroup of G.*

Our final preliminary lemma tells us that what looks like $\mathbb{R}^k$ is actually $\mathbb{R}^k$.

Lemma 14.11. *Suppose G is a connected locally compact Hausdorff topological group containing no infinite compact subgroup and locally isomorphic* *to $\mathbb{R}^k$ in the sense that there exists a neighbourhood V of 0 and a homeomorphism ϕ from the unit ball $B(\mathbf{0}, 1)$ of $\mathbb{R}^k$ to V such that*

$$\phi(\mathbf{x} + \mathbf{y}) = \phi(\mathbf{x}) + \phi(\mathbf{y})$$

for all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x} + \mathbf{y} \in B(\mathbf{0}, 1)$. Then G is (topological group isomorphic) to $\mathbb{R}^k$.

Lemma 14.12. *If f is a continuous open map of a locally compact Hausdorff space X onto a Hausdorff space Y and if K is a compact subset of Y then we can find a compact subset C of X with $f(C) = K$.*

We can now prove Theorem 14.6. Our proof is summarised in the following two lemmas.

Lemma 14.13. *If G is a Hausdorff locally compact Abelian group then we can find an open (so closed) subgroup H such that H is compactly generated and contains no open subgroup of infinite index.*

Lemma 14.14. *If H is a compactly generated Hausdorff Abelian group with no open subgroup of infinite order then $H = W \oplus \mathbb{R}^n$ with W a compact group.*

It may be helpful to run throught the proofs in the particular cases $G = \mathbb{R}$, $G = \mathbb{R} \times \mathbb{T} \times \mathbb{Z}$ and $G = 0, 1^{\mathbb{R}}$.

15 Final remarks

A different approach which reflects the original way in these results were discovered is given in the book of Hewitt and Ross [4]. First one obtains a more powerful version of the structure theorem.

Theorem 15.1. *If G is a compactly generated Hausdorff topological group then*

$$G \cong F \times W \times \mathbb{Z}^l \times \mathbb{R}^k$$

where W is a compact group and F is a discrete group.

Then one establishes Pontryagin duality for compact and discrete groups and uses this to obtain the full duality theorem. The book of Hewitt and Ross contains proper attributions of the various theorems. (In particular, what we call Pontryagin duality should really be called Van Kampen-Pontryagin duality.)

As a research tool the theory presented in this course was perhaps too successful for its own good. It shows that, for many purposes, $\mathbb{R}$, $\mathbb{T}$, D^∞ and their duals are typical locally compact Abelian groups and if we understand these cases it is not hard to extend results to the general case.

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