Metric and Topological Spaces

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August 17, 2015

Small print The syllabus for the course is defined by the Faculty Board Schedules (which are minimal for lecturing and maximal for examining). What is presented here contains some results which it would not, in my opinion, be fair to set as book-work although they could well appear as problems. In addition, I have included a small amount of material which appears in other 1B courses. I should **very much** appreciate being told of any corrections or possible improvements and might even part with a small reward to the first finder of particular errors. These notes are written in LaTeX 2_ε and should be available in tex, ps, pdf and dvi format from my home page

http://www.dpmms.cam.ac.uk/~twk/

I can send some notes on the exercises in Sections 16 and 17 to supervisors by e-mail.

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1 Preface

Within the last sixty years, the material in this course has been taught at Cambridge in the fourth (postgraduate), third, second and first years or left to students to pick up for themselves. Under present arrangements, students may take the course either at the end of their first year (before they have met metric spaces in analysis) or at the end of their second year (after they have met metric spaces).

Because of this, the first third of the course presents a rapid overview of metric spaces (either as revision or a first glimpse) to set the scene for the main topic of topological spaces.

The first part of these notes states and discusses the main results of the course. Usually, each statement is followed by directions to a proof in the final part of these notes. Whilst I do not expect the reader to find all the proofs by herself, I do ask that she *tries* to give a proof herself before looking one up. Some of the more difficult theorems have been provided with hints as well as proofs.

In my opinion, the two sections on compactness are the deepest part of the course and the reader who has mastered the proofs of the results therein is well on the way to mastering the whole course.

May I repeat that, as I said in the small print, I welcome corrections and comments.

The reader should be acquainted with the convention that, if we have a function $f : X \to Y$, then f is associated with two further *set-valued* functions

$$f : \mathcal{P}(X) \to \mathcal{P}(Y) \text{ and } f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$$

(here $\mathcal{P}(Z)$ denotes the collection of subsets of Z) given by

$$f(A) = \{f(a) : a \in A\} \text{ and } f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

We shall mainly be interested in f^{-1} since this is better behaved as a set-valued function than f.

Exercise 1.1. *We use the notation just introduced.*

(i) Let $X = Y = \{1, 2, 3, 4\}$ *and* $f(1) = 1$, $f(2) = 1$, $f(3) = 4$, $f(4) = 3$. *Identify*

$$f^{-1}(\{1\}),\ f^{-1}(\{2\}) \textit{ and } f^{-1}(\{3, 4\}).$$

(ii) If $U_\theta \subseteq Y$ *for all* $\theta \in \Theta$, *show that*

$$f^{-1}\left(\bigcup_{\theta\in\Theta} U_\theta\right) = \bigcup_{\theta\in\Theta} f^{-1}(U_\theta) \textit{ and } f^{-1}\left(\bigcap_{\theta\in\Theta} U_\theta\right) = \bigcap_{\theta\in\Theta} f^{-1}(U_\theta).$$

Show also that $f^{-1}(Y) = X$, $f^{-1}(\varnothing) = \varnothing$ *and that, if* $U \subseteq Y$,

$$f^{-1}(Y \setminus U) = X \setminus f^{-1}(U).$$

(iii) If $V_\theta \subseteq X$ *for all* $\theta \in \Theta$, *show that*

$$f\left(\bigcup_{\theta\in\Theta} V_\theta\right) = \bigcup_{\theta\in\Theta} f(V_\theta)$$

and observe that $f(\varnothing) = \varnothing$.

(iv) By finding appropriate X, Y, f *and* $V, V_1, V_2 \subseteq X$, *show that we may have*

$$f(V_1 \cap V_2) \neq f(V_1) \cap f(V_2),\ f(X) \neq Y \textit{ and } f(X \setminus V) \neq Y \setminus f(V).$$

Solution. The reader should not have much difficulty with this, but if necessary, she can consult page 64. □

2 What is a metric?

If I wish to travel from Cambridge to Edinburgh, then I may be interested in one or more of the following numbers.

(1) The distance, in kilometres, from Cambridge to Edinburgh 'as the crow flies'.

(2) The distance, in kilometres, from Cambridge to Edinburgh by road.

(3) The time, in minutes, of the shortest journey from Cambridge to Edinburgh by rail.

(4) The cost, in pounds, of the cheapest journey from Cambridge to Edinburgh by rail.

Each of these numbers is of interest to someone and none of them is easily obtained from another. However, they do have certain properties in common which we try to isolate in the following definition.

Definition 2.1. *Let X be a set*[1] *and $d : X^2 \to \mathbb{R}$ a function with the following properties:*

(i) $d(x, y) \geq 0$ for all $x, y \in X$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iv) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$. (This is called the triangle inequality *after the result in Euclidean geometry that the sum of the lengths of two sides of a triangle is at least as great as the length of the third side.)*

Then we say that d is a metric *on X and that (X, d) is a* metric space.

You should imagine the author muttering under his breath

'(i) Distances are always positive.

(ii) Two points are zero distance apart if and only if they are the same point.

(iii) The distance from A to B is the same as the distance from B to A.

(iv) The distance from A to B via C is at least as great as the distance from A to B directly.'

Exercise 2.2. *If $d : X^2 \to \mathbb{R}$ is a function with the following properties:*

[1] We thus allow $X = \varnothing$. This is purely a question of taste. If we did not allow this possibility, then, every time we defined a metric space (X, d), we would need to prove that X was non-empty. If we do allow this possibility, and we prefer to reason about non-empty spaces, then we can begin our proof with the words 'If X is empty, then the result is vacuously true, so we may assume that X is non-empty.' (Of course, the result may be false for $X = \varnothing$, in which case the statement of the theorem must include the condition $X \neq \varnothing$.)

(ii) $d(x, y) = 0$ *if and only if* $x = y$*,*

(iii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*,*

(iv) $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in X$*,*

show that d *is a metric on* X*.*

[*Thus condition (i) of the definition is redundant.*]

Solution. See page 66 for a solution. □

Exercise 2.3. *Let* X *be the set of towns on the British railway system. Consider the* d *corresponding to the examples (1) to (4) and discuss informally whether conditions (i) to (iv) apply.*

[*An open ended question like this will be more useful if tackled in a spirit of good will.*]

Exercise 2.4. *Let* $X = \{a, b, c\}$ *with* a*,* b *and* c *distinct. Write down functions* $d_j : X^2 \to \mathbb{R}$ *satisfying condition (i) of Definition 2.1 such that:*

(1) d_1 *satisfies conditions (ii) and (iii) but not (iv).*

(2) d_2 *satisfies conditions (iii) and (iv) and* $d_2(x, y) = 0$ *implies* $x = y$*, but it is not true that* $x = y$ *implies* $d_2(x, y) = 0$*.*

(3) d_3 *satisfies conditions (iii) and (iv) and* $x = y$ *implies* $d_3(x, y) = 0$*. but it is not true that* $d_3(x, y) = 0$ *implies* $x = y$*.*

(4) d_4 *satisfies conditions (ii) and (iv) but not (iii).*

You should verify your statements.

Solution. See page 67. □

Other axiom grubbing exercises are given as Exercise 16.1 and 17.1.

Exercise 2.5. *Let* X *be a set and* $\rho : X^2 \to \mathbb{R}$ *a function with the following properties.*

(i) $\rho(x, y) \geq 0$ *for all* $x, y \in X$*.*

(ii) $\rho(x, y) = 0$ *if and only if* $x = y$*.*

(iv) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ *for all* $x, y, z \in X$*.*

Show that, if we set $d(x, y) = \rho(x, y) + \rho(y, x)$*, then* (X, d) *is a metric space.*

3 Examples of metric spaces

We now look at some examples. The material from Definition 3.1 to Theorem 3.10 inclusive is covered in detail in Analysis II. You have met (or you will meet) the concept of a normed vector space both in algebra and analysis courses.

Definition 3.1. *Let V be a vector space over $\mathbb{F}$ (with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and $N : V \to \mathbb{R}$ a map such that, writing $N(\mathbf{u}) = \|\mathbf{u}\|$, the following results hold.*

(i) $\|\mathbf{u}\| \geq 0$ for all $\mathbf{u} \in V$.

(ii) If $\|\mathbf{u}\| = 0$, then $\mathbf{u} = \mathbf{0}$.

(iii) If $\lambda \in \mathbb{F}$ and $\mathbf{u} \in V$, then $\|\lambda\mathbf{u}\| = |\lambda|\|\mathbf{u}\|$.

(iv) [Triangle law.] If $\mathbf{u}$, $\mathbf{v} \in V$, then $\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u} + \mathbf{v}\|$.

Then we call $\|\ \|$ a norm *and say that $(V, \|\ \|)$ is a* normed vector space*.*

Exercise 3.2. *By putting $\lambda = 0$ in Definition 3.1 (iii), show that $\|\mathbf{0}\| = 0$.*

Any normed vector space can be made into a metric space in a natural way.

Lemma 3.3. *If $(V, \|\ \|)$ is a normed vector space, then the condition*

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

defines a metric d on V.

Proof. The easy proof is given on page 67. □

The concept of an inner product occurs both in algebra and in many physics courses.

Definition 3.4. *Let V be a vector space over $\mathbb{R}$ and $M : V \times V \to \mathbb{R}$ a map such that, writing $M(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$, the following results hold for $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$, $\lambda \in \mathbb{R}$.*

(i) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$.

(ii) If $\langle \mathbf{u}, \mathbf{u} \rangle = 0$, then $\mathbf{u} = \mathbf{0}$.

(iii) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

(iv) $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$.

(v) $\langle \lambda\mathbf{u}, \mathbf{v} \rangle = \lambda\langle \mathbf{u}, \mathbf{v} \rangle$.

Then we call $\langle\ ,\ \rangle$ an inner product *and say that $(V, \langle\ ,\ \rangle)$ is an* inner product space*.*

Lemma 3.5. *Let $(V, \langle\ ,\ \rangle)$ be an inner product space. If we write $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2}$ (taking the positive root), then the following results hold.*

(i) (The Cauchy–Schwarz inequality) If $\mathbf{u}$, $\mathbf{v} \in V$, then

$$\|\mathbf{u}\|\|\mathbf{v}\| \geq |\langle \mathbf{u}, \mathbf{v} \rangle|.$$

(ii) $(V, \|\ \|)$ is a normed vector space.

Proof. The standard proofs are given on page 68. □

Lemma 3.6. *If we work on $\mathbb{R}^n$ made into a vector space in the usual way, then*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j y_j$$

is an inner product.

Proof. Direct verification. $\square$

We call the norm

$$\|\mathbf{x}\|_2 = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2},$$

derived from this inner product the *Euclidean norm* (or sometimes just 'the usual norm'). Although several very important norms are derived from inner products most are not.

Lemma 3.7. *(The parallelogram law) Using the hypotheses and notation of Lemma 3.5, we have*

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

Proof. Direct computation . $\square$

We need one more result before we can unveil a collection of interesting norms.

Lemma 3.8. *Suppose that $a < b$, that $f : [a, b] \to \mathbb{R}$ is continuous and $f(t) \geq 0$ for all $t \in [a, b]$. Then, if*

$$\int_a^b f(t)\, dt = 0,$$

it follows that $f(t) = 0$ for all $t \in [a, b]$.

Proof. See page 69. $\square$

Exercise 3.9. *Show that the result of Lemma 3.8 is false if we replace 'f continuous' by 'f Riemann integrable'.*

Solution. See page 69 if necessary. $\square$

Theorem 3.10. *Suppose that $a < b$ and we consider the space $C([a, b])$ of continuous functions $f : [a, b] \to \mathbb{R}$ made into a vector space in the usual way.*

(i) The equation

$$\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$$

defines an inner product on $C([a, b])$. We write

$$\|f\|_2 = \left(\int_a^b f(t)^2\,dt\right)^{1/2}$$

for the derived norm.

(ii) The equation

$$\|f\|_1 = \int_a^b |f(t)|\,dt$$

defines a norm on $C([a, b])$. This norm does not obey the parallelogram law.

(iii) The equation

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|.$$

defines a norm on $C([a, b])$. This norm does not obey the parallelogram law.

Proof. The routine proofs are given on page 69. □

However, not all metrics can be derived from norms. Here is a metric that turns out to be more important and less peculiar than it looks at first sight.

Definition 3.11. *If X is a set and we define $d : X^2 \to \mathbb{R}$ by*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y,\\ 1 & \text{if } x \neq y,\end{cases}$$

then d is called the discrete metric *on X.*

Lemma 3.12. *The discrete metric on X is indeed a metric.*

Proof. The easy proof is given on page 72. □

Exercise 3.13. *(We deal with the matter somewhat better in Exercise 5.7)*
(i) If V is a vector space over $\mathbb{R}$ and d is a metric derived from a norm in the manner described above, show that, if $\mathbf{u} \in V$ we have $d(\mathbf{0}, 2\mathbf{u}) = 2d(\mathbf{0}, \mathbf{u})$.

(ii) If V is non-trivial (i.e. not zero-dimensional) vector space over $\mathbb{R}$ and d is the discrete metric on V, show that d cannot be derived from a norm on V.

You should test any putative theorems on metric spaces on both $\mathbb{R}^n$ with the Euclidean metric and $\mathbb{R}^n$ with the discrete metric.

Exercise 3.14. [The counting metric.] *If E is a finite set and $\mathcal{E}$ is the collection of subsets of E, we write* $\operatorname{card} C$ *for the number of elements in C and*

$$d(A, B) = \operatorname{card} A \triangle B.$$

Show that d is a metric on $\mathcal{E}$. The reader may be inclined to dismiss this metric as uninteresting but it plays an important role (as the Hamming metric*) in the Part II course* Codes and Cryptography*.*

Here are two metrics which are included simply to show that metrics do not have to be as simple as the ones above. I shall use them as examples once or twice, but they do not form part of standard mathematical knowledge and you do not have to learn their definition.

Definition 3.15. *(i) If we define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by*

$$d(\mathbf{u}, \mathbf{v}) = \begin{cases} \|\mathbf{u}\|_2 + \|\mathbf{v}\|_2, & \text{if } \mathbf{u} \neq \mathbf{v}, \\ 0 & \text{if } \mathbf{u} = \mathbf{v}, \end{cases}$$

then d is called the British Rail express metric*. (To get from A to B travel via London.)*

(ii) If we define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(\mathbf{u}, \mathbf{v}) = \begin{cases} \|\mathbf{u} - \mathbf{v}\|_2 & \text{if } \mathbf{u} \text{ and } \mathbf{v} \text{ are linearly dependent,} \\ \|\mathbf{u}\|_2 + \|\mathbf{v}\|_2 & \text{otherwise,} \end{cases}$$

then d is called the British Rail stopping metric*. (To get from A to B travel via London unless A and B are on the same London route.)*

(Recall that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if $\mathbf{u} = \lambda\mathbf{v}$ for some real λ and/or $\mathbf{v} = \mathbf{0}$.)

Exercise 3.16. *Show that the British Rail express metric and the British Rail stopping metric are indeed metrics.*

Solution. On page 72 we show that the British Rail stopping metric is indeed a metric. The British Rail express metric can be dealt with similarly. □

However, fascinating as exotic metrics may be, the reader should reflect on the number of different *useful* metric spaces that exist. We have $\mathbb{R}^n$ with the usual Euclidean norm, $C([a, b])$ with the three norms described in

Theorem 3.10, the counting (Hamming) metric, the p-adic metric (used in number theory) described in Exercise 16.23, the metric space described in Exercise 16.21 (a model for coin tossing) and many others. The notion of a metric provides a common thread, suggesting appropriate theorems and proofs.

4 Continuity and open sets for metric spaces

Some definitions and results transfer essentially unchanged from classical analysis on $\mathbb{R}$ to metric spaces. Recall the classical definition of continuity.

Definition 4.1. [Old definition.] *A function $f : \mathbb{R} \to \mathbb{R}$ is called* continuous *if, given $t \in \mathbb{R}$ and $\epsilon > 0$, we can find a $\delta(t, \epsilon) > 0$ such that*

$$|f(t) - f(s)| < \epsilon \text{ whenever } |t - s| < \delta(t, \epsilon).$$

It is not hard to extend this definition to our new, wider context.

Definition 4.2. [New definition.] *Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is called* continuous *if, given $t \in X$ and $\epsilon > 0$, we can find a $\delta(t, \epsilon) > 0$ such that*

$$\rho(f(t), f(s)) < \epsilon \text{ whenever } d(t, s) < \delta(t, \epsilon).$$

It may help you grasp this definition if you read '$\rho(f(t), f(s))$' as 'the distance from $f(t)$ to $f(s)$ in Y' and '$d(t, s)$' as 'the distance from t to s in X'.

Lemma 4.3. [The composition law.] *If (X, d) and (Y, ρ) and (Z, σ) are metric spaces and $g : X \to Y$, $f : Y \to Z$ are continuous, then so is the composition fg.*

Proof. This is identical to the one we met in classical analysis. If needed, details are given on page 73. □

Exercise 4.4. *Let $\mathbb{R}$ and $\mathbb{R}^2$ have their usual (Euclidean) metric.*

(i) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous. Show that the map $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous.

(ii) Show that the map $M : \mathbb{R}^2 \to \mathbb{R}$ given by $M(x, y) = xy$ is continuous.

(iii) Use the composition law to show that the map $m : \mathbb{R}^2 \to \mathbb{R}$ given by $m(x, y) = f(x)g(y)$ is continuous.

Solution. See page 73. □

Exercise 4.4 may look perverse at first sight, but, in fact, we usually show functions to be continuous by considering them as compositions of simpler functions rather than using the definition directly. Think about

$$x \mapsto \log\left(2 + \sin\frac{1}{1+x^2}\right).$$

If you are interested, we continue the chain of thought in Exercise 16.2. If you are not interested or are mildly confused by all this, just ignore this paragraph.

Just as there are 'well behaved' and 'badly behaved' functions between spaces, so there are 'well behaved' and 'badly behaved' subsets of spaces. In classical analysis and analysis on metric spaces, the notion of continuous function is sufficiently wide to give us a large collection of interesting functions and sufficiently narrow to ensure reasonable behaviour[2]. In introductory analysis we work on $\mathbb{R}$ with the Euclidean metric and only consider subsets in the form of intervals. Once we move to $\mathbb{R}^2$ with the Euclidean metric, it becomes clear that there is no appropriate analogue to intervals. (We want appropriate rectangles to be well behaved, but we also want to talk about discs and triangles and blobs.)

Cantor identified two particular classes of 'well behaved' sets. We start with open sets.

Definition 4.5. *Let (X, d) be a metric space. We say that a subset E is* open *in X if, whenever $e \in E$, we can find a $\delta > 0$ (depending on e) such that*

$$x \in E \text{ whenever } d(x, e) < \delta.$$

Suppose we work in $\mathbb{R}^2$ with the Euclidean metric. If E is an open set then any point $\mathbf{e}$ in E is the centre of a disc of strictly positive radius all of whose points lie in E. If we are sufficiently short sighted, every point that we can see from $\mathbf{e}$ lies in E. This property turns out to be a key to many proofs in classical analysis (remember that in the proof of Rolle's theorem it was vital that the maximum did not lie at an end point) and complex analysis (where we examine functions analytic *on an open set*).

Here are a couple of simple examples of an open set and a simple example of a set which is not open.

Example 4.6. *(i) Let (X, d) be a metric space. If $r > 0$, then*

$$B(x, r) = \{y : d(x, y) < r\}$$

[2] Sentences like this are not mathematical statements, but many mathematicians find them useful.

is open.

(ii) If we work in $\mathbb{R}^n$ with the Euclidean metric, then the one point set $\{\mathbf{x}\}$ is not open.

(iii) If (X, d) is a discrete metric space, then

$$\{x\} = B(x, 1/2)$$

and all subsets of X are open.

Proof. See page 74. □

We call $B(\mathbf{x}, r)$ the *open ball* with centre $\mathbf{x}$ and radius r. The following result is very important for the course, but is also very easy to check.

Theorem 4.7. *If (X, d) is a metric space, then the following statements are true.*

(i) The empty set $\varnothing$ and the space X are open.

(ii) If U_α is open for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha$ is open. (In other words, the union of open sets is open.)

(iii) If U_j is open for all $1 \leq j \leq n$, then $\bigcap_{j=1}^{n} U_j$ is open.

Proof. See page 75. □

It is important to realise that we place no restriction on the size of A in (ii). In particular, A could be uncountable. However, conclusion (iii) cannot be extended.

Example 4.8. *Let us work in $\mathbb{R}^n$ with the usual metric. Then $B(\mathbf{x}, 1/j)$ is open, but $\bigcap_{j=1}^{\infty} B(\mathbf{x}, 1/j) = \{\mathbf{x}\}$ is not.*

Proof. See Example 4.6. □

There is a remarkable connection between the notion of open sets and continuity.

Theorem 4.9. *Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X whenever U is open in Y.*

Proof. See page 76. □

Note that the theorem does not work 'in the opposite direction'.

Example 4.10. *Let $X = \mathbb{R}$ and d be the discrete metric. Let $Y = \mathbb{R}$ and ρ be the usual (Euclidean) metric.*

(i) If we define $f : X \to Y$ by $f(x) = x$, then f is continuous but there exist open sets U in X such that $f(U)$ is not open.

(ii) If we define $g : Y \to X$ by $g(y) = y$, then g is not continuous but $g(V)$ is open in X whenever V is open in Y.

Proof. Very easy, but see page 76 if you need. ☐

The message of this example is reinforced by the more complicated Exercise 16.3.

Observe that Theorem 4.9 gives a very neat proof of the composition law.

Theorem 4.3. *If (X, d), (Y, ρ), (Z, σ) are metric spaces and $g : X \to Y$, $f : Y \to Z$ are continuous, then so is the composition fg.*

New proof. If U is open in Z, then, by continuity, $f^{-1}(U)$ is open in Y and so, by continuity, $(fg)^{-1}(U) = g^{-1}\big(f^{-1}(U)\big)$ is open in X. Thus fg is continuous. ☐

This confirms our feeling that the ideas of this section are on the right track.

We finish with an exercise, which may be omitted at first reading, but which should be done at some time since it gives examples of what open sets can look like.

Exercise 4.11. *Consider $\mathbb{R}^2$. For each of the British rail express and British rail stopping metrics:*

(i) Describe the open balls. (Consider both large and small radii.)

(ii) Describe the open sets as well as you can. (There is a nice description for the British rail express metric.) Give reasons for your answers.

Solution. See page 77. ☐

5 Closed sets for metric spaces

The second class of well behaved sets identified by Cantor were the closed sets. In order to define closed sets in metric spaces, we need a notion of limit. Fortunately, the classical definition generalises without difficulty.

Definition 5.1. *Consider a sequence x_n in a metric space (X, d). If $x \in X$ and, given $\epsilon > 0$, we can find an integer $N \geq 1$ (depending on ϵ) such that*

$$d(x_n, x) < \epsilon \text{ for all } n \geq N,$$

then we say that $x_n \to x$ as $n \to \infty$ and that x is the limit *of the sequence x_n.*

Lemma 5.2. *Consider a metric space (X, d). If a sequence x_n has a limit, then that limit is unique.*

Proof. The simple proof is given on page 78. Just as in the next exercise, it suffices to follow the 'first course in analysis' proof with minimal changes. □

Exercise 5.3. *Consider two metric spaces* (X, d) *and* (Y, ρ). *Show that a function* $f : X \to Y$ *is continuous if and only if, whenever* $x_n \in X$ *and* $x_n \to x$ *as* $n \to \infty$, *we have* $f(x_n) \to f(x)$

Solution. See page 78, if necessary. □

Exercise 5.4. *In this exercise we consider the identity map between a space and itself when we equip the space with* different *metrics. We look at the three norms (and their associated metrics) defined on* $C([0,1])$ *in Theorem 3.10.*

Define $j_{\alpha,\beta} : \big(C([0,1]), \|\ \|_\alpha\big) \to \big(C([0,1]), \|\ \|_\beta\big)$ *by* $j_{\alpha,\beta}(f) = f$.

(i) Show that $j_{\infty,1}$ *and* $j_{\infty,2}$ *are continuous but* $j_{1,\infty}$ *and* $j_{2,\infty}$ *are not.*

(ii) By using the Cauchy–Schwarz inequality $|\langle f, g\rangle| \leq \|f\|_2\|g\|_2$ *with* $g = 1$, *or otherwise, show that* $j_{2,1}$ *is continuous. Show that* $j_{1,2}$ *is not.* [*Hint: Consider functions of the form* $f_{R,K}(x) = K\max\{0, 1 - Rx\}$.]

Solution. See page 79, if necessary. □

Definition 5.5. *Let* (X, d) *be a metric space. A set* F *in* X *is said to be* closed *if, whenever* $x_n \in F$ *and* $x_n \to x$ *as* $n \to \infty$, *it follows that* $x \in F$.

The following exercises are easy, but instructive.

Exercise 5.6. *(i) If* (X, d) *is any metric space, then* X *and* $\varnothing$ *are both open and closed.*

(ii) If we consider $\mathbb{R}$ *with the usual metric and take* $b > a$, *then* $[a, b]$ *is closed but not open,* (a, b) *is open but not closed and* $[a, b)$ *is neither open nor closed.*

Exercise 5.7. *(i) If* (X, d) *is a metric space with discrete metric* d, *then all subsets of* X *are both open and closed.*

(ii) If V *is a vector space over* $\mathbb{R}$ *and* ρ *is a metric derived from a norm, show that the one point sets* $\{\mathbf{x}\}$ *are not open in this metric.*

(iii) Deduce that the discrete metric d *on the vector space* V *cannot be derived from a norm on* V.

It is easy to see why closed sets will be useful in those parts of analysis which involve taking limits. The reader will recall theorems in elementary analysis (for example the boundedness of continuous functions) which were true for closed intervals, but not for other types of intervals.

Life is made much easier by the very close link between the notions of closed and open sets given by our next theorem.

Theorem 5.8. *Let (X, d) be a metric space. A set F in X is closed if and only if its complement is open.*

Proof. There is a proof on page 80. $\square$

We can now deduce properties of closed sets from properties of open sets by complementation. In particular, we have the following complementary versions of Theorems 4.7 and 4.9

Theorem 5.9. *If (X, d) is a metric space, then the following statements are true.*
(i) The empty set $\varnothing$ and the space X are closed.
(ii) If F_α is closed for all $\alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is closed. (In other words the intersection of closed sets is closed.)
(iii) If F_j is closed for all $1 \leq j \leq n$, then $\bigcup_{j=1}^n F_j$ is closed.

Proof. See page 80. $\square$

Theorem 5.10. *Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed in X whenever F is closed in Y.*

Proof. See page 81. $\square$

6 Topological spaces

We now investigate general objects which have the structure described by Theorem 4.7.

Definition 6.1. *Let X be a set and τ a collection of subsets of X with the following properties.*
(i) The empty set $\varnothing \in \tau$ and the space $X \in \tau$.
(ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
(iii) If $U_j \in \tau$ for all $1 \leq j \leq n$, then $\bigcap_{j=1}^n U_j \in \tau$.
Then we say that τ is a topology *on X and that (X, τ) is a* topological space.

Theorem 6.2. *If (X, d) is a metric space, then the collection of open sets forms a topology.*

Proof. This is Theorem 4.7. $\square$

If (X, d) is a metric space we call the collection of open sets the topology *induced* by the metric.

If (X, τ) is a topological space we extend the notion of open set by calling the members of τ *open sets*. The discussion above ensures what computer scientists call 'downward compatibility'.

Just as group theory deals with a collection of objects together with an operation of 'multiplication' which follows certain rules, so we might say that topology deals with a collection τ of objects (subsets of X) under the two operations of 'union' and 'intersection' following certain rules. A remarkable application of this philosophy is provided by Exercise 17.7. However, many mathematicians simply use topology as a language which emphasises certain aspects of $\mathbb{R}^n$ and other metric spaces.

Exercise 6.3. *If (X, d) is a metric space with the discrete metric, show that the induced topology consists of all the subsets of X.*

We call the topology consisting of all subsets of X the *discrete* topology on X.

Exercise 6.4. *If X is a set and $\tau = \{\varnothing, X\}$, then τ is a topology.*

We call $\{\varnothing, X\}$ the *indiscrete* topology on X.

Exercise 6.5. *(i) If F is a finite set and (F, d) is a metric space, show that the induced topology is the discrete topology.*

(ii) If F is a finite set with more than one point, show that the indiscrete topology is not induced by any metric.

You should test any putative theorems on topological spaces on the discrete topology and the indiscrete topology, $\mathbb{R}^n$ with the topology derived from the Euclidean metric and $[0, 1]$ with the topology derived from the Euclidean metric.

The following exercise is tedious but instructive (the tediousness is the instruction).

Exercise 6.6. *Write $\mathcal{P}(Y)$ for the collection of subsets of Y. If X has three elements, how many elements does $\mathcal{P}\big(\mathcal{P}(X)\big)$ have?*

How many topologies are there on X?

Solution. See page 81. □

The idea of downward compatibility suggests 'turning Theorem 4.9 in a definition'.

Definition 6.7. *Let (X, τ) and (Y, σ) be topological spaces. A function $f : X \to Y$ is said to be* continuous *if and only if $f^{-1}(U)$ is open in X whenever U is open in Y.*

Theorem 4.9 tells us that, if (X, d) and (Y, ρ) are metric spaces, the notion of a continuous function $f : X \to Y$ is the same whether we consider the metrics or the topologies derived from them.

The proof of Theorem 4.3 given on page 13 carries over unchanged to give the following generalisation.

Theorem 6.8. *If (X, τ), (Y, σ), (Z, μ) are topological spaces and $g : X \to Y$, $f : Y \to Z$ are continuous, then so is the composition fg.*

Downward compatibility suggests the definition of a closed set for a topological space based on Theorem 5.8.

Definition 6.9. *Let (X, τ) be a topological space. A set F in X is said to be* closed *if its complement is open.*

Theorem 5.8 tells us that if (X, d) is a metric space the notion of a closed set is the same whether we consider the metric or the topology derived from it.

Just as in the metric case, we can deduce properties of closed sets from properties of open sets by complementation. In particular, the same proofs as we gave in the metric case give the following extensions of Theorems 5.9 and 5.10

Theorem 6.10. *If (X, τ) is a topological space, then the following statements are true.*

(i) The empty set $\varnothing$ and the space X are closed.

(ii) If F_α is closed for all $\alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is closed. (In other words, the intersection of closed sets is closed.)

(iii) If F_j is closed for all $1 \leq j \leq n$, then $\bigcup_{j=1}^{n} F_j$ is closed.

Theorem 6.11. *Let (X, τ) and (Y, σ) be topological spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed in X whenever F is closed in Y.*

7 Interior and closure

The next section is short, not because the ideas are unimportant, but because they are so useful that the reader will meet them over and over again in other courses.

Definition 7.1. *Let (X,τ) be a topological space and A a subset of X. We write*

$$\operatorname{Int} A = \bigcup\{U \in \tau : U \subseteq A\} \text{ and } \operatorname{Cl} A = \bigcap\{F \text{ closed} : F \supseteq A\}$$

and call $\operatorname{Cl} A$ *the* closure *of A and* $\operatorname{Int} A$ *the* interior *of A.*

Simple complementation, which I leave to the reader, shows how closely the two notions of closure and interior are related. (Recall that $A^c = X \setminus A$, the complement of A.)

Lemma 7.2. *With the notation of Definition 7.1*

$$(\operatorname{Cl} A^c)^c = \operatorname{Int} A \text{ and } (\operatorname{Int} A^c)^c = \operatorname{Cl} A.$$

There are other useful ways of viewing $\operatorname{Int} A$ and $\operatorname{Cl} A$.

Lemma 7.3. *Let (X,τ) be a topological space and A a subset of X.*
(i) $\operatorname{Int} A = \{x \in A : \exists\, U \in \tau \text{ with } x \in U \subseteq A\}$.
(ii) $\operatorname{Int} A$ *is the unique $V \in \tau$ such that $V \subseteq A$ and, if $W \in \tau$ and $V \subseteq W \subseteq A$, then $V = W$. (Informally,* $\operatorname{Int} A$ *is the largest open set contained in A.)*

Proof. The easy proof is given on page 82. □

Exercise 7.4. *Consider $\mathbb{R}$ with its usual topology (i.e. the one derived from the Euclidean norm). We look at the open interval $I = (0,1)$. Show that if F is closed and $F \subseteq (0,1)$, there is a closed G with $F \subseteq G \subseteq (0,1)$ and $G \neq F$. (Thus there is no largest closed set contained in $(0,1)$.)*

Solution. See page 83 if necessary. □

Simple complementation, which I leave to the reader, gives the corresponding results for closure.

Lemma 7.5. *Let (X,τ) be a topological space and A a subset of X.*
(i) $\operatorname{Cl} A = \{x \in X : \forall U \in \tau \text{ with } x \in U, \text{ we have } A \cap U \neq \varnothing\}$.
(ii) $\operatorname{Cl} A$ *is the unique closed set G such that $G \supseteq A$ and, if F is closed with $G \supseteq F \supseteq A$, then $F = G$. (Informally,* $\operatorname{Cl} A$ *is the smallest closed set containing A.)*

Exercise 7.6. *Prove Lemma 7.5 directly without using Lemma 7.3.*

Sometimes, when touring an ancient college, you may be shown a 14th century wall which still plays an important part in holding up the building. The next lemma goes back to Cantor and the very beginnings of topology. (It would then have been a definition rather than a lemma.)

Lemma 7.7. *Let (X, d) be a metric space and A a subset of X. Then $\operatorname{Cl} A$ consists of all those x such that we can find $x_n \in A$ with $d(x, x_n) \to 0$. (In old fashioned terminology, the closure of A is its set of* closure points[3]*.)*

Proof. The easy proof is given on page 83. □

The idea of closure is strongly linked to the idea of a dense subset.

Definition 7.8. *Let (X, τ) be a topological space and F a closed subset of X. We say that $A \subseteq X$ is a* dense *subset of F if $\operatorname{Cl} A = F$.*

In some sense A is a 'skeleton' of F and we may hope to prove results about F by first proving them on the dense subset A and then extending the result by 'density'. Sometimes this idea works (see, for example, part (ii) of Exercise 7.9) and sometimes it does not (see, for example, part (iii) of Exercise 7.9). When it does work, this is very powerful technique.

Exercise 7.9. *(i) Let (X, τ) be a topological space and (Y, d) a metric space. If f, $g : X \to Y$ are continuous show that the set*

$$\{x \in X \,:\, f(x) = g(x)\}$$

is closed.

(ii) Let (X, τ) be a topological space and (Y, d) a metric space[4]. If f, $g : X \to Y$ are continuous and $f(x) = g(x)$ for all $x \in A$, where A is dense in X, show that $f(x) = g(x)$ for all $x \in X$.

(iii) Consider the unit interval $[0, 1]$ with the Euclidean metric and $A = [0, 1] \cap \mathbb{Q}$ with the inherited metric. Exhibit, with proof, a continuous map $f : A \to \mathbb{R}$ (where $\mathbb{R}$ has the standard metric) such that there does not exist a continuous map $\tilde{f} : [0, 1] \to \mathbb{R}$ with $\tilde{f}(x) = f(x)$ for all $x \in A$.

Solution. There is a solution on Page 83. □

8 More on topological structures

Two groups are the same for the purposes of group theory if they are (group) isomorphic. Two vector spaces are the same for the purposes of linear algebra if they are (vector space) isomorphic. When are two topological spaces (X, τ)

[3] I strongly advise caution in employing terms like 'limit point', 'accumulation point', 'adherent point' and 'closure point' since both the literature and your lecturer are confused about what they mean. If an author uses one of these terms, check what definition they are using. If you wish to use these terms, define them explicitly.

[4] Exercise 9.7 gives an improvement of parts (i) and (ii).

and (Y, σ) the same for the purposes of topology? In other words, when does there exist a bijection between X and Y for which open sets correspond to open sets, and the grammar of topology (things like union and inclusion) is preserved? A little reflection shows that the next definition provides the answer we want.

Definition 8.1. *We say that two topological spaces (X, τ) and (Y, σ) are* homeomorphic *if there exists a bijection $\theta : X \to Y$ such that θ and θ^{-1} are continuous. We call θ a* homeomorphism.

The following exercise acts as useful revision of concepts learnt last year.

Exercise 8.2. *Show that homeomorphism is an equivalence relation on topological spaces.*

Homeomorphism implies equivalence *for the purposes of topology.*

Exercise 8.3. *Suppose that (X, d) and (Y, ρ) are metric spaces and $f : X \to Y$ is a homeomorphism. Show that*

$$d(x_n, x) \to 0 \Leftrightarrow \rho(f(x_n), f(x)) \to 0.$$

Thus the limit structure of a metric space is a topological property.

To give an interesting example of a property which is not preserved by homeomorphism, we introduce a couple of related ideas which are fundamental to analysis on metric spaces, but which will only be referred to occasionally in this course.

Definition 8.4. *(i) If (X, d) is a metric space, we say that a sequence x_n in X is* Cauchy *if, given $\epsilon > 0$, we can find an $N_0(\epsilon)$ with*

$$d(x_n, x_m) < \epsilon \text{ whenever } n,\ m \geq N_0(\epsilon).$$

(ii) We say that a metric space (X, d) is complete *if every Cauchy sequence converges.*

Example 8.5. *Let $X = \mathbb{R}$ and let d be the usual metric on $\mathbb{R}$. Let $Y = (0, 1)$ (the open interval with end points 0 and 1) and let ρ be the usual metric on $(0, 1)$. Then (X, d) and (Y, ρ) are homeomorphic as topological spaces, but (X, d) is complete and (Y, ρ) is not.*

Proof. See page 84. □

We say that 'completeness is not a topological property'. Exercise 17.32 shows that there exist metric spaces which are not homeomorphic to any complete metric space.

In group theory, we usually prove that two groups are isomorphic by constructing an explicit isomorphism and that two groups are not isomorphic by finding a group property exhibited by one but not by the other. Similarly, in topology, we usually prove that two topological spaces are homeomorphic by constructing an explicit homeomorphism and that two topological spaces are not homeomorphic by finding a topological property exhibited by one but not by the other. Later in this course we will meet some topological properties like being Hausdorff and compactness and you will be able to tackle Exercise 16.17.

We also want to be able to construct new topological spaces from old. To do this we we make use of a simple, but useful, lemma.

Lemma 8.6. *Let X be a space and let $\mathcal{H}$ be a collection of subsets of X. Then there exists a unique topology $\tau_{\mathcal{H}}$ such that*
(i) $\tau_{\mathcal{H}} \supseteq \mathcal{H}$, and
(ii) if τ is a topology with $\tau \supseteq \mathcal{H}$, then $\tau \supseteq \tau_{\mathcal{H}}$.

Proof. The proof, which follows the standard pattern for such things, is given on page 85. □

We call $\tau_{\mathcal{H}}$ the smallest (or coarsest) topology containing $\mathcal{H}$.

Lemma 8.7. *Suppose that A is non-empty, the spaces (X_α, τ_α) are topological spaces and we have maps $f_\alpha : X \to X_\alpha$ $[\alpha \in A]$. Then there is a smallest topology τ on X for which the maps f_α are continuous.*

Proof. A topology σ on X makes all the f_α continuous if and only if it contains

$$\mathcal{H} = \{f_\alpha^{-1}(U) \, : \, U \in \tau_\alpha, \ \alpha \in A\}.$$

Now apply Lemma 8.6. □

Recall that, if $Y \subseteq X$, then the inclusion map $j : Y \to X$ is defined by $j(y) = y$ for all $y \in Y$.

Definition 8.8. *If (X, τ) is a topological space and $Y \subseteq X$, then the* subspace topology *τ_Y on Y* induced *by τ is the smallest topology on Y for which the inclusion map is continuous.*

Lemma 8.9. *If (X, τ) is a topological space and $Y \subseteq X$, then the subspace topology τ_Y on Y is the collection of sets $Y \cap U$ with $U \in \tau$.*

Proof. The very easy proof is given on page 85. □

Exercise 8.10. *(i) If (X, τ) is a topological space and $Y \subseteq X$ is open, show that the subspace topology τ_Y on Y is the collection of sets $U \in \tau$ with $U \subseteq Y$.*

(ii) Consider $\mathbb{R}$ with the usual topology τ (that is, the topology derived from the Euclidean metric). If $Y = [0, 1]$, show that $[0, 1/2) \in \tau_Y$ but $[0, 1/2) \notin \tau$.

Exercise 8.11. *Let (X, d) be a metric space, Y a subset of X and d_Y the metric d restricted to Y (formally, $d_Y : Y^2 \to \mathbb{R}$ is given by $d_Y(x, y) = d(x, y)$ for x, $y \in Y$). Then if we give X the topology induced by d, the subspace topology on Y is identical with the topology induced by d_Y.*
[*This is an exercise in stating the obvious.*]

Next recall that if X and Y are sets the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are given by

$$\pi_X(x, y) = x,$$
$$\pi_Y(x, y) = y.$$

Definition 8.12. *If (X, τ) and (Y, σ) are topological spaces, then the* product topology *μ on $X \times Y$ is the smallest topology on $X \times Y$ for which the projection maps π_X and π_Y are continuous.*

Lemma 8.13. *Let (X, τ) and (Y, σ) be topological spaces and λ the product topology on $X \times Y$. Then $O \in \lambda$ if and only if, given $(x, y) \in O$, we can find $U \in \tau$ and $V \in \sigma$ such that*

$$(x, y) \in U \times V \subseteq O.$$

Proof. See page 86. □

Exercise 14.9 gives a very slightly different treatment of the matter.

Exercise 8.14. *Suppose that (X, τ) and (Y, σ) are topological spaces and we give $X \times Y$ the product topology μ. Now fix $x \in X$ and give $E = \{x\} \times Y$ the subspace topology μ_E. Show that the map $k : (Y, \sigma) \to (E, \mu_E)$ given by $k(y) = (x, y)$ is a homeomorphism.*

Solution. The proof is a direct application of Lemma 8.13. See page 87 if necessary. □

The next remark is useful for proving results like those in Exercise 8.16.

Lemma 8.15. *Let τ_1 and τ_2 be two topologies on the same space X.*

(i) We have $\tau_1 \subseteq \tau_2$ if and only if, given $x \in U \in \tau_1$, we can find $V \in \tau_2$ such that $x \in V \subseteq U$.

(ii) We have $\tau_1 = \tau_2$ if and only if, given $x \in U \in \tau_1$, we can find $V \in \tau_2$ such that $x \in V \subseteq U$ and, given $x \in U \in \tau_2$, we can find $V \in \tau_1$ such that $x \in V \subseteq U$.

Proof. The easy proof is given on Page 87 □

Exercise 8.16. *Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let τ be the product topology on $X_1 \times X_2$ where X_j is given the topology induced by d_j $[j = 1, 2]$.*

Define $\rho_k : (X_1 \times X_2)^2 \to \mathbb{R}$ by

$$\begin{aligned}
\rho_1((x,y),(u,v)) &= d_1(x,u),\\
\rho_2((x,y),(u,v)) &= d_1(x,u) + d_2(y,v),\\
\rho_3((x,y),(u,v)) &= \max(d_1(x,u), d_2(y,v)),\\
\rho_4((x,y),(u,v)) &= (d_1(x,u)^2 + d_2(y,v)^2)^{1/2}.
\end{aligned}$$

Establish that ρ_1 is not a metric and that ρ_2, ρ_3 and ρ_4 are. Show that each of the ρ_j with $2 \leq j \leq 4$ induces the product topology τ on $X_1 \times X_2$.

It is easy to extend our definitions and results to any finite product of topological spaces[5]. In fact, it is not difficult to extend our definition to the product of an infinite collection of topological spaces, but I feel that it is important for the reader to concentrate on first thoroughly understanding the finite product case and I have relegated the infinite case to an exercise (Exercise 16.7).

We conclude this section by looking briefly at the quotient topology. This will not play a major part in our course and the reader should not worry too much about it.

If $\sim$ is an equivalence relation on a set X, then we know from previous courses that it gives rise to equivalence classes

$$[x] = \{y \in X : y \sim x\}.$$

There is a natural map q from X to the space $X/\sim$ of equivalence classes given by $q(x) = [x]$. When we defined the subspace and product topologies, we used natural maps from the new spaces to the old spaces. Here, we have a natural map from the old space to the new, so our definition has to take a different form.

[5] Once you are confident with the material you may wish to look at Exercise 17.11, but this exercise is confusing for the beginner and trivial to the expert.

Exercise 8.17. *Let* $X = \{1, 2, 3\}$ *and* $\theta = \{\varnothing, \{1\}, \{2\}, X\}$. *Check that there does not exist a topology* $\tau_1 \subseteq \theta$ *such that, if* $\tau \subseteq \theta$ *is a topology, then* $\tau \subseteq \tau_1$. *(Thus there does not exist a largest topology contained in* θ*.)*

However, since intersection and union behave well under inverse mappings, it is easy to check the following statement.

Lemma 8.18. *Let* (X, τ) *be a topological space and* Y *a set. If* $f : X \to Y$ *is a map and we write*

$$\sigma = \{U \subseteq Y \,:\, f^{-1}(U) \in \tau\},$$

then σ *is a topology on* Y *such that*

(i) $f : (X, \tau) \to (Y, \sigma)$ *is continuous and*

(ii) if θ *is a topology on* Y *with* $f : (X, \tau) \to (Y, \theta)$ *continuous, then* $\theta \subseteq \sigma$.

Lemma 8.18 allows us to make the following definition.

Definition 8.19. *Let* (X, τ) *be a topological space and* $\sim$ *an equivalence relation on* X. *Write* q *for the map from* X *to the quotient space* $X/\sim$ *given by* $q(x) = [x]$. *Then we call the largest topology* σ *on* $X/\sim$ *for which* q *is continuous that is to say*

$$\sigma = \{U \subseteq X/\!\sim\colon\, q^{-1}(U) \in \tau\}$$

the quotient topology

The following is just a restatement of the definition.

Lemma 8.20. *Under the assumptions and with the notation of Definition 8.19, the quotient topology consists of the sets* U *such that*

$$\bigcup_{[x]\in U} [x] \in \tau.$$

Later we shall give an example (Exercise 11.7) of a nice quotient topology. Exercise 16.24, which requires ideas from later in the course, is an example of really nasty quotient topology.

In general, the quotient topology can be extremely unpleasant (basically because equivalence relations form a very wide class) and although nice equivalence relations sometimes give very useful quotient topologies, you should always think before using one. Exercise 16.9 gives some further information.

9 Hausdorff spaces

When we work in a metric space, we make repeated use of the fact that, if $d(x, y) = 0$, then $x = y$. The metric is 'powerful enough to separate points'. The indiscrete topology, on the other hand, clearly cannot separate points.

When Hausdorff first crystallised the modern idea of a topological space, he included an extra condition to ensure 'separation of points'. It was later discovered that topologies without this extra condition could be useful, so it is now considered separately.

Definition 9.1. *A topological space (X, τ) is called* Hausdorff *if, whenever $x, y \in X$ and $x \neq y$, we can find $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.*

In the English educational system, it is traditional to draw U and V as little huts containing x and y and to say that x and y are 'housed off from each other'.

The next exercise requires a one line answer, but you should write that line down.

Exercise 9.2. *Show that, if (X, d) is a metric space, then the derived topology is Hausdorff.*

Although we defer the discussion of neighbourhoods in general to towards the end of the course, it is natural to introduce the following locution here.

Definition 9.3. *If (X, τ) is a topological space and $x \in U \in \tau$, we call U an* open neighbourhood *of x.*

Exercise 9.4. *If (X, τ) is a topological space, then a subset A of X is open if and only if every point of A has an open neighbourhood $U \subseteq A$.*

Proof. The easy proof is given on page 88. □

Lemma 9.5. *If (X, τ) is a Hausdorff space, then the one point sets $\{x\}$ are closed.*

Proof. The easy proof is given on page 88. □

The following exercise shows that the converse to Lemma 9.5 is false and that, if we are to acquire any intuition about topological spaces, we will need to study a wide range of examples.

Exercise 9.6. *Let X be infinite (we could take $X = \mathbb{Z}$ or $X = \mathbb{R}$). We say that a subset E of X lies in τ if either $E = \varnothing$ or $X \setminus E$ is finite. Show that τ is a topology and that every one point set $\{x\}$ is closed, but that (X, τ) is not Hausdorff.*

What happens if X is finite?

Solution. See page 88. □

Exercise 9.7. *Prove Exercise 7.9 (i) and (ii) with '(Y, d) a metric space' replaced by '(Y, σ) a Hausdorff topological space'.*

It is easy to give examples of topologies which are not derived from metrics. It is somewhat harder to give examples of Hausdorff topologies which are not derived from metrics. An important example is given in Exercise 16.10.

The next two lemmas are very useful.

Lemma 9.8. *If (X, τ) is a Hausdorff topological space and $Y \subseteq X$, then Y with the subspace topology is also Hausdorff.*

Proof. The easy proof is given on page 89. □

Lemma 9.9. *If (X, τ) and (Y, σ) are Hausdorff topological spaces, then $X \times Y$ with the product topology is also Hausdorff.*

Proof. The proof is easy (but there is one place where you can make a silly mistake). See page 89. □

Exercise 16.9 shows that, even when the original topology is Hausdorff, the resulting quotient topology need not be.

10 Compactness

Halmos says somewhere that if an idea is used once it is a trick, if used twice it is a method, if used three times a theorem but if used four times it becomes an axiom.

Several important theorems in analysis hold for closed bounded intervals. Heine used a particular idea to prove one of these. Borel isolated the idea as a theorem (the Heine–Borel theorem), essentially Theorem 10.6 below. Many treatments of analysis (for example, Hardy's *Pure Mathematics*) use the Heine–Borel theorem as a basic tool. The notion of compactness represents the last stage in the Halmos progression.

Definition 10.1. *A topological space* (X, τ) *is called* compact *if, whenever we have a collection* U_α *of open sets* $[\alpha \in A]$ *with* $\bigcup_{\alpha \in A} U_\alpha = X$*, we can find a finite subcollection* $U_{\alpha(1)}, U_{\alpha(2)}, \dots, U_{\alpha(n)}$ *with* $\alpha(j) \in A$ $[1 \le j \le n]$ *such that* $\bigcup_{j=1}^n U_{\alpha(j)} = X$.

Definition 10.2. *If* (X, τ) *is a topological space, then a subset* Y *is called* compact *if the subspace topology on* Y *is compact.*

The reader should have no difficulty in combining these two definitions to come up with the following restatement,

Lemma 10.3. *If* (X, τ) *is a topological space, then a subset* Y *is compact if, whenever we have a collection* U_α *of open sets* $[\alpha \in A]$ *with* $\bigcup_\alpha U_\alpha \supseteq Y$*, we can find a finite subcollection* $U_{\alpha(1)}, U_{\alpha(2)}, \dots, U_{\alpha(n)}$ *with* $\alpha(j) \in A$ $[1 \le j \le n]$ *such that* $\bigcup_{j=1}^n U_{\alpha(j)} \supseteq Y$.

In other words, 'a set is compact if any cover by open sets has a finite subcover'.

The reader is warned that compactness is a subtle property which requires time and energy to master[6]. (At the simplest level, a substantial minority of examinees fail to get the definition correct.) Up to this point most of the proofs in this course have been simple deductions from definitions. Several of our theorems on compactness go much deeper and have quite intricate proofs.

Here are some simple examples of compactness and non-compactness.

Exercise 10.4. *(i) Show that, if* X *is finite, every topology on* X *is compact.*

(ii) Show that the discrete topology on a set X *is compact if and only if* X *is finite.*

(iii) Show that the indiscrete topology is always compact.

(iv) Show that the topology described in Exercise 9.6 is compact.

(v) Let X *be uncountable (we could take* $X = \mathbb{R}$*). We say that a subset* A *of* X *lies in* τ *if either* $A = \varnothing$ *or* $X \setminus A$ *is countable. Show that* τ *is a topology but that* (X, τ) *is not compact.*

Solution. There is a partial solution for parts (iv) and (v) on page 89. □

The next lemma will serve as a simple exercise on compactness but is also important in its own right.

[6] My generation only reached compactness after a long exposure to the classical Heine–Borel theorem.

Lemma 10.5. *Suppose that (X, d) is a compact metric space (that is to say, the topology induced by the metric is compact).*

(i) Given any $\delta > 0$, we can find a finite set of points E such that $X = \bigcup_{e \in E} B(e, \delta)$.

(ii) X has a countable dense subset.

Proof. See 90. □

Observe that $\mathbb{R}$ with the usual metric has a countable dense subset but is not compact.

We now come to our first major theorem.

Theorem 10.6. [The Heine–Borel Theorem.] *Let $\mathbb{R}$ be given its usual topology (that is to say the topology derived from the usual Euclidean metric). Then the closed bounded interval $[a, b]$ is compact.*

Proof. I give a hint on page 62 and a proof on 91. An alternative proof, which is much less instructive, is given on page 34. □

Lemma 10.3 gives the following equivalent statement.

Theorem 10.7. *Let $[a, b]$ be given its usual topology (that is to say the topology derived from the usual Euclidean metric). Then the derived topology is compact.*

We now have a couple of very useful results.

Theorem 10.8. *A closed subset of a compact set is compact. [More precisely, if E is compact and F closed in a given topology, then, if $F \subseteq E$, it follows that F is compact.]*

Proof. This is easy if you look at it the right way. See page 91. □

Theorem 10.9. *If (X, τ) is Hausdorff, then every compact set is closed.*

Proof. This is harder, though it becomes easier if you realise that you must use the fact that τ is Hausdorff (see Example 10.11 below). There is a hint on page 63 and a proof on page 92. □

Exercise 10.10. *Why does Theorem 10.9 give an immediate proof of Lemma 9.5?*

Example 10.11. *Give an example of a topological space (X, τ) and a compact set in X which is not closed.*

Proof. There is a topological space with two points which will do. See page 92. □

Combining the Heine–Borel theorem with Theorems 10.8 and 10.9 and a little thought, we get a complete characterisation of the compact subsets of $\mathbb{R}$ (with the standard topology).

Theorem 10.12. *Consider $(\mathbb{R}, \tau)$ with the standard (Euclidean) topology. A set E is compact if and only if it is closed and bounded (that is to say, there exists a M such that $|x| \leq M$ for all $x \in E$).*

Proof. The easy proof is given on page 92. □

In Example 4.10 we saw that the continuous image of an open set need not be open. It also easy to see that the continuous image of a closed set need not be closed.

Exercise 10.13. *Let $\mathbb{R}$ have the usual metric. Give an example of a continuous injective function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R})$ is not closed.*

Hint. Look at the solution of Example 8.5 if you need a hint. □

However, the continuous image of a compact set is always compact.

Theorem 10.14. *Let (X, τ) and (Y, σ) be topological spaces and $f : X \to Y$ a continuous function. If K is a compact subset of X, then $f(K)$ is a compact subset of Y.*

Proof. This is easy if you look at it the right way. See page 93. □

This result has many delightful consequences. Recall, for example, that the quotient topology $X/\sim$ is defined in such a way that the quotient map $q : X \to X/\sim$ is continuous. Since $q(X) = X/\sim$, Theorem 10.14 gives us a positive property of the quotient topology.

Theorem 10.15. *Let (X, τ) be a compact topological space and $\sim$ an equivalence relation on X. Then the quotient topology on $X/\sim$ is compact.*

The next result follows at once from our characterisation of compact sets for the real line with the usual topology.

Theorem 10.16. *Let $\mathbb{R}$ have the usual metric. If K is a closed and bounded subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous, then $f(K)$ is closed and bounded.*

This gives a striking extension of one of the crowning glories of a first course in analysis.

Theorem 10.17. *Let $\mathbb{R}$ have the usual metric. If K is a closed and bounded subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous, then f is bounded and attains its bounds.*

Proof. The straightforward proof is given on page 93. ☐

Theorem 10.17 is complemented by the following observation.

Exercise 10.18. *Let $\mathbb{R}$ have the usual metric.*

(i) If K is subset of $\mathbb{R}$ with the property that, whenever $f : K \to \mathbb{R}$ is continuous, f is bounded, show that that K is closed and bounded.

(ii) If K is subset of $\mathbb{R}$ with the property that, whenever $f : K \to \mathbb{R}$ is continuous and bounded, then f attains its bounds show that K is closed and bounded.

Proof. See page 94. ☐

Exercise 17.3 gives a stronger result, but will be easier to tackle when the reader has done the section on sequential compactness (Section 12). Theorem 10.17 has the following straightforward generalisation whose proof is left to the reader.

Theorem 10.19. *If K is a compact space and $f : K \to \mathbb{R}$ is continuous then f is bounded and attains its bounds.*

We also have the following useful result.

Theorem 10.20. *Let (X, τ) be a compact and (Y, σ) a Hausdorff topological space. If $f : X \to Y$ is a continuous bijection, then it is a homeomorphism.*

Proof. There is a hint on page 63 and a proof on page 94. ☐

Theorem 10.20 is illuminated by the following almost trivial remark.

Lemma 10.21. *Let τ_1 and τ_2 be topologies on the same space X. The identity map*

$$\iota : (X, \tau_1) \to (X, \tau_2)$$

from X with topology τ_1 to X with topology τ_2 given by $\iota(x) = x$ is continuous if and only if $\tau_1 \supseteq \tau_2$.

Theorem 10.22. *Let τ_1 and τ_2 be topologies on the same space X.*

(i) If $\tau_1 \supseteq \tau_2$ and τ_1 is compact, then so is τ_2.

(ii) If $\tau_1 \supseteq \tau_2$ and τ_2 is Hausdorff, then so is τ_1.

(iii) If $\tau_1 \supseteq \tau_2$, τ_1 is compact and τ_2 is Hausdorff, then $\tau_1 = \tau_2$.

Proof. The routine proof is given on page 95. ☐

The reader may care to recall that 'Little Bear's porridge was neither too hot nor too cold but just right'.

With the hint given by the previous theorem it should be fairly easy to do do the next exercise.

Exercise 10.23. *(i) Give an example of a Hausdorff space (X, τ) and a compact Hausdorff space (Y, σ) together with a continuous bijection $f : X \to Y$ which is not a homeomorphism.*

(ii) Give an example of a compact Hausdorff space (X, τ) and a compact space (Y, σ) together with a continuous bijection $f : X \to Y$ which is not a homeomorphism.

Solution. See page 95. □

We shall give a (not terribly convincing) example of the use of Theorem 10.20 in our proof of Exercise 11.7.

The reader may have gained the impression that compact Hausdorff spaces form an ideal backdrop for continuous functions to the reals. Later work shows that the impression is absolutely correct, but it must be remembered that many important spaces (including the real line with the usual topology) are not compact.

11 Products of compact spaces

The course contains one further major theorem on compactness.

Theorem 11.1. *The product of two compact spaces is compact. (More formally, if (X, τ) and (Y, σ) are compact topological spaces and λ is the product topology, then $(X \times Y, \lambda)$ is compact.)*

Proof. There is a very substantial hint on page 63 and a proof on page 95. □

Tychonov showed that the general product of compact spaces is compact (see the note to Exercise 16.7) so Theorem 11.1 is often referred to as Tychonov's theorem.

The same proof, or the remark that the subspace topology of a product topology is the product topology of the subspace topologies (see Exercise 16.11), gives the closely related result.

Theorem 11.2. *Let (X, τ) and (Y, σ) be topological spaces and let λ be the product topology. If K is a compact subset of X and L is a compact subset of Y, then $K \times L$ is a compact in λ.*

We know (see Exercise 8.16) that the topology on $\mathbb{R}^2$ derived from the Euclidean metric is the same as the product topology when we give $\mathbb{R}$ the topology derived from the Euclidean metric. Theorem 10.7 thus has the following corollary.

Theorem 11.3. *$[a, b] \times [c, d]$ with its usual (Euclidean) topology is compact.*

The arguments of the previous section carry over to give results like the following[7].

Theorem 11.4. *Consider $\mathbb{R}^2$ with the standard (Euclidean) topology. A set E is compact if and only if it is closed and bounded (that is to say, there exists a M such that $\|\mathbf{x}\| \leq M$ for all $\mathbf{x} \in E$).*

Theorem 11.5. *Let $\mathbb{R}^2$ have the usual metric. If K is a closed and bounded subset of $\mathbb{R}^2$ and $f : K \to \mathbb{R}$ is continuous, then f is bounded and attains its bounds.*

Exercise 11.6. *Let $\mathbb{R}^2$ have the usual metric. If K is a subset of $\mathbb{R}^2$ with the property that, whenever $f : K \to \mathbb{R}$ is continuous, then f is bounded, show that K is closed and bounded. Let $\mathbb{R}^2$ have the usual metric. If K is a subset of $\mathbb{R}^2$ with the property that, whenever $f : K \to \mathbb{R}$ is continuous, and bounded, then f attains its bounds, show that K is closed and bounded.*

The generalisation to $\mathbb{R}^n$ is left to the reader.

The next exercise brings together many of the themes of this course. The reader should observe that we *know* what we want the circle to look like. This exercise *checks* that defining the circle via quotient maps gives us what we want.

Exercise 11.7. *Consider the complex plane with its usual metric. Let*

$$\partial D = \{z \in \mathbb{C} : |z| = 1\}$$

and give ∂D the subspace topology τ. Give $\mathbb{R}$ its usual topology and define an equivalence relation $\sim$ by $x \sim y$ if $x - y \in \mathbb{Z}$. We write $\mathbb{R}/\sim = \mathbb{T}$ and give

[7] Stated more poetically by Conway.

> If E's closed and bounded, says Heine–Borel,
> And also Euclidean, then we can tell
> That, if it we smother
> With a large open cover,
> There's a finite refinement as well.

$\mathbb{T}$ the quotient topology. The object of this exercise is to show that ∂D and $\mathbb{T}$ are homeomorphic.

(i) Verify that $\sim$ is indeed an equivalence relation.

(ii) Show that, if we define $f : \mathbb{R} \to \partial D$ by $f(x) = \exp(2\pi i x)$, then $f(U)$ is open whenever U is open.

(iii) If $q : \mathbb{R} \to \mathbb{T}$ is the quotient map $q(x) = [x]$ show that $q(x) = q(y)$ if and only if $f(x) = f(y)$. Deduce that $q\big(f^{-1}(\{\exp(2\pi i x)\})\big) = [x]$ and that the equation $F(\exp(2\pi i x)) = [x]$ gives a well defined bijection $F : \partial D \to \mathbb{T}$.

(iv) Show that $F^{-1}(V) = f\big(q^{-1}(V)\big)$ and deduce that F is continuous.

(v) Show that $\mathbb{T}$ is Hausdorff and explain why ∂D is compact. Deduce that F is a homeomorphism.

Solution. See page 96. □

12 Compactness in metric spaces

When we work in $\mathbb{R}$ (or, indeed, in $\mathbb{R}^n$) with the usual metric, we often use the theorem of Bolzano–Weierstrass that every sequence in a bounded closed set has a subsequence with a limit in that set. It is also easy to see that closed bounded sets are the only subsets of $\mathbb{R}^n$ which have the property that every sequence in the set has a subsequence with a limit in that set. This suggests a series of possible theorems some of which turn out to be false.

Example 12.1. *Give an example of a metric space (X, d) which is bounded (in the sense that there exists an M with $d(x, y) \leq M$ for all x, $y \in X$) but for which there exist sequences with no convergent subsequence.*

Solution. We can find such a space within our standard family of examples. See page 98. □

Fortunately we do have a very neat and useful true theorem.

Definition 12.2. *A metric space (X, d) is said to be* sequentially compact *if every sequence in X has a convergent subsequence.*

Theorem 12.3. *A metric space is sequentially compact if and only if it is compact.*

We prove the if and only if parts separately. The proof of the if part is quite simple when you see how.

Theorem 12.4. *If the metric space (X, d) is compact, then it is sequentially compact.*

Proof. There is a hint on page 63 and a proof on page 98 □

Here is a simple but important consequence.

Theorem 12.5. *If the metric space (X,d) is compact, then d is complete.*

Proof. The easy proof is given on page 99. It uses a remark of independent interest given below as Lemma 12.6. □

Lemma 12.6. *Let (X,d) be a metric space. If a subsequence of a Cauchy sequences converges, then the series converges.*

Proof. The easy proof is given on page 99. □

Observe that $\mathbb{R}$ with the usual Euclidean metric is complete but not compact.

The only if part of Theorem 12.3 is more difficult to prove (but also, in my opinion, less important). We start by proving a result of independent interest.

Lemma 12.7. *Suppose that (X,d) is a sequentially compact metric space and that the collection U_α with $\alpha \in A$ is an open cover of X. Then there exists a $\delta > 0$ such that, given any $x \in X$, there exists an $\alpha(x) \in A$ such that the open ball $B(x,\delta) \subseteq U_{\alpha(x)}$.*

Proof. There is a hint on page 64 and a proof on page 99. □

We now prove the required result.

Theorem 12.8. *If the metric space (X,d) is sequentially compact, it is compact.*

Proof. There is a hint on page 64 and a proof on page 100. □

This gives an alternative, but less instructive, proof of the theorem of Heine–Borel.

Alternative proof of Theorem 10.6. By the Bolzano–Weierstrass theorem, $[a,b]$ is sequentially compact. Since we are in a metric space, it follows that $[a,b]$ is compact. □

If you prove a theorem on metric spaces using sequential compactness it is good practice to try and prove it directly by compactness. (See, for example, Exercise 16.16.)

The reader will hardly need to be warned that this section dealt only with metric spaces. Naive generalisations to general topological spaces are likely to be meaningless or false.

13 Connectedness

This section deals with a problem which the reader will meet (or has met) in her first complex variable course. Here is a similar problem that occurs on the real line. Suppose that U is an open subset of $\mathbb{R}$ (in the usual topology) and $f : U \to \mathbb{R}$ is a differentiable function with $f'(u) = 0$ for all $u \in U$. We would like to conclude that f is constant, but the example $U = (-2, -1) \cup (1, 2)$, $f(u) = 1$ if $u > 0$, $f(u) = -1$ if $u < 0$ shows that the general result is false. What extra condition should we put on U to make the result true?

After some experimentation, mathematicians have come up with the following idea.

Definition 13.1. *A topological space (Y, σ) is said to be* disconnected *if we can find non-empty open sets U and V such that $U \cup V = Y$ and $U \cap V = \varnothing$. A space which is not disconnected is called* connected.

Definition 13.2. *If E is a subset of a topological space (X, τ), then E is called* connected *(respectively* disconnected*) if the subspace topology on E is connected (respectively disconnected).*

The definition of a subspace topology gives the following alternative characterisation which the reader may prefer.

Lemma 13.3. *If E is a subset of a topological space (X, τ), then E is disconnected if and only if we can find open sets U and V such that $U \cup V \supseteq E$, $U \cap V \cap E = \varnothing$, $U \cap E \neq \varnothing$ and $V \cap E \neq \varnothing$*

We now look at another characterisation of connectedness which is very useful but requires a little preliminary work

Lemma 13.4. *Let (X, τ) be a topological space and A a set. Let Δ be the discrete topology on A. The following statements about a function $f : X \to A$ are equivalent.*

(i) If $x \in X$ we can find a $U \in \tau$ with $x \in U$ such that f is constant on U.

(ii) If $x \in A$, $f^{-1}(\{x\}) \in \tau$

(iii) The map $f : (X, \tau) \to (A, \Delta)$ is continuous.

Proof. Immediate. □

If the conditions of Lemma 13.4 apply, we say that f is *locally constant*.

Theorem 13.5. *If A contains at least two points, then a topological space (X, τ) is connected if and only if every locally constant function $f : X \to A$ is constant.*

Proof. The proof given on page 100 shows how easy it is to use connectedness. □

We have answered the question which began this section.

Since $\mathbb{Z}$ and $\{0, 1\}$ have the discrete topology when considered as subspaces of $\mathbb{R}$ with the usual topology, we have the following corollary.

Lemma 13.6. *(i) A topological space (X, τ) is connected if and only if every continuous integer valued function $f : X \to \mathbb{R}$ (where $\mathbb{R}$ has its usual topology) is constant.*

(ii) A topological space (X, τ) is connected if and only if every continuous function $f : X \to \mathbb{R}$ (where $\mathbb{R}$ has its usual topology) which only takes the values 0 or 1 is constant.

The following deep result is now easy to prove.

Theorem 13.7. *If we give $\mathbb{R}$ the usual topology, then the intervals $[a, b]$ are connected.*

Proof. Observe that if $f : [a, b] \to \mathbb{R}$ is continuous then if $f(x) = 1$ and $f(y) = 0$ the intermediate value theorem tells us that there is some z between x and y such that $f(z) = 1/2$. (For a more direct alternative see Exercise 17.4.) □

The reader will find it instructive to use Lemma 13.6 (ii) to prove parts (i) and (iii) of the next exercise.

Exercise 13.8. *Prove the following results.*

(i) If (X, τ) and (Y, σ) are topological spaces, E is a connected subset of X and $g : E \to Y$ is continuous, then $g(E)$ is connected. (More briefly, the continuous image of a connected set is connected.)

(ii) If (X, τ) is a connected topological space and $\sim$ is an equivalence relation on X, then $X/\sim$ with the quotient topology is connected.

(iii) If (X, τ) and (Y, σ) are connected topological spaces, then $X \times Y$ with the product topology is connected.

(iv) If (X, τ) is a connected topological space and E is a subset of X, then it does not follow that E with the subspace topology is connected.

Solution. See page 101. □

The next lemma will be required shortly.

Lemma 13.9. *Let E be a subset of a topological space (X, τ). If E is connected so is $\mathrm{Cl}\, E$.*

Proof. See page 101. □

The following lemma outlines a very natural development.

Lemma 13.10. *We work in a topological space (X, τ).*

(i) Let $x_0 \in X$. If $x_0 \in E_\alpha$ and E_α is connected for all $\alpha \in A$, then $\bigcup_{\alpha \in A} E_\alpha$ is connected.

(ii) Write $x \sim y$ if there exists a connected set E with $x, y \in E$. Then $\sim$ is an equivalence relation.

(iii) The equivalence classes $[x]$ are connected.

(iv) If F is connected and $F \supseteq [x]$, then $F = [x]$.

Proof. If you need more details, see page 102. □

The sets $[x]$ are known as the *connected components* of (X, τ).

Applying Lemma 13.9 with $E = [x]$ we get the following result.

Lemma 13.11. *The connected components of a topological space are closed. If there are only finitely many components then they are open.*

Exercise 16.21 provides an important example of a topological space in which the connected components consist of single points. These components are not open.

Connectedness is related to another, older, concept.

Definition 13.12. *Let (X, τ) be a topological space. We say that $x, y \in X$ are* path-connected *if (when $[0, 1]$ is given its standard Euclidean topology) there exists a continuous function $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.*

Of course, γ is referred to as a *path* from x to y.

Lemma 13.13. *If (X, τ) is a topological space and we write $x \sim y$ if x is path-connected to y, then $\sim$ is an equivalence relation.*

Proof. This just a question of getting the notation under control. There is a proof on page 103. □

We say that a topological space is *path-connected* if every two points in the space are path-connected.

The following theorem is often useful.

Theorem 13.14. *If a topological space is path-connected, then it is connected.*

Proof. This is not hard. There is a proof on page 104. □

Exercise 13.15. *Show that the bounded connected subsets of* $\mathbb{R}$ *(with the usual topology) are the intervals. (By intervals we mean sets of the form* $[a, b]$, $[a, b)$, $(a, b]$ *and* (a, b) *with* $a \leq b$. *Note that* $[a, a] = \{a\}$, $(a, a) = \varnothing$.*)*

Describe, without proof, all the connected subsets of $\mathbb{R}$.

Solution. See page 104. □

The converse of Theorem 13.14 is false (see Example 13.17 below) but there is one very important case where connectedness implies path-connectedness.

Theorem 13.16. *If we give* $\mathbb{R}^n$ *the usual topology, then any open set* Ω *which is connected is path-connected.*

Proof. There is a hint on page 64 and a proof on page 105. □

The following example shows that, even in $\mathbb{R}^2$, we cannot remove the condition Ω open.

Example 13.17. *We work in* $\mathbb{R}^2$ *with the usual topology. Let*

$$E_1 = \{(0, y) : |y| \leq 1\} \text{ and } E_2 = \{(x, \sin 1/x) : 0 < x \leq 2/\pi\}$$

and set $E = E_1 \cup E_2$.

(i) Sketch E.

(ii) Explain why E_1 *and* E_2 *are path-connected and show that* E *is closed and connected.*

(iii) Suppose, if possible, that $\mathbf{x} : [0, 1] \to E$ *is continuous and* $\mathbf{x}(0) = (1, \sin 1)$, $\mathbf{x}(1) = (0, 0)$. *Explain why we can find* $0 < t_1 < t_2 < t_3 < \dots$ *such that* $x(t_j) = \big((j + \frac{1}{2})\pi\big)^{-1}$. *By considering the behaviour of* t_j *and* $y(t_j)$, *obtain a contradiction.*

(iv) Deduce that E *is not path-connected.*

Proof. Parts (ii) to (iv) are done on page 105. □

Paths play an important role in complex analysis and algebraic topology.

14 The language of neighbourhoods

One of the lines of thought involved in the birth of analytic topology was initiated by Riemann. We know that many complicated mathematical structures can be considered as a space which locally looks like a simpler space. Thus the surface of the globe we live on is sphere but we consider it locally as a plane (ie like $\mathbb{R}^2$). The space we live in looks locally like $\mathbb{R}^3$ but its global structure could be very different. For example, Riemann says 'Space would necessarily be finite if ... [we] ascribed to it a constant curvature, as long as that curvature had a positive value, however small.' [Riemann's discussion *On the Hypotheses which lie at the Foundations of Geometry* is translated and discussed in the second volume of Spivak's *Differential Geometry.*]

Unfortunately the mathematical language of his time was not broad enough to allow the expression of Riemann's insights. If we are given a particular surface such as sphere, it is easy, starting with the complete structure, to see what 'locally' and 'resembles' might mean, but, in general, we seem to be stuck in a vicious circle with 'locally' only meaningful when the global structure is known and the global structure only knowable when the meaning of 'locally' is known.

The key to the problem was found by Hilbert who, in the course of his investigations into the axiomatic foundations of geometry, produced an axiomatisation of the notion of neighbourhood in the Euclidean plane $\mathbb{R}^2$. By developing Hilbert's ideas, Weyl obtained what is essentially the modern definition of a Riemann surface (this object, which looks locally like $\mathbb{C}$, was another brilliant creation of Riemann).

However, although the notion of an abstract space with an abstract notion of closeness was very much in the air, there were a large number of possible candidates for such an abstraction. It was the achievement of Hausdorff to see in Hilbert's work the general notion of a neighbourhood.

Although Hausdorff defined topologies in terms of neighbourhoods, it appears to be technically easier to define topologies in terms of open sets as we have done in this course. However, topologists still use the notion of neighbourhoods.

We have already defined an open neighbourhood of x to be an open set containing x. We now give the more general definition.

Definition 14.1. *Let (X, τ) be a topological space. If $x \in X$, we say that N is a* neighbourhood *of x if we can find $U \in \tau$ with $x \in U \subseteq N$.*

The reader may check her understanding by proving the following easy lemmas.

Lemma 14.2. *Let (X, τ) be a topological space. Then $U \in \tau$ if and only if, given $x \in U$, we can find a neighbourhood N of x with $N \subseteq U$.*

Proof. The easy proof is given on page 107. □

Lemma 14.3. *Let (X, τ) and (Y, σ) be topological spaces. Then $f : X \to Y$ is continuous if and only if, given $x \in X$ and M a neighbourhood of $f(x)$ in Y, we can find a neighbourhood N of x with $f(N) \subseteq M$.*

Proof. The easy proof is given on page 107. □

Exercise 14.4. *(i) If (X,d) is a metric space, show that N is a neighbourhood of x if and only we can find an $\epsilon > 0$ such that the open ball $B(x,\epsilon) \subseteq N$.*

(ii) Consider $\mathbb{R}$ with the usual topology. Give an example of a neighbourhood which is not an open neighbourhood. Give an example of an unbounded neighbourhood. Give an example of a neighbourhood which is not connected.

Here is another related way of looking at topologies which we have not used explicitly, but which can be useful.

Definition 14.5. *Let X be a set. A collection $\mathcal{B}$ of subsets is called a* basis *if the following conditions hold.*

(i) $\bigcup_{B\in\mathcal{B}} B = X$.

(ii) If B_1, $B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ we can find a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

Lemma 14.6. *Let X be a set and $\mathcal{B}$ a collection of subsets of X. Let $\tau_{\mathcal{B}}$ be the collection of sets U such that, whenever $x \in U$ we can find a $B \in \mathcal{B}$ such that $x \in B \subset U$.*

Then $\tau_{\mathcal{B}}$ is a topology if and only if $\mathcal{B}$ is a basis.

Proof. The routine proof is given on page 107. □

Definition 14.7. *If $\mathcal{B}$ is a basis and $\tau_{\mathcal{B}}$ is as in Lemma 14.6, we say that $\mathcal{B}$ is a* basis[8] *for τ_B.*

Exercise 14.8. *Consider $\mathbb{R}^2$ with the Euclidean norm. Show that the open discs*

$$B(\mathbf{q}, 1/n) = \{\mathbf{x} : \|\mathbf{x} - \mathbf{q}\| < 1/n\}$$

with $\mathbf{q} \in \mathbb{Q}^2$ and $n \geq 1$, $n \in \mathbb{Z}$ form a countable basis $\mathcal{B}$ for the Euclidean topology. Is it true that the intersection of two elements of $\mathcal{B}$ lies in $\mathcal{B}$? Give reasons.

Exercise 14.9. *Let (X,τ) and (Y,σ) be topological spaces. Show that*

$$\mathcal{B} = \{U \times V : U \in \tau, V \in \sigma\}$$

is a basis and check, using Lemma 8.13, that it generates the product topology.

We end the course with a warning. Just as it is possible to define continuous functions in terms of neighbourhoods so it is possible to define convergence in terms of neighbourhoods. This works well in metric spaces.

[8]Since $\mathcal{B} \subseteq \tau_B$ we sometimes call $\mathcal{B}$ a 'basis of open neighbourhoods'.

Lemma 14.10. *If (X, d) is a metric space, then $x_n \to x$, if and only if given N a neighbourhood of x, we can find an n_0 (depending on N) such that $x_n \in N$ for all $n \geq n_0$.*

Proof. Immediate. □

However, things are not as simple in general topological spaces.

Definition 14.11. [WARNING. Do not use this definition without reading the commentary that follows.] *Let (X, τ) be a topological space. If $x_n \in X$ and $x \in X$ then we say $x_n \to x$ if, given N a neighbourhood of x we can find n_0 (depending on N) such that $x_n \in N$ for all $n \geq n_0$.*

Any hopes that limits of sequences will behave as well in general topological spaces are dashed by the following example.

Example 14.12. *Let $X = \{a, b\}$ with $a \neq b$. If we give X the indiscrete topology, then, if we set $x_n = a$ for all n, we have $x_n \to a$ and $x_n \to b$.*

Thus limits need not be unique.

Of course, it is possible to persist in spite of this initial shock, but the reader will find that she cannot prove the links between limits of sequences and topology that we would wish to be true. This failure is not the reader's fault. Deeper investigations into set theory reveal that sequences are inadequate tools for the study of topologies which have neighbourhood systems which are 'large in the set theoretic sense'. (Exercise 17.31 represents an attempt to show what this means.) It turns out that the deeper study of set theory reveals not only the true nature of the problem but also solutions via *nets* (a kind of generalised sequence) or *filters* (preferred by the majority of mathematicians).

15 Final remarks and books

Because the notion of a topological space is so general it applies to vast collection of objects. Many useful results apply only to some subcollection and this means that the subject contains many counterexamples to show that such and such a condition is required for a certain theorem to be true.

To the generality of mankind, the longer and more complicated a piece of mathematics appears to be, the more impressive it is. Mathematicians know that the simpler a proof or a counterexample is, the easier it is to check, understand and use. Just as it is worth taking time to see if a proof can be made simpler, so it is worth taking time to see if there is a simpler counterexample for the purpose in hand.

When searching for a counterexample we may start by looking at $\mathbb{R}$ and $\mathbb{R}^n$ with the standard metrics and subspaces like $\mathbb{Q}$, $[a, b]$, (a, b) and $[a, b)$. Then we might look at the discrete and indiscrete topologies on a space. It is often worth looking at possible topologies on spaces with a small number of points (typically 3).

As her experience grows, the reader will have a much wider range of spaces to think about. Some like those of Exercises 16.6, 17.2, and 16.21 are very useful in their own right. Some, like that of Exercise 17.20, merely provide object lessons in how strange topologies can be,

If the reader looks at a very old book on *general* (or *analytic*) topology, she may find both the language and the contents rather different from what she is used to. In 1955, Kelley wrote a book *General Topology* [1] which stabilised the content and notation which might be expected in advanced course on the subject.

Texts like [3] (now in a very cheap Dover reprint[9]) and [2] (out of print) which extracted a natural elementary course quickly appeared and later texts followed the established pattern. Both [3] and [2] are short and sweet. With luck, they should be in your college library. The book of Sutherland [4] has the possible advantage of being written for a British audience and the certain advantage of being in print.

Many books on Functional Analysis, Advanced Analysis, Algebraic Topology and Differential Geometry cover the material in this course and then go on to develop it in the directions demanded by their particular subject.

References

[1] Kelley, J. L., *General Topology*, Princeton N. J,. Van Nostrand, 1955. [Reissued by Springer in 1975 and Ishi Press in 2008.]

[2] Mansfield, M. J., *Introduction to Topology*, Princeton N. J., Van Nostrand, 1963.

[3] Mendelson, B., *Introduction to Topology*, Boston Mass., Allyn and Bacon, 1962. [Now available in a Dover reprint, New York, Dover, 1990]

[4] Sutherland, W. A., *Introduction to Metric and Topological Spaces*, Oxford, OUP, 1975. (Now in second edition.)

[9] October, 2012.

16 Exercises

Exercise 16.1. *Let X be a set and $d : X^2 \to \mathbb{R}$ a function with the following properties.*

(i)′ $d(x,x) = 0$ for all $x \in X$.

(ii)′ $d(x,y) = 0$ implies $x = y$.

(iv)′ $d(y,x) + d(y,z) \geq d(x,z)$ for all x, y, $z \in X$.

Show that d is a metric on X.

Exercise 16.2. *Let $\mathbb{R}^N$ have its usual (Euclidean) metric.*

(i) Suppose that $f_j : \mathbb{R}^{n_j} \to \mathbb{R}^{m_j}$ is continuous for $1 \leq j \leq k$. Show that the map $f : \mathbb{R}^{n_1+n_2+\ldots+n_k} \to \mathbb{R}^{m_1+m_2+\ldots+m_k}$ given by

$$f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) = (f_1(\mathbf{x}_1), f_2(\mathbf{x}_2), \ldots, f_k(\mathbf{x}_k))$$

is continuous.

(ii) Show that the map $U : \mathbb{R}^n \to \mathbb{R}^{kn}$ given by

$$U(\mathbf{x}) = (\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x})$$

is continuous.

(iii) Suppose that $g_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ is continuous for $1 \leq j \leq k$. Use the composition law to show that the map $g : \mathbb{R}^n \to \mathbb{R}^{m_1+m_2+\ldots+m_k}$ given by

$$g(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_k(\mathbf{x}))$$

is continuous.

(iv) Show that the maps $A, B : \mathbb{R}^2 \to \mathbb{R}$ given by $A(x,y) = x+y$, $B(x,y) = xy$ are continuous.

(v) Use the composition law repeatedly to show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \sin\left(\frac{xy}{x^2+y^2+1}\right)$$

is continuous. (You may use results about maps $g : \mathbb{R} \to \mathbb{R}$
[If you have difficulty with (v), try smaller subproblems. For example, can you show that $(x,y) \mapsto x^2 + y^2$ is continuous?]

Exercise 16.3. *Consider $\mathbb{R}$ with the ordinary Euclidean metric.*

(i) We know that $\sin : \mathbb{R} \to \mathbb{R}$ is continuous. Show that, if $U = \mathbb{R}$, then U is open, but $\sin U$ is not.

(ii) We define a function $f : \mathbb{R} \to \mathbb{R}$ as follows. If $x \in \mathbb{R}$, set $\langle x \rangle = x - [x]$ and write

$$\langle x \rangle = .x_1 x_2 x_3 \ldots$$

as a decimal, choosing the terminating form in case of ambiguity. If $x_{2n+1} = 0$ for all sufficiently large n, let N be the least integer such that $x_{2n+1} = 0$ for all $n \geq N$, and set

$$f(x) = (-1)^N \sum_{j=1}^{\infty} x_{2N+2j} 10^{N-j}.$$

We set $f(x) = 0$ otherwise.

Show that if U is a non-empty open set, $f(U) = \mathbb{R}$ and so $f(U)$ is open. Show that f is not continuous.

Exercise 16.4. *Let (X, d) be a metric space and let $r > 0$. Show that*

$$\overline{B(x,r)} = \{y : d(x,y) \leq r\}$$

is a closed set:

(a) By using the definition of a closed set in terms of limits.

(b) By showing that the complement of $\overline{B(x,r)}$ is open.

We call $\overline{B(x,r)}$ the closed ball *centre x and radius r.*

Exercise 16.5. *Prove Theorems 5.9 and 5.10 directly from the definition of a closed set in terms of limits without using open sets.*

Exercise 16.6. *(i) Let (X, d) be a metric space. Show that*

$$\rho(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

defines a new metric on X.

(ii) Show that, in (i), d and ρ have the the same open sets.

(iii) Suppose that d_1, d_2, ... are metrics on X. Show that

$$\theta(x,y) = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(x,y)}{1 + d_n(x,y)}$$

defines a metric θ on X.

Exercise 16.7. *(i) Suppose that A is non-empty and that (X_α, τ_α) is a topological space. Explain what is meant by saying that τ is the smallest topology on $\prod_{\alpha \in A} X_\alpha$ for which each of the projection maps $\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$ is continuous and explain why we know that it exists. We call τ the product topology.*

(ii) Show that $U \in \tau$ if and only if, given $x \in U$, we can find $U_\alpha \in \tau_\alpha$ $[\alpha \in A]$ such that

$$x \in \prod_{\alpha \in A} U_\alpha \subseteq U$$

and $U_\alpha = X_\alpha$ for all but finitely many of the α.

(iii) By considering $A = [0,1]$ and taking each (X_α, τ_α) to be a copy of $\mathbb{R}$, show that the following condition defines a topology σ on the space $\mathbb{R}^{[0,1]}$ of functions $f : [0,1] \to \mathbb{R}$. A set $U \in \sigma$ if and only if, given any $f_0 \in U$, there exists an $\epsilon > 0$ and $x_1, x_2, \ldots, x_n \in [0,1]$ such that

$$\{f \in \mathbb{R}^{[0,1]} : |f(x_j) - f_0(x_j)| < \epsilon \text{ for all } 1 \le j \le n\} \subseteq U.$$

[The reader who cannot see the point of this topology is in good, but mistaken, company. The great topologist Alexandrov recalled that when Tychonov (then aged only 20) produced this definition 'His chosen ... definition seemed not only unexpected but perfectly paradoxical. [I remember] with what mistrust [I] met Tychonov's proposed definition. How was it possible that a topology induced by means of such enormous neighbourhoods, which are only distinguished from the whole space by a finite number of the coordinates, could catch any of the essential characteristics of a topological product?' However, Tychonov's choice was justified by its consequences, in particular, the generalisation (by Tychonov) of Theorem 11.1 to show that the (Tychonov) product of compact spaces is compact. This theorem called Tychonov's theorem *is one of the most important in modern analysis.*

In common with many of the most brilliant members of the Soviet school, Tychonov went on to work in a large number of branches of pure and applied mathematics. His best known work includes a remarkable paper on solutions of the heat equation[10].]

Exercise 16.8. [The Kuratowski problem[11]] *We work in a topological space (X, τ).*

(i) If A is a subset of X show that $x \in \operatorname{Cl} A \setminus \operatorname{Int} A$ if and only if, whenever $x \in U \in \tau$, we have $U \cap A \neq \varnothing$ and $U \cap A^c \neq \varnothing$.

(ii) Find a set A of $\mathbb{R}$ with the usual topology such that A, $\operatorname{Cl} A$, $\operatorname{Int} A \operatorname{Cl}$ and $\operatorname{Cl} \operatorname{Int} \operatorname{Cl} A$ are all distinct.

(iii) Show that if A is any subset of X then

$$\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} A))) = \operatorname{Int}(\operatorname{Cl} A).$$

[10] A substantial part of Volume 22, Number 2 of *Russian Mathematical Surveys* 1967 is devoted to Tychonov and his work. The quotation from Alexandrov is taken from there.

[11] So called because Kuratowski solved it.

(iv) Deduce that, starting from a set A, the operations of taking interior and closure in various orders can produce at most seven different sets (including A itself).

(v) Find a subset B of $\mathbb{R}$ with the usual topology such that the operations of taking closures and interiors in various orders produce exactly seven different sets.

Exercise 16.9. *Consider $\mathbb{R}$ with the usual (Euclidean) topology. Let $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Show that $\sim$ is an equivalence relation. Show that $\mathbb{R}/\sim$ is uncountable but that the quotient topology on $\mathbb{R}/\sim$ is the indiscrete topology.*

Exercise 16.10. *(i) If (X, σ) is a topology derived from a metric, show that, given $x \in X$, we can find open sets U_j $[1 \leq j]$ such that $\{x\} = \bigcap_{j=1}^{\infty} U_j$.*

(ii) Show, by verifying the conditions for a topological space directly (so you may not quote Exercise 16.7), that the following condition defines a topology τ on the space $\mathbb{R}^{[0,1]}$ of functions $f : [0,1] \to \mathbb{R}$. A set $U \in \tau$ if and only if, given any $f_0 \in U$, there exists an $\epsilon > 0$ and $x_1, x_2, \ldots, x_n \in [0,1]$ such that

$$\{f \in \mathbb{R}^{[0,1]} : |f(x_j) - f_0(x_j)| < \epsilon \textit{ for } 1 \leq j \leq n\} \subseteq U.$$

(iii) Show that the topology τ is Hausdorff but cannot be derived from a metric.

Exercise 16.11. *Let (X, τ) and (Y, σ) be topological spaces with subsets E and F. Let the subspace topology on E be τ_E and the subspace topology on F be σ_F. Let the product topology on $X \times Y$ derived from τ and σ be λ and let the product topology on $E \times F$ derived from τ_E and σ_F be μ. Show that μ is the subspace topology on $E \times F$ derived from λ.*

Exercise 16.12. *(i) Let $\mathcal{H}_i$ be a collection of subsets of X_i and let τ_i be the smallest topology on X_i containing $\mathcal{H}_i$ $[i = 1, 2]$. If $f : X_1 \to X_2$ has the property that $f^{-1}(H) \in \mathcal{H}_1$ whenever $H \in \mathcal{H}_2$, show that f is continuous (with respect to the topologies τ_1 and τ_2).*

(ii) Suppose that (X, τ) and (Y, σ) are topological space and we give $X \times Y$ the product topology. If (Z, λ) is a topological space, show that $f : Z \to X \times Y$ is continuous if and only if $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f : Z \to Y$ are continuous.

(iii) Let $\mathbb{R}$ have the usual topology (induced by the Euclidean metric) and let $\mathbb{R}^2$ have the product topology (which we know to be the usual topology induced by the Euclidean metric). Define

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$$

Show that, if we define $h_x(y) = g_y(x) = f(x,y)$ for all $(x,y) \in \mathbb{R}^2$, then the function $h_x : \mathbb{R} \to \mathbb{R}$ is continuous for each $x \in \mathbb{R}$ and the function $g_y : \mathbb{R} \to \mathbb{R}$ is continuous for each $y \in \mathbb{R}$. Show, however, that f is not continuous.

Exercise 16.13. *In complex variable theory we encounter 'uniform convergence on compacta'. This question illustrates the basic idea in the case of $C(\Omega)$ the space of* continuous *functions $f : \Omega \to \mathbb{C}$ where*

$$\Omega = \{z \in \mathbb{C} : |z| < 1\}.$$

(i) Show, by means of an example, that an $f \in C(\Omega)$ need not be bounded on Ω.

(ii) Explain why

$$d_n(f,g) = \sup_{|z| \leq 1-1/n} |f(z) - g(z)|$$

exists and is finite for each $n \geq 1$ and all $f,\ g \in C(\Omega)$. Show that d_n satisfies the triangle law and symmetry, but give an example of a pair of functions $f,\ g \in C(\Omega)$ with $f \neq g$ yet $d_n(f,g) = 0$.

(iii) Show that

$$d(f,g) = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(f,g)}{1 + d_n(f,g)}$$

exists and is finite for all $f,\ g \in C(\Omega)$.

(iv) Show that d is a metric on $C(\Omega)$.

[*If you require a hint, do Exercise 16.6 (i).*]

Exercise 16.14. *(i) Find the connected components of*

$$\{0\} \cup \bigcup \{1/n \,:\, n \geq 1,\ n \in \mathbb{Z}\}$$

with the usual metric.

Which are open in the subspace topology and which are not? Give reasons.

(ii) Is it true that the interior of a connected set is always connected? Give a proof or a counterexample.

Exercise 16.15. *(i) If (X,τ) and (Y,σ) are topological spaces, E is a path-connected subset of X and $g : E \to Y$ is continuous, show that $g(E)$ is path-connected. (More briefly, the continuous image of a path-connected set is path-connected.)*

(ii) If (X,τ) is a path-connected topological space and $\sim$ is an equivalence relation on X, show that $X/\!\sim$ with the quotient topology is path-connected.

(iii) If (X, τ) and (Y, σ) are path-connected topological spaces, show that $X \times Y$ with the product topology is path-connected.

(iv) If (X, τ) is a path-connected topological space and E is a subset of X, show that it does not follow that E with the subspace topology is path-connected.

Exercise 16.16. *Suppose that (X, d) is a compact metric space, (Y, ρ) is a metric space and $f : X \to Y$ is continuous. Explain why, given $\epsilon > 0$, we can find, for each $x \in X$, a $\delta_x > 0$ such that, if $d(x, y) < 2\delta_x$, it follows that $\rho(f(x), f(y)) < \epsilon/2$. By considering the open cover $B(x, \delta_x)$ and using compactness, show that there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \epsilon$. (In other words, a continuous function from a compact metric space to a metric space is uniformly continuous.)*

Exercise 16.17. *Which of the following spaces are homeomorphic and which are not? Give reasons.*

(i) $\mathbb{R}$ with the usual topology.

(ii) $\mathbb{R}$ with the discrete topology.

(iii) $\mathbb{Z}$ with the discrete topology.

(iv) $[0, 1]$ with the usual topology.

(v) $(0, 1)$ with the usual topology.

[*This is rather feeble question but in this short course we have not found enough topological properties to distinguish between some clearly distinguishable topological spaces. We return to this matter in Exercise 16.18.*]

Exercise 16.18. *Suppose that $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are continuous maps with $f(0) = -1$, $f(1) = 2$, $g(0) = 0$ and $g(1) = 1$. Show that*

$$f([0, 1]) \cap g([0, 1]) \neq \varnothing$$

(In other words, the two paths must cross.)

Show that $\mathbb{R}$ and $\mathbb{R}^2$ with the usual topologies are not homeomorphic. Are $[0, 1]$ and the circle

$$\{z \in \mathbb{C} : |z| = 1\}$$

homeomorphic and why?

(But are $\mathbb{R}^2$ and $\mathbb{R}^3$ homeomorphic? Questions like this form the beginning of modern algebraic topology.)

Exercise 16.19. *Which of the following statements are true and which false. Give a proof or counter-example.*

(i) If a topological space (X, τ) is connected then the only sets which are both open and closed are X and $\varnothing$.

(ii) If every set in a topological space (X,τ) is open or closed (or both) then τ is the discrete topology.

(iii) Every open cover of $\mathbb{R}$ with the usual topology has a countable subcover.

(iv) Suppose that τ and σ are topologies on a space X with $\sigma \supseteq \tau$. Then, if (X,τ) is connected, so is (X,σ).

(v) Suppose that τ and σ are topologies on a space X with $\sigma \supseteq \tau$. Then, if (X,σ) is connected, so is (X,τ).

Exercise 16.20. [The finite intersection property]

(i) (This result is almost trivial but very useful.) Show that a topological space (X,τ) is compact if and only if it has the following property.

If $\mathcal{F}$ is a collection of closed sets with the 'finite intersection property'

$$F_1,\ F_2,\ \ldots,\ F_n \in \mathcal{F} \Rightarrow \bigcap_{j=1}^{n} F_j \neq \varnothing,$$

then

$$\bigcap_{F\in\mathcal{F}} F \neq \varnothing.$$

(ii) We work in $\mathbb{R}$ with the usual metric. Give an example of of sequence of non-empty bounded open sets O_j such that

$$O_1 \supseteq O_2 \supseteq O_3 \supseteq \ldots,\ \textit{but}\ \bigcap_{j=1}^{\infty} O_j = \varnothing.$$

Give an example of of sequence of non-empty closed sets F_j such that

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots,\ \textit{but}\ \bigcap_{j=1}^{\infty} F_j = \varnothing.$$

.

Exercise 16.21. *Consider the space of sequences of zeros and ones $X = \{0,1\}^{\mathbb{N}}$. Let us set*

$$d(\mathbf{x},\mathbf{y}) = 2^{-n}$$

if $x_j = y_j$ for $1 \leq j \leq n-1$, $x_n \neq y_n$ and take $d(\mathbf{x},\mathbf{x}) = 0$.

(i) Show that d is a metric.

(ii) Show that (X,d) is complete.

(iii) Show that (X,d) is compact.

(iv) Show that no point in (X,d) is isolated (that is to say, no one point set $\{\mathbf{x}\}$ is open).

(v) Show that the connected components of (X, d) *are the one point sets.*

(vi) Show that $X \times X$ *with the product topology is homeomorphic to* X*.* [*The space just described may look nasty at first sight, but is, in fact, both elegant and useful.*]

Exercise 16.22. [Bases of neighbourhoods.] *(i) Let* (X, τ) *be a topological space. Write* $\mathcal{N}_x$ *for the set of neighbourhoods of* $x \in X$*. Prove the following results.*

(1) $\mathcal{N}_x \neq \varnothing$.

(2) If $N \in \mathcal{N}_x$*, then* $x \in N$.

(3) If $N,\ M \in \mathcal{N}_x$*, then* $N \cap M \in \mathcal{N}_x$.

(4) If $N \in \mathcal{N}_x$ *and* $M \supseteq N$*, then* $M \in \mathcal{N}_x$.

(5) If $N \in \mathcal{N}_x$ *then there exists an* $U \in \mathcal{N}_x$ *such that* $U \subseteq N$ *and* $U \in \mathcal{N}_y$ *for all* $y \in U$.

(ii) Suppose that X *is a set such that each* $x \in X$ *is associated with a collection* $\mathcal{N}_x$ *of subsets of* X*. If conditions (1) to (4) of part (i) hold, show that the family* τ *of sets* U *such that, if* $x \in U$*, then we can find an* $N \in \mathcal{N}_x$ *with* $N \subseteq U$ *is a topology on* X*. If, in addition, condition (5) holds show that* $\mathcal{N}_x$ *is a collection of* τ*-neighbourhoods of* x *for each* $x \in X$.

Exercise 16.23. *(The p-adic metric.) Suppose that* p *is a prime. If* $m,\ n \in \mathbb{Z}$ *we set* $d(m, n) = 0$ *if* $m = n$ *and, otherwise,* $d(m, n) = \frac{1}{r+1}$ *where* p^r *divides* $m - n$*, but* p^{r+1} *does not. Show that* d *is a metric on* $\mathbb{Z}$.

Now let $p = 5$*. Show that the sequence* 2014, 20014, 200014, … *tends to a limit in the metric. Show that the sequence* $5^n + 5^{n-1} + \ldots + 5 + 1$ *is Cauchy but does not converge.*

Exercise 16.24. *Consider* $\mathbb{R}^2$ *with the usual Euclidean topology. Let*

$$E = \{(x, -1) : x \in \mathbb{R}\} \cup \{(x, 1) : x \in \mathbb{R}\}$$

and give E *the subspace topology.*

Define a relation $\sim$ *on* E *by taking*

$$\begin{aligned} (x, y) &\sim (x, y) && \text{for all } (x, y) \in E \\ (x, y) &\sim (x, -y) && \text{for all } (x, y) \in E \text{ with } x \neq 0. \end{aligned}$$

Show that that $\sim$ *is an equivalence relation on* E.

Now give $E/\!\sim$ *the quotient topology. Show that if* $[(x, y)] \in E/\!\sim$ *we can find an open neighbourhood* U *of* $[(x, y)]$ *which is homeomorphic to* $\mathbb{R}$*. Show, however, that* $E/\!\sim$ *is not Hausdorff.*

[*This nasty example shows that 'looks nice locally' is not sufficient to give 'looks nice globally'. It is good start to a course in differential geometry to ask what extra conditions are required to make sure that a space that 'looks locally like a line' 'looks globally like a line or a circle'.*]

17 More exercises

There is an ancient superstition in Cambridge that 12 exercises are necessary and sufficient to learn six hours of lectures. If the reader does not share this superstition she may find the following exercises useful.

Exercise 17.1. *There is an apocryphal story of a Phd student who wrote a thesis on anti-metric spaces.*

(a) Let X be a set and $D : X^2 \to \mathbb{R}$ a function with the following properties.

(i) $D(x,y) \geq 0$ for all x, $y \in X$.

(ii) $D(x,y) = 0$ if and only if $x = y$.

(iii) $D(x,y) = D(y,x)$ for all x, $y \in X$.

(iv) $D(x,y) + D(y,z) \leq D(x,z)$ for all x, y, $z \in X$.

Show that X contains at most one point.

(b) Let X be a set and $P : X^2 \to \mathbb{R}$ a function with the following properties.

(i) $P(x,y) \geq 0$ for all x, $y \in X$.

(ii) $P(x,y) = 0$ if and only if $x = y$.

(iii) $P(x,y) = P(y,x)$ for all x, $y \in X$.

(iv) $P(x,y) + P(y,z) \leq P(x,z)$ for all x, y, $z \in X$ with $x \neq z$. How many points can X contain? Give reasons for your answer[12].

Exercise 17.2. *Let $q \geq 1$. Let l^q be the set of sequences of real numbers $\mathbf{a} = (a_1, a_2, \ldots)$ with $\sum_{j=1}^\infty |a_j|^q$ convergent. We write $\|\mathbf{a}\|_q = \left(\sum_{j=1}^\infty |a_j|^q\right)^{1/q}$.*

(i) If $\mathbf{a}$ is a sequence and we write

$$\mathbf{a}[N] = (a_1, a_2, \ldots, a_N, 0, 0, \ldots)$$

show that $\mathbf{a} \in l^q$ if and only if $\|\mathbf{a}[N]\|_q$ is bounded and that, if $\mathbf{a} \in l^q$, then

$$\|\mathbf{a}[N]\|_q \to \|\mathbf{a}\|_q$$

as $N \to \infty$.

(ii) Show, using (i), that l^1 and l^2 are real vector spaces, that $\| \, \|_1$ is a norm on l^1 and that $\| \, \|_2$ is a norm on l^2.

(iii) Show that $l^2 \supseteq l^1$.

(iv) Show that the identity map

$$\iota : (l^1, \| \, \|_2) \to (l^1, \| \, \|_1)$$

[12] *First Examiner*: All hail blithe spirit. Art thou bird or heavenly dancer?
Second Examiner: Answer yes or no. Give reasons for your answer.

(that is to say from l^1 with the subspace norm derived from $\| \|_2$ to l^1 with the norm $\| \|_1$) is not continuous.

Show that the identity map

$$\iota : (l^1, \| \|_1) \to (l^1, \| \|_2)$$

is continuous.

(v) If $f(a_1, a_2, \ldots) = (a_1, a_2/2, a_3/3, \ldots)$, show that

$$f : (l^1, \| \|_2) \to (l^1, \| \|_1)$$

is well defined and continuous. Is

$$f : (l^1, \| \|_1) \to (l^1, \| \|_2)$$

well defined and continuous. Give reasons

Exercise 17.3. *(i) Suppose that (X, d) is a metric space and (x_n) is a sequence in X with no convergent subsequence. Let*

$$f_n(x) = \max\big\{0, 1 - 2^n d(x_n, x)\big\}.$$

Explain why f_n is continuous.

Show that, for each $t \in X$, we can find a $\delta(t) > 0$ and an $N(t) \geq 1$ such that

$$f_n(x) = 0 \text{ for all } x \text{ with } d(x, t) < \delta(t)$$

and all $n \geq N(t)$. Deduce that, if $\lambda_n \in \mathbb{R}$ the equation

$$f(t) = \sum_{n=1}^{\infty} \lambda_n f_n(t)$$

is a well defined continuous function.

(ii) Deduce that, if (Y, ρ) is a metric space such that every continuous function $f : Y \to \mathbb{R}$ is bounded, then (Y, ρ) is compact.

Show further that, if (Y, ρ) is a metric space such that every continuous bounded function $f : Y \to \mathbb{R}$ attains its bounds, then (Y, ρ) is compact.

Exercise 17.4. *(The intermediate value theorem via Heine–Borel.)*

Suppose that $f : [0, 1] \to \mathbb{R}$ is a continuous function only taking the values 0 and 1. Explain why, given $x \in [0, 1]$, we can find a $\delta_x > 0$ such that f is constant on $[0, 1] \cap (x - \delta_x, x + \delta_x)$. By using compactness, show that f is constant. Deduce that $[0, 1]$ is connected.

Now suppose $g : [0, 1] \to \mathbb{R}$ is a continuous function with $g(0) < c < g(1)$. By considering $g^{-1}((-\infty, c))$ and $g^{-1}((c, \infty))$, use the result of the first paragraph to show that there exists a $t \in [0, 1]$ with $g(t) = c$.

Exercise 17.5. *(Very short if you use results from Section 13.) Show that any topological space can be written as the disjoint union of sets $A \in \mathcal{A}$ such that each $A \in \mathcal{A}$ has the following two properties.*

(a) If x, $y \in A$, then x and y are path-connected.

(b) If $x \in A$, $y \notin A$, then x and y are not path-connected.

The sets $A \in \mathcal{A}$ are called the path-connected components of the space.

Suppose that $\mathcal{A}$ is finite. Is it true that every path-connected component is closed? Give a proof or a counterexample.

Exercise 17.6. *Suppose that d_1 and d_2 are metrics on a space X. Show that it is a sufficient condition for them to generate the same topology that there exists a $K \geq 1$ with*

$$Kd_1(x,y) \geq d_2(x,y) \geq K^{-1}d_1(x,y)$$

for all x, $y \in X$

By considering d defined by $d(x,y) = |x-y|^{1/2}$ for x, $y \in \mathbb{R}$, or otherwise, show that the condition is not necessary.

Show, however, that, if $\| \ \|_A$ and $\| \ \|_B$ are norms on a vector space V, then it is a necessary and sufficient condition for them to generate the same topology that there exists a $K \geq 1$ with

$$K\|\mathbf{x}\|_A \geq \|\mathbf{x}\|_B \geq K^{-1}\|\mathbf{x}\|_A$$

for all $\mathbf{x} \in V$.

Exercise 17.7. *There are many proofs that there exist an infinity of primes. Here is a remarkable one published by Fürstenberg in 1955 when he was still an undergraduate.*

Consider $\mathbb{Z}$. Let $\mathcal{A}$ be the collection of arithmetic progressions

$$\{an + b \,:\, n \in \mathbb{Z}\}$$

with $a \neq 0$ and let τ be the collection of unions of sets in $\mathcal{A}$ together with $\mathbb{Z}$.

(i) Show that τ is a topology.

(ii) Show that every $A \in \mathcal{A}$ is closed in the topology τ.

(iii) If $\mathcal{A}^*$ consists of the arithmetic progressions

$$\{np \,:\, n \in \mathbb{Z}\}$$

with p prime, identify

$$\mathbb{Z} \setminus \bigcup_{A \in \mathcal{A}^*} A.$$

(iv) Suppose, if possible, there are only finitely many primes. Use parts (ii) and (iii) to obtain a contradiction.

(v) Applaud.

Exercise 17.8. *Show, by means of an example, that the following statement may be false.*

If E is subset of $\mathbb{R}$, with the usual topology, then there exists a unique open set V such that

(a) $V \supseteq E$,

(b) if U is a open set with $U \supseteq E$, then $U \supseteq V$.

Exercise 17.9. *Show that the following statements about a topological space (Y, σ) are equivalent.*

(i) (Y, σ) is Hausdorff.

(ii) If $Y \times Y$ is given the product topology, the diagonal

$$\Delta = \{(y, y) : y \in Y\}$$

is closed.

(iii) For any topological space (X, τ) and any continuous functions f, $g : X \to Y$, the set

$$\{x \in X : f(x) = g(x)\}$$

is closed.

Use the equivalence of (i) and (iii) to produce an alternative proof of the result of Exercise 9.7.

Exercise 17.10. *We work on $\mathbb{R}$. Let τ_1 be the collection of sets which are unions of half open intervals $[a, b)$ (including $\varnothing$). Let τ_2 be the collection of subsets of $\mathbb{R}$ such that either $E = \varnothing$ or $\mathbb{R} \setminus E$ is finite.*

(i) Show that τ_1 and τ_2 are topologies.

(ii) Is τ_1 Hausdorff?

(iii) Is τ_2 Hausdorff?

(iv) Is τ_1 compact?

(v) Is τ_2 compact?

(vi) Is τ_1 connected?

(vii) Is τ_2 connected?

(viii) Is the identity map $\iota : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$ continuous?

(ix) Is the identity map $\iota : (\mathbb{R}, \tau_2) \to (\mathbb{R}, \tau_1)$ continuous?

Exercise 17.11. *Let (X, τ_X), (Y, τ_Y), (Z, τ_Z) be topological spaces. Suppose that we give $X \times Y$ the product topology $\tau_{X \times Y}$ derived from τ_X and τ_Y, $(X \times Y) \times Z$ the product topology $\tau_{X \times Y}$ derived from $\tau_{X \times Y}$ and τ_Z and so on.*

(i) Show that $(X \times Y, \tau_{X \times Y})$ and $(Y \times X, \tau_{Y \times X})$ are homeomorphic.

(ii) Show that $((X \times Y) \times Z, \tau_{(X \times Y) \times Z})$ and $(X \times (Y \times Z), \tau_{X \times (Y \times Z)})$ are homeomorphic.

Exercise 17.12. *Give three short proofs of the fact that* $E = [0,1] \cap \mathbb{Q}$ *is not compact (using the usual metric) based on the the following considerations,*

(i) $[0,1] \cap E$ *is not a a closed subset of* $\mathbb{R}$.

(ii) $\cup_{j=1}^{\infty}(-1, 2^{-1/2} - 1/j) \cup (2^{-1/2} + 1/j, 2) \supseteq E$.

(iii) Since E *is countable we can write*

$$E = \{q_1, q_2, \ldots\}$$

Now observe that

$$E \subseteq \cup_{j=1}^{\infty}(q_j - 2^{-j-4}, q_j + 2^{-j-4})$$

yet $\sum_{j=1}^{n} 2^{-j-3} < 1/2$ *for all* $n \geq 1$.

Does there exist an infinite compact subset of the rationals (with the usual metric)? Give reasons.

Exercise 17.13. *Consider the space* $C_{\mathbb{R}}([0,1])$ *of continuous functions* $f : [0,1] \to \mathbb{R}$ *(with the usual metrics). Show that*

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|$$

is a well defined norm on $C_{\mathbb{R}}([0,1])$. *The general principle of uniform convergence which you meet in Analysis II tells you that this norm is complete.*

Show that

$$B = \{f : \|f\|_\infty \leq 1\}$$

is closed and bounded but not compact.
[*Contrast the theorem of Heine–Borel.*]

Exercise 17.14. *(Traditional) Fairyland may be considered as a perfectly flat, endless plane. Good Queen Ermentrude has planted an infinite forest of trees in such a way that, wherever she looks from her throne, she sees a tree. A troop of renegade beavers decide to gnaw down all but a finite set of trees so that it remains true that wherever the Queen looks from her throne, she sees a tree. Can they always do this? (Queen Ermentrude is open minded and only plants open trees.)*

Can the beavers always gnaw down an an infinite set of trees in such a way that their operations are invisible from the throne?

Suppose that the trees are planted so that there are only finitely many within any given distance of the throne. Can the beavers gnaw down all but a finite set of trees whilst remaining invisible from the throne?

Give reasons.

Exercise 17.15. **[The one point compactification]** *Let (X, τ) be a topological space (which may or may not be compact).*

(i) Write $X^* = X \cup \{\infty\}$. (Note that ∞ is just an object which is not in X. We could follow Hilbert and take it to be 'beer mug'.)

Let τ^* be the collection of sets $E \subseteq X^*$ such that either $E \in \tau$ or $E = (X \setminus K) \cup \{\infty\}$, where K is a closed compact subset of X. Show that τ^* is topology on X^* and (X^*, τ^*) is compact. Show that the subspace topology τ^*_X induced on X by τ^* is τ.

(ii) (A well known variation.) Let $\tilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ (so, this time, we add a beer mug and a sherry glass).

Show that the collection $\mathcal{E}$ of sets of the form

$$[-\infty, a) = \{-\infty\} \cup \{x \in \mathbb{R} : x < a\},\ (b, \infty] = \{\infty\} \cup \{x \in \mathbb{R} : b < x\}$$

together with the open intervals (a, b) form the basis for a compact topology $\tilde{\tau}$ on $\tilde{\mathbb{R}}$.

Exercise 17.16. *Let P_1, P_2, $\ldots$, P_n be distinct points in $\mathbb{R}^2$ and $A = \{P_1, P_2, \ldots, P_n\}$. Let $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{x} = \mathbf{y}$ or $\mathbf{x}, \mathbf{y} \in A$. Show that $\sim$ is an equivalence relation on $\mathbb{R}^2$.*

If τ is the usual Euclidean topology on $\mathbb{R}^2$, show that the topology $\tau/\sim$ on $\mathbb{R}^2/\sim$ can be derived from a metric.
[*Hint: The metric usually chosen is called the London underground metric.*]

Exercise 17.17. *Let X and Y be non-empty topological spaces, and give $X \times Y$ the product topology. Show that*

$$(x, y) \sim (u, v) \Leftrightarrow y = v$$

is an equivalence relation on $X \times Y$ Show that, as one might hope, the space $X \times Y/\sim$ with the quotient topology is homeomorphic to Y.

Exercise 17.18. *If (X, d) is a metric space, $x \in X$ and E is non-empty subset of X, we set*

$$f_E(x) = d(x, E) = \inf\{d(x, e) : e \in E\}.$$

(i) Show that the map f_E from (X, d) to $\mathbb{R}$ with its usual metric is continuous.

(ii) Show that E is closed if and only if $d(x, E) > 0$ for all $x \notin E$.

(iii) By using the functions f_{E_1} and f_{E_2}, or otherwise, show that, if E_1 and E_2 are disjoint closed subsets of X, then there exists a continuous function $f : X \to \mathbb{R}$ with the properties that $1 \geq f(x) \geq 0$ for all $x \in X$ and

$$f(x) = \begin{cases} 1 & \textit{if } x \in E_1, \\ 0 & \textit{if } x \in E_2. \end{cases}$$

Deduce that we can find disjoint open sets U_1 and U_2 such that $U_1 \supseteq E_1$ and $U_2 \supseteq E_2$.

Exercise 17.19. *(This continues on from parts (i) and (ii) of Exercise 17.18.)*

(i) We work in a metric space (X, d). Consider two non-empty disjoint sets E and G. If E is compact and G is closed, show that there exists a $\delta > 0$ such that

$$d(e, g) \geq \delta$$

for all $e \in E$ and $g \in G$.

(ii) Find two non-empty disjoint closed sets E and G in $\mathbb{R}$ with the usual metric such that

$$\inf_{e \in E, g \in G} |e - g| = 0.$$

Exercise 17.20. *We work on $\mathbb{R}$. Let τ consist of all sets of the form $U \cup S$ where U is an open set for the usual Euclidean topology and S is a subset of the irrationals.*

(i) Show that τ is a topology. (It is called the 'scattered topology'.)

(ii) Show that τ is Hausdorff.

(iii) Show that $\{x\}$ is open if and only if x is irrational.

Exercise 17.21. *Let (X, τ) be a topological space and E and F subsets with the subspace topologies τ_E, τ_F. Suppose that $E \cup F = X$, that (Y, σ) is another topological space and $g : X \to Y$ a function. Suppose that $g|_E : (E, \tau_E) \to (Y, \sigma)$ and $g|_F : (F, \tau_F) \to (Y, \sigma)$ are continuous.*

Which of the following statements are always true and which may be false? Give proofs or counterexamples.

(i) If E and F are open, then $g : (X, \tau) \to (Y, \sigma)$ is continuous.

(ii) If E and F are closed, then $g : (X, \tau) \to (Y, \sigma)$ is continuous.

(iii) If E is open and $F = X \setminus E$, then $g : (X, \tau) \to (Y, \sigma)$ is continuous.

Exercise 17.22. *(Requires the idea of uniform convergence from Analysis II.) This example of a space filling curve due to Liu Wen is simple rather than pretty.*

Let $\delta_k = [k/10, (k+1)/10]$ for $0 \leq k \leq 9$. Let $f, g : [0, 1] \to \mathbb{R}$ be continuous functions satisfying the following conditions:

$$f(t) = \begin{cases} 0 & \textit{when } t \in \delta_1 \cup \delta_3 \\ 1 & \textit{when } t \in \delta_5 \cup \delta_7 \end{cases} \quad g(t) = \begin{cases} 0 & \textit{when } t \in \delta_1 \cup \delta_5 \\ 1 & \textit{when } t \in \delta_3 \cup \delta_7 \end{cases}$$

and $f(0) = f(1) = 0$, $g(0) = g(1) = 0$. Sketch such a function.

Set $F(t+n) = f(t)$, $G(t+n) = g(t)$ for all $t \in [0,1]$ and $n \in \mathbb{Z}$. Explain why, if we set

$$\phi(t) = \sum_{k=1}^{\infty} 2^{-k} F(10^{k-1}t),\ \psi(t) = \sum_{k=1}^{\infty} 2^{-k} G(10^{k-1}t),$$

the map $t \mapsto \big(\phi(t), \psi(t)\big)$ is a continuous map of $[0,1]$ to $[0,1]^2$ (with the usual metrics).

If

$$x = \sum_{j=1}^{\infty} x_j 2^{-j} \text{ and } y = \sum_{j=1}^{\infty} y_j 2^{-j}$$

with x_j, $y_j \in \{0,1\}$, find $t_j \in \{1,3,5,7\}$ such that, writing

$$t = \sum_{j=1}^{\infty} t_j 10^{-j},$$

we have $\big(\phi(t), \psi(t)\big) = (x,y)$.

Conclude that there is a continuous surjective map from $[0,1]$ to $[0,1]^2$.

Exercise 17.23. *We use the standard Euclidean metrics. Show that there does not exist a continuous injection $f : [0,1]^2 \to [0,1]$.*
[Hint: Let $E = [0,1]^2 \setminus \{\mathbf{a}\}$ for some fixed $\mathbf{a}$ and consider $f|_E$.]

Exercise 17.24. *We use the standard metric. Show that there does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$x \in \mathbb{Q} \Leftrightarrow f(x) \notin \mathbb{Q}.$$

Does there exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$x \in \mathbb{Q} \Leftrightarrow g(x) \in \mathbb{Q}\ ?$$

Give reasons for your answer.

Exercise 17.25. *Which, if any, of the following subsets of $\mathbb{R}^2$ with the usual topology are connected? Give reasons for your answer.*

(i) $A = \{(x,y) : x \in \mathbb{Q}\}$.

(ii) $A = \{(x,y) : x \in \mathbb{Q}\} \cup \{(x,y) : y \in \mathbb{Q}\}$.

Exercise 17.26. *Consider a compact metric space (X,d). Show that there exists a K such that $d(x,y) \leq K$ for all x, $y \in X$. If E is a non-empty subset of of X, we define the diameter $\Delta(E)$ of E by*

$$\Delta(E) = \sup_{(x,y) \in E} d(x,y).$$

Show that if $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of X, then there exists a $\delta > 0$ such that every non-empty subset E with $\Delta E < \delta$ lies in some U_λ.

Exercise 17.27. *We work in a metric space (X, d). Suppose that E_1, E_2, $\ldots$ are connected sets with $E_1 \supseteq E_2 \supseteq \ldots$. Show that, if the E_j are compact, $\bigcap_{j=1}^{\infty} E_j$ is connected.*
[*Hint: You may find Exercise 17.18 (iii) useful.*]

Give an example in $\mathbb{R}^2$ with the usual Euclidean topology to show that the result may fail if we replace 'compact' by 'closed'.

Exercise 17.28. *In this question you may quote the result that the product of two compact spaces is compact, but no other result on product topologies.*

Suppose that (X, τ), (Y, σ) are topological spaces and we give $X \times Y$ the product topology ρ.

(i) Show that, if $x \in X$, then

$$\big\{\{y \in Y \,:\, (x, y) \in U\} \,:\, U \in \rho\big\} = \sigma.$$

(ii) Give an example with X and Y each consisting of 2 points of a topology η on $X \times Y$ such that

$$\big\{\{x \in X \,:\, (x, y) \in U\} \,:\, U \in \eta\big\} = \sigma$$

for each $y \in Y$ and

$$\big\{\{y \in Y \,:\, (x, y) \in U\} \,:\, U \in \eta\big\} = \tau$$

for each $x \in X$, but $\eta \neq \rho$.

(iii) Prove the following results.

(a) ρ Hausdorff $\Leftrightarrow \tau, \sigma$ Hausdorff.

(b) ρ compact $\Leftrightarrow \tau, \sigma$ compact.

(c) ρ connected $\Leftrightarrow \tau, \sigma$ connected.

(d) ρ path-connected $\Leftrightarrow \tau, \sigma$ path-connected.

Exercise 17.29. *(i) Consider a metric space (X, d). If X has a countable dense subset show that so does every subset of X (for the subspace topology). (See Exercise 17.30 (vii) for why this can not be extended to topological spaces.)*

(ii) Consider a metric space (Y, ρ). If Y has a countable dense subset show that the associated topology has a countable basis.

(iii) Let X be an uncountable space. Check that the collection τ of sets consisting of $\varnothing$ and all sets U with $X \setminus U$ finite is a topology. (We call τ the cofinite topology.) Show that any countable subset is dense but that τ does not have countable basis.

Exercise 17.30. *Show that the collection of half open intervals $[a, b)$ form a basis. Consider the 'half open topology' τ_H on $\mathbb{R}$ is generated by this basis.*

(i) Show that τ_H is Hausdorff

(ii) Show that the connected components of $(\mathbb{R}, \tau_H)$ are the one point sets $\{x\}$.

(iii) Show that $[a, b]$ with $a < b$ is not compact in τ_H.

(iv) Consider $\mathbb{R}^2$ with the product topology σ_H obtained from τ_H. Show that $\mathbb{R} \times \mathbb{R}$ has a countable dense subset.

(v) Show that the subspace topology on $Z = \{(x, -x) : x \in \mathbb{R}\}$ derived from σ_H is discrete.

(vi) Use Exercise 17.29 (ii) to show that σ_H is not derived from a metric. Deduce that τ_H is not derived from a metric.

(vii) Observe that parts (iv) and (vi) together show that Exercise 17.29 (i) cannot be extended from metric to general topological spaces. (I owe this remark to Mr M. J. Colbrook.)

Exercise 17.31. *Consider the collection X_* of all functions $f : [0, 1] \to \mathbb{R}$ with $f(x) > 0$ for $x > 0$, $f(0) = 0$ and $f(x) \to 0$ as $x \to 0$. We take $X = X_* \cup \{f_0\}$ where f_0 is the zero function defined by $f_0(x) = 0$ for all $x \in [0, 1]$. If $g \in X_*$, write*

$$U_g = \{f \in X : f(x)/g(x) \to 0 \text{ as } x \to 0\}.$$

Show that, given g_1, $g_2 \in X_*$, we can find a $g_3 \in X_*$ such that

$$U_{g_3} \subseteq U_{g_1} \cap U_{g_2}.$$

Conclude that, if τ consists of $\varnothing$ together with all those sets V such that $V \supseteq U_g$ for some $g \in X_*$, then τ is a topology on X. Show that

$$\bigcap_{g \in X_*} U_g = \{f_0\}.$$

Now suppose $g_j \in X^*$. If we set $g(0) = 0$ and

$$g(t) = n^{-1} \min_{1 \le j \le n} g_j(t) \text{ for } t \in \big((n+1)^{-1}, n^{-1}\big],$$

show that $g \in X^*$ and $g_j \notin U_g$. Conclude that, although every open neighbourhood of f_0 contains infinitely many points and the intersection of the open neighbourhoods of f_0 is the one point set $\{f_0\}$, there is no sequence g_j with $g_j \neq f_0$ such that $g_j \to f_0$.

[*If you just accept this result without thought, it is not worth doing the question. You should compare and contrast the metric case. I would say that*

f_0 is 'surrounded by too many neighbourhood-shells to be approached by a sequence', but the language of the course is inadequate to make this thought precise.

I am told that the ancient Greek geometers used a similar counterexample for a related purpose.]

Exercise 17.32. *(i) Show that the following two statements about a metric space (X, d) are equivalent.*

(A) There is a complete metric ρ on X which induces the same topology as d.

(B) There is a complete metric space (Y, θ) which is homeomorphic to (X, d).

(ii) Consider $\mathbb{Q}$ with the usual metric d and a metric ρ which induces the same topology as d. Write $\mathbb{Q} = \{q_1, q_2, \ldots\}$. Let $y_0 = 0$, $r_0 = 1$. Show that we can find inductively $y_n \in \mathbb{Q}$ and $r_n > 0$ such that $r_n \leq 2^{-n}$ and

(a) $\rho(x, y_{n+1}) \leq r_{n+1} \Rightarrow \rho(x, y_n) \leq r_n$,

(b) $\rho(q_{n+1}, y_{n+1}) \geq 2r_{n+1}$.

(iii) Continuing with (ii), show that the y_n form a Cauchy sequence for ρ which does not converge.

(iv) Deduce that $(\mathbb{Q}, d)$ is not homeomorphic to a complete metric space.

If the reader is not exhausted, the next exercise provides a nice complement to this one.

Exercise 17.33. *This question is included for the lecturer's own amusement, but is quite a good revision question on metric spaces.*

(i) If (X, d) is a complete metric space, show that a subset E is complete under the restriction metric if and only if E is closed.

(ii) By considering

$$\{(x, 1/x) : x \in \mathbb{R}\}$$

as a subset of $\mathbb{R}^2$, or otherwise, show that $\rho(x, y) = \sqrt{(x-y)^2 + (x^{-1} - y^{-1})^2}$ is a complete metric on $\mathbb{R} \setminus \{0\}$.

Show that $(\mathbb{R} \setminus \{0\}, \rho)$ is homeomorphic to $(\mathbb{R} \setminus \{0\}, d)$ with the usual metric d.

(Note that all of this is very similar to Example 8.5.)

(iii) Enumerate the rationals $\mathbb{Q}$ as q_1, q_2, q_3, ... Define

$$\rho_n(x, y) = \min\{2^{-n}, \rho(x - q_n, y - q_n)\}$$

for x, $y \in \mathbb{R} \setminus \{q_n\}$.

Write $\mathbb{J} = \mathbb{R} \setminus \mathbb{Q}$ and set

$$\kappa(x, y) = \sum_{n=1}^{\infty} \rho_n(x, y)$$

for x, $y \in \mathbb{J}$. Explain why κ is a well defined metric on $\mathbb{J}$.

(iv) Show that $(\mathbb{J}, \kappa)$ is homeomorphic to $(\mathbb{J}, d)$ with the usual metric d.

Show that $(\mathbb{J}, \kappa)$ is a complete metric space. (Contrast this with the conclusion of Exercise 17.32.)

Exercise 17.34. *We get interesting results when we allow for an interplay between algebra and topology. Consider a* topological group*, that is to say a group G together with a topology τ on G such that (if we give $G \times G$ the associated product topology) the multiplication function $m : G \times G \to G$ (given by $m(x, y) = xy$) and the inverse function $j : G \to G$ (given by $j(x) = x^{-1}$) are continuous. Typical examples include the matrix groups such as $SO(\mathbb{R}^3)$ and $U(\mathbb{C}^3)$.*

(i) (Homogeneity) Show that that, if u, $v \in G$, there exists a homeomorphism $\phi : G \to G$ with $\phi(u) = v$.

(ii) Show that G is Hausdorff if and only if $\{e\}$ is closed.

(iii) If $\{e\}$ is closed, show that the diagonal

$$\Delta G = \{(x, x) \,:\, x \in G\}$$

is a closed subgroup of $G \times G$ (i.e both a subgroup and closed in the product topology).

(iv) If $\{e\}$ is closed, show that the centre

$$Z(G) = \{g \,:\, gh = hg \ \forall h \in G\}$$

is a closed normal subgroup.

(v) Suppose that H is a subgroup of G. Consider the collection X of cosets of H. Show that, if we give X the natural quotient topology, the map $\pi : G \to X$ given by $\pi(g) = gH$ is open (that is to say π maps open sets to open sets).

(vi) Show that X, as given in (v), is Hausdorff if and only if H is closed in G.

18 Some hints

Theorem 10.6. [The Heine–Borel Theorem.] *Let $\mathbb{R}$ be given its usual (Euclidean) topology. Then the closed bounded interval $[a, b]$ is compact.*

Hint. Suppose that $\mathcal{C}$ is an open cover of $[a, b]$. If $\mathcal{C}_1$ is a finite subcover of $[a, c]$ and $\mathcal{C}_2$ is a finite subcover of $[c, b]$, then $\mathcal{C}_1 \cup \mathcal{C}_2$ is a finite cover of $[a, b]$. We can use this as a basis for a lion hunting (bisection) argument.

[Return to page 28 or go to a full proof on 91.] □

Theorem 10.9. *If (X, τ) is Hausdorff, then every compact set is closed.*

Hint. Let K be a compact set. If $x \notin K$, then, given any $k \in K$, we know that $k \neq x$ and so, since X is Hausdorff, we can find open sets U_k and V_k such that

$$k \in V_k,\ x \in U_k \text{ and } V_k \cap U_k = \varnothing.$$

Now use compactness.

[Return to page 28 or go to a full proof on 92.] □

Theorem 10.20. *Let (X, τ) be a compact and (Y, σ) a Hausdorff topological space. If $f : X \to Y$ is a continuous bijection, then it is a homeomorphism.*

Hint. Observe that we need only show that $f(K)$ is closed whenever K is closed.

[Return to page 30 or go to a full proof on 94.] □

Theorem 11.1. *The product of two compact spaces is compact. (More formally, if (X, τ) and (Y, σ) are compact topological spaces and λ is the product topology, then $(X \times Y, \lambda)$ is compact.)*

Hint. Let $\{O_\alpha\}_{\alpha \in A}$ be an open cover for $X \times Y$. Then given $(x, y) \in X \times Y$ we can find $U_{x,y} \in \tau$, $V_{x,y} \in \sigma$ and $\alpha(x, y) \in A$ such that

$$(x, y) \in U_{x,y} \times V_{x,y} \subseteq O_{\alpha(x,y)}.$$

Now show that, for each $x \in X$, we can find a positive integer $n(x)$ and $y(x, j) \in Y$ $[1 \leq j \leq n(x)]$ such that

$$\bigcup_{j=1}^{n(x)} V_{x,y(x,j)} = Y.$$

Now consider the $U_x = \bigcap_{j=1}^{n(x)} U_{x,y(x,j)}$.

[Return to page 31 or go to a full proof on 95.] □

Theorem 12.4. *If the metric space (X, d) is compact, it is sequentially compact.*

Hint. Let x_n be a sequence in X. If it has no convergent subsequence, then, for each $x \in X$, we can find a $\delta(x) > 0$ and an $N(x)$ such that $x_n \notin B(x, \delta(x))$ for all $n \geq N(x)$.

[Return to page 33 or go to a full proof on page 98.] □

Lemma 12.7. *Suppose that (X, d) is a sequentially compact metric space and that the collection U_α with $\alpha \in A$ is an open cover of X. Then there exists a $\delta > 0$ such that, given any $x \in X$, there exists an $\alpha(x) \in A$ such that the open ball $B(x, \delta) \subseteq U_{\alpha(x)}$.*

Hint. Suppose the first sentence is true and the second sentence false. Then, for each $n \geq 1$, we can find an x_n such that $B(x_n, 1/n) \nsubseteq U_\alpha$ for all $\alpha \in A$.

[Return to page 34 or go to a full proof on page 99.] □

Theorem 12.8. *If the metric space (X, d) is sequentially compact, it is compact.*

Hint. Let $(U_\alpha)_{\alpha \in A}$ be an open cover and let δ be defined as in Lemma 12.7. The $B(x, \delta)$ form a cover of X. If they have no finite subcover then, given x_1, x_2, $\dots x_n$, we can find an $x_{n+1} \notin \bigcup_{j=1}^{n} B(x_j, \delta)$.

[Return to page 34 or go to a full proof on page 100.] □

Theorem 13.16. *If we give $\mathbb{R}^n$ the usual topology, then any open set Ω which is connected is path-connected.*

Hint. Pick $\mathbf{x} \in \Omega$, let U be the set of all points in Ω which are path-connected to $\mathbf{x}$ and let V be the set of all points in Ω which are not. We need to prove that U and V are open and to do this we make use of the fact that any point in an open ball is path-connected to the centre of the ball.

[Return to page 38 or go to a full proof on page 105.] □

19 Some proofs

Exercise 1.1. *We use the notation just introduced.*

(i) Let $X = Y = \{1, 2, 3, 4\}$ and $f(1) = 1$, $f(2) = 1$, $f(3) = 4$, $f(4) = 3$. Identify

$$f^{-1}(\{1\}),\ f^{-1}(\{2\}) \text{ and } f^{-1}(\{3, 4\}).$$

(ii) If $U_\theta \subseteq Y$ for all $\theta \in \Theta$, show that

$$f^{-1}\left(\bigcap_{\theta \in \Theta} U_\theta\right) = \bigcap_{\theta \in \Theta} f^{-1}(U_\theta) \text{ and } f^{-1}\left(\bigcup_{\theta \in \Theta} U_\theta\right) = \bigcup_{\theta \in \Theta} f^{-1}(U_\theta).$$

Show also that $f^{-1}(Y) = X$, $f^{-1}(\varnothing) = \varnothing$ and that, if $U \subseteq Y$,

$$f^{-1}(Y \setminus U) = X \setminus f^{-1}(U).$$

(iii) If $V_\theta \subseteq X$ for all $\theta \in \Theta$ show that

$$f\left(\bigcup_{\theta\in\Theta} V_\theta\right) = \bigcup_{\theta\in\Theta} f(V_\theta)$$

and observe that $f(\varnothing) = \varnothing$.

(iv) By finding appropriate X, Y, f and $V, V_1, V_2 \subseteq X$, show that we may have

$$f(V_1 \cap V_2) \neq f(V_1) \cap f(V_2),\ f(X) \neq Y \text{ and } f(X \setminus V) \neq Y \setminus f(V).$$

Solution. (i) We have

$$f^{-1}(\{1\}) = \{1,2\},\ f^{-1}(\{2\}) = \varnothing,\ f^{-1}(\{3,4\}) = \{3,4\}.$$

(ii) We have

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{\theta\in\Theta} U_\theta\right) &\Leftrightarrow f(x) \in \bigcap_{\theta\in\Theta} U_\theta \\ &\Leftrightarrow f(x) \in U_\theta \text{ for all } \theta \in \Theta \\ &\Leftrightarrow x \in f^{-1}(U_\theta) \text{ for all } \theta \in \Theta \\ &\Leftrightarrow x \in \bigcap_{\theta\in\Theta} f^{-1}(U_\theta) \end{aligned}$$

and

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{\theta\in\Theta} U_\theta\right) &\Leftrightarrow f(x) \in \bigcup_{\theta\in\Theta} U_\theta \\ &\Leftrightarrow f(x) \in U_\theta \text{ for some } \theta \in \Theta \\ &\Leftrightarrow x \in f^{-1}(U_\theta) \text{ for some } \theta \in \Theta \\ &\Leftrightarrow x \in \bigcup_{\theta\in\Theta} f^{-1}(U_\theta). \end{aligned}$$

Thus

$$f^{-1}\left(\bigcap_{\theta\in\Theta} U_\theta\right) = \bigcap_{\theta\in\Theta} f^{-1}(U_\theta) \text{ and } f^{-1}\left(\bigcup_{\theta\in\Theta} U_\theta\right) = \bigcup_{\theta\in\Theta} f^{-1}(U_\theta).$$

Trivially

$$x \in f^{-1}(Y) \Leftrightarrow f(x) \in Y \Leftrightarrow x \in X$$

and

$$x \in f^{-1}(\varnothing) \Leftrightarrow f(x) \in \varnothing \Leftrightarrow x \in \varnothing$$

so $f^{-1}(Y) = X$ and $f^{-1}(\varnothing) = \varnothing$.

Finally, if $U \subseteq Y$.

$$\begin{aligned} x \in f^{-1}(Y \setminus U) &\Leftrightarrow f(x) \in Y \setminus U \Leftrightarrow f(x) \notin U \\ &\Leftrightarrow x \notin f^{-1}(U) \Leftrightarrow x \in X \setminus f^{-1}(U) \end{aligned}$$

so $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$.

(iii) We have

$$\begin{aligned} y \in f\left(\bigcup_{\theta\in\Theta} V_\theta\right) &\Leftrightarrow \text{there exists an } x \in \bigcup_{\theta\in\Theta} V_\theta \text{ with } f(x) = y \\ &\Leftrightarrow \text{there exists a } \theta \in \Theta \text{ and an } x \in V_\theta \text{ with } f(x) = y \\ &\Leftrightarrow \text{there exists a } \theta \in \Theta \text{ with } y \in f(V_\theta) \\ &\Leftrightarrow y \in \bigcup_{\theta\in\Theta} f(V_\theta). \end{aligned}$$

Thus

$$f\left(\bigcup_{\theta\in\Theta} V_\theta\right) = \bigcup_{\theta\in\Theta} f(V_\theta).$$

We have $f(\varnothing) = \varnothing$ vacuously.

(iv) Take $X = Y = \{1,2,3\}$, $f(1) = 1$, $f(2) = 2$, $f(3) = 1$, $V = V_1 = \{1,2\}$ and $V_2 = \{2,3\}$. Then

$$f(V_1 \cap V_2) = f(\{2\}) = \{2\} \neq \{1,2\} \cap \{1,2\} = f(V_1) \cap f(V_2),$$
$$f(X) = \{1,2\} \neq \{1,2,3\} = Y \text{ and} f(X \setminus V) = f(\{3\}) = \{1\} \neq \{3\} = Y \setminus \{1,2\} = Y \setminus V.$$

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Exercise 2.2. *If $d : X^2 \to \mathbb{R}$ is a function with the following properties:*

(ii) $d(x,y) = 0$ if and only if $x = y$,

(iii) $d(x,y) = d(y,x)$ for all x, $y \in X$,

(iv) $d(x,y) + d(y,z) \geq d(x,z)$ for all x, y, $z \in X$,

show that d is a metric on X.

Solution. Setting $z = x$ in condition (iv) and using (iii) and (ii), we have

$$2d(x,y) = d(x,y) + d(y,x) \geq d(x,x) = 0$$

so $d(x,y) \geq 0$.

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Exercise 2.4. *Let $X = \{a,\, b,\, c\}$ with a, b and c distinct. Write down functions $d_j : X^2 \to \mathbb{R}$ satisfying condition (i) of Definition 2.1 such that*

(1) d_1 satisfies conditions (ii) and (iii) but not (iv).

(2) d_2 satisfies conditions (iii) and (iv) but it is not true that $x = y$ implies $d(x, y) = 0$.

(3) d_3 satisfies conditions (iii) and (iv) and $x = y$ implies $d_3(x, y) = 0$. but it is not true that $d_3(x, y) = 0$ implies $x = y$.

(4) d_4 satisfies conditions (ii) and (iv) but not (iii).

You should verify your statements.

Solution. Here are some possible choices.

(1) Take $d_1(x, x) = 0$ for all $x \in X$, $d_1(a, b) = d_1(b, a) = d_1(a, c) = d_1(c, a) = 1$ and $d_1(b, c) = d_1(c, b) = 3$. Conditions (ii) and (iii) hold by inspection, but

$$d_1(b, a) + d_1(a, c) = 2 < 3 = d_1(b, c).$$

(2) Take $d_2(x, x) = 1$ and $d_2(x, y) = 2$ if $x \neq y$. Condition (ii) fails and condition (iii) holds by inspection. We observe that

$$d_2(x, y) + d_2(y, z) \geq 1 + 1 = 2 \geq d_2(x, z)$$

so the triangle law holds.

(3) Take $d_2(x, y) = 0$ for all $x,\, y \in X$.

(4) Take $d_4(x, x) = 0$ for all $x \in X$, $d_4(a, b) = d_4(b, a) = d_4(a, c) = d_4(c, a) = 1$ and $d_1(b, c) = 1$, $d_1(c, b) = \frac{5}{4}$. Conditions (ii) holds, and condition (iii) fails by inspection and

$$\begin{aligned}
d(x, y) + d(y, z) &= d(x, y) = d(x, z) \geq d(x, z) && \text{if } y = z,\\
d(x, y) + d(y, z) &= d(y, z) = d(x, z) \geq d(x, z) && \text{if } x = y,\\
d(x, y) + d(y, z) &\geq 1 + 1 = 2 \geq \tfrac{5}{4} \geq d(x, z) && \text{otherwise,}
\end{aligned}$$

so the triangle law holds.

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Lemma 3.3. *If $(V, \|\ \|)$ is a normed vector space, then the condition*

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

defines a metric d on V.

Proof. We observe that

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| \geq 0$$

and

$$d(\mathbf{u},\mathbf{u}) = \|\mathbf{0}\| = \|0\mathbf{0}\| = |0|\|\mathbf{0}\| = 0\|\mathbf{0}\| = 0.$$

Further, if $d(\mathbf{u},\mathbf{v}) = 0$, then $\|\mathbf{u}-\mathbf{v}\| = 0$ so $\mathbf{u}-\mathbf{v} = \mathbf{0}$ and $\mathbf{u} = \mathbf{v}$. We also observe that

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|(-1)(\mathbf{v}-\mathbf{u})\| = |-1|\|\mathbf{v}-\mathbf{u}\| = d(\mathbf{v},\mathbf{u})$$

and

$$\begin{aligned} d(\mathbf{u},\mathbf{v}) + d(\mathbf{v},\mathbf{w}) &= \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\| \\ &\geq \|(\mathbf{u}-\mathbf{v}) + (\mathbf{v}-\mathbf{w})\| \\ &= \|\mathbf{u}-\mathbf{w}\| = d(\mathbf{u},\mathbf{w}). \end{aligned}$$

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Lemma 3.5. *Let $(V, \langle\ ,\ \rangle)$ be an inner product space. If we write $\|\mathbf{u}\| = \langle \mathbf{u},\mathbf{u}\rangle^{1/2}$ (taking the positive root), then the following results hold.*

(i) (The Cauchy–Schwarz inequality) If $\mathbf{u}$, $\mathbf{v} \in V$, then

$$\|\mathbf{u}\|\|\mathbf{v}\| \geq |\langle \mathbf{u},\mathbf{v}\rangle|.$$

(ii) $(V, \|\ \|)$ is a normed vector space.

Proof. (i) If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ the result is immediate, so we may assume that $\mathbf{u}$, $\mathbf{v} \neq \mathbf{0}$.

Now observe that, if $\lambda \in \mathbb{R}$,

$$\begin{aligned} 0 \leq \langle \mathbf{u}+\lambda\mathbf{v}, \mathbf{u}+\lambda\mathbf{v}\rangle &= \langle \mathbf{u},\mathbf{u}\rangle + 2\lambda\langle \mathbf{u},\mathbf{v}\rangle + \lambda^2\langle \mathbf{v},\mathbf{v}\rangle \\ &= \left(\lambda\langle \mathbf{v},\mathbf{v}\rangle^{1/2} + \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\langle \mathbf{v},\mathbf{v}\rangle^{1/2}}\right)^2 - \frac{\langle \mathbf{u},\mathbf{v}\rangle^2}{\langle \mathbf{v},\mathbf{v}\rangle} + \langle \mathbf{u},\mathbf{u}\rangle. \end{aligned}$$

Taking

$$\lambda = -\frac{\langle \mathbf{u},\mathbf{v}\rangle}{(\langle \mathbf{v},\mathbf{v}\rangle\langle \mathbf{v},\mathbf{v}\rangle)^{1/2}}$$

we obtain

$$\langle \mathbf{u},\mathbf{u}\rangle - \frac{\langle \mathbf{u},\mathbf{v}\rangle^2}{\langle \mathbf{v},\mathbf{v}\rangle} \geq 0$$

and this yields the desired inequality.

(ii) Observe that using the Cauchy–Schwarz lemma

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle &= \|\mathbf{u}\|^2 + 2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 \end{aligned}$$

and the triangle inequality follows on taking square roots.

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Lemma 3.8. *Suppose that $a < b$, that $f : [a, b] \to \mathbb{R}$ is continuous and $f(t) \geq 0$ for all $t \in [a, b]$. Then, if*

$$\int_a^b f(t)\, dt = 0,$$

it follows that $f(t) = 0$ for all $t \in [a, b]$.

Proof. Suppose that the conditions of the first sentence hold and $f(x) > 0$ for some $x \in [a, b]$. By continuity, we can find a δ with $1 > \delta > 0$ such that

$$|f(t) - f(x)| \leq f(x)/2 \text{ for all } t \in [a, b] \cap [x - \delta, x + \delta]$$

and so

$$f(t) \geq f(x)/2 \text{ for all } t \in [a, b] \cap [x - \delta, x + \delta].$$

It follows that

$$\int_a^b f(t)\, dt \geq \int_{t \in [a,b] \cap [x-\delta, x+\delta]} f(t)\, dt \geq \delta f(x)/2 > 0.$$

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Exercise 3.9. *Show that the result of Lemma 3.8 is false if we replace 'f continuous' by 'f Riemann integrable'.*

Solution. Let $a = 0$, $b = 1$. Set $f(t) = 0$ if $t \neq 1/2$ and set $f(1/2) = 1$.
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Theorem 3.10. *Suppose that $a < b$ and we consider the space $C([a, b])$ of continuous functions $f : [a, b] \to \mathbb{R}$ made into a vector space in the usual way.*

(i) The equation

$$\langle f, g \rangle = \int_a^b f(t)g(t)\, dt$$

defines an inner product on $C([a, b])$. We write

$$\|f\|_2 = \left(\int_a^b f(t)^2\, dt \right)^{1/2}$$

for the derived norm.

(ii) The equation

$$\|f\|_1 = \int_a^b |f(t)|\, dt$$

defines a norm on $C([a,b])$. *This norm does not obey the parallelogram law.*

(iii) The equation

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|.$$

defines a norm on $C([a,b])$. *This norm does not obey the parallelogram law.*

Proof. (i) We have

$$\langle f, f\rangle = \int_a^b f(t)^2\,dt \geq 0.$$

If $\langle f, f\rangle = 0$, then $\int_a^b f(t)^2\,dt = 0$ and, by Lemma 3.8, $f(t)^2 = 0$ for all t so $f(t) = 0$ for all t and $f = 0$.

We have

$$\langle f, g\rangle = \int_a^b f(t)g(t)\,dt = \int_a^b g(t)f(t)\,dt = \langle g, f\rangle$$

$$\langle f+g, h\rangle = \int_a^b (f(t)+g(t))h(t)\,dt = \int_a^b f(t)h(t)\,dt + \int_a^b g(t)h(t)\,dt = \langle f, h\rangle + \langle g, h\rangle$$

$$\langle \lambda f, g\rangle = \int_a^b \lambda f(t)g(t)\,dt = \lambda\int_a^b f(t)g(t)\,dt = \lambda\langle f, g\rangle,$$

so we have an inner product.

(ii) Observe that

$$\|f\|_1 = \int_a^b |f(t)|\,dt \geq 0$$

and that, if $\|f\|_1 = 0$, then

$$\int_a^b |f(t)|\,dt = 0,$$

so, by Lemma 3.8, $|f(t)| = 0$ for all t so $f(t) = 0$ for all t and $f = 0$.

Further

$$\|\lambda f\|_1 = \int_a^b |\lambda||f(t)|\,dt = |\lambda|\int_a^b |f(t)|\,dt = |\lambda|\|f\|_1$$

and, since $|f(t)+g(t)| \leq |f(t)| + |g(t)|$, we have

$$\|f+g\|_1 = \int_a^b |f(t)+g(t)|\,dt \leq \int_a^b |f(t)| + |g(t)|\,dt = \|f\|_1 + \|g\|_1,$$

so we have a norm.

If we take $a = 0$, $b = 1$,

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le 1/4 \\ 1/2 - t & \text{if } 1/4 \le t \le 1/2 \\ 0 & \text{if } 1/2 \le t \le 1 \end{cases}$$

and $g(t) = f(1-t)$, then

$$\|f+g\|_1^2+\|f-g\|_1^2 = (1/8)^2+(1/8)^2 = 1/32 \neq 2((1/16)^2+(1/16)^2) = 2(\|f\|_1^2+\|g\|_1^2),$$

so the parallelogram equality fails.

(iii) Observe that $|f(t)| \ge 0$ so

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)| \ge 0$$

that

$$\|f\|_\infty = 0 \Rightarrow \sup_{t\in[a,b]} |f(t)| = 0 \Rightarrow |f(t)| = 0 \ \forall t \Rightarrow f = 0$$

that

$$\|\lambda f\|_\infty = \sup_{t\in[a,b]} |\lambda f(t)| = \sup_{t\in[a,b]} |\lambda||f(t)| = |\lambda| \sup_{t\in[a,b]} |f(t)| = \lambda\|f\|_\infty$$

and, since $|f(t)+g(t)| \le |f(t)| + |g(t)|$,

$$\begin{aligned} \|f+g\|_\infty &= \sup_{t\in[a,b]} |f(t)+g(t)| \le \sup_{t\in[a,b]} (|f(t)| + |g(t)|) \\ &\le \sup_{t,s\in[a,b]} (|f(t)| + |g(s)|) = \|f\|_\infty + \|g\|_\infty, \end{aligned}$$

so we are done.

If we take $a = 0$, $b = 1$,

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le 1/4 \\ 1/2 - t & \text{if } 1/4 \le t \le 1/2 \\ 0 & \text{if } 1/2 \le t \le 1 \end{cases}$$

and $g(t) = f(1-t)$, then

$$\|f+g\|_\infty^2+\|f-g\|_\infty^2 = (1/4)^2+(1/4)^2 \neq 2((1/4)^2+(1/4)^2) = 2(\|f\|_\infty^2+\|g\|_\infty^2),$$

so the parallelogram equality fails.

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Lemma 3.12. *The discrete metric on X is indeed a metric.*

Proof. The only non-evident condition is the triangle law. But

$$\begin{aligned} d(x,y)+d(y,z) &= d(x,y) = d(x,z) \geq d(x,z) && \text{if } y=z,\\ d(x,y)+d(y,z) &= d(y,z) = d(x,z) \geq d(x,z) && \text{if } x=y,\\ d(x,y)+d(y,z) &\geq 1+1=2\geq 1\geq d(x,z) && \text{otherwise.} \end{aligned}$$

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Exercise 3.16. *Show that the British Rail express metric and the British Rail stopping metric are indeed metrics.*

Solution. We show that the British Rail stopping metric is indeed a metric. The case of the British Rail express metric is left to the reader.

Let d be the British rail stopping metric on $\mathbb{R}^2$. It is easy to see that $d(\mathbf{u},\mathbf{v}) \geq 0$ and that $d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$. Since $\mathbf{u}$ and $\mathbf{u}$ are linearly dependent,

$$d(\mathbf{u},\mathbf{u}) = \|\mathbf{u}-\mathbf{u}\|_2 = \|\mathbf{0}\|_2 = 0.$$

If $d(\mathbf{u},\mathbf{v}) = 0$, then we know that at least one of the following statements is true

(1) $\|\mathbf{u}-\mathbf{v}\|_2 = 0$ and so $\mathbf{u}-\mathbf{v}=\mathbf{0}$,

(2) $\|\mathbf{u}\|_2 + \|\mathbf{v}\|_2 = 0$ and so $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 0$, whence $\mathbf{u}=\mathbf{v}=\mathbf{0}$.

In either case $\mathbf{u}=\mathbf{v}$ as required.

It only remains to prove the triangle inequality. Observe that, if $\mathbf{v}$ and $\mathbf{w}$ are not linearly dependent,

$$d(\mathbf{u},\mathbf{v}) + d(\mathbf{v},\mathbf{w}) \geq \|\mathbf{u}-\mathbf{v}\|_2 + \|\mathbf{v}\|_2 + \|\mathbf{w}\|_2 \geq \|\mathbf{u}\|_2 + \|\mathbf{w}\|_2 \geq d(\mathbf{u},\mathbf{w}).$$

By similar reasoning

$$d(\mathbf{u},\mathbf{v}) + d(\mathbf{v},\mathbf{w}) \geq d(\mathbf{u},\mathbf{w})$$

if $\mathbf{u}$ and $\mathbf{v}$ are not linearly dependent. Finally, if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent and $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent, then $\mathbf{u}$ and $\mathbf{w}$ are linearly dependent so

$$d(\mathbf{u},\mathbf{v}) + d(\mathbf{v},\mathbf{w}) = \|\mathbf{u}-\mathbf{v}\|_2 + \|\mathbf{v}-\mathbf{w}\|_2 \geq \|\mathbf{u}-\mathbf{w}\|_2 = d(\mathbf{u},\mathbf{w}).$$

Thus the triangle law holds.

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Lemma 4.3. [The composition law.] *If (X, d) and (Y, ρ) and (Z, σ) are metric spaces and $g : X \to Y$, $f : Y \to Z$ are continuous, then so is the composition fg.*

Proof. Let $\epsilon > 0$ be given and let $x \in X$. Since f is continuous, we can find a $\delta_1 > 0$ (depending on ϵ and $fg(x) = f(g(x))$ with

$$\sigma(f(g(x)), f(y)) < \epsilon \text{ whenever } \rho(g(x), y) < \delta_1.$$

Since g is continuous, we can find a $\delta_2 > 0$ such that

$$\rho(g(x), g(t)) < \delta_1 \text{ whenever } d(x, t) < \delta_2.$$

We now have

$$\sigma(f(g(x)), f(g(t))) < \epsilon \text{ whenever } d(x, t) < \delta_2$$

as required.

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Exercise 4.4. *Let $\mathbb{R}$ and $\mathbb{R}^2$ have their usual (Euclidean) metric.*

(i) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous. Show that the map $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(f, g)(x, y) = (f(x), g(y))$ is continuous.

(ii) Show that the map $M : \mathbb{R}^2 \to \mathbb{R}$ given by $M(x, y) = xy$ is continuous.

(iii) Use the composition law to show that the map $m : \mathbb{R}^2 \to \mathbb{R}$ given by $m(x, y) = f(x)g(y)$ is continuous.

Solution. (i) Let $(x, y) \in \mathbb{R}^2$. Given $\epsilon > 0$, we can find $\delta_1 > 0$ such that

$$|f(x) - f(s)| < \epsilon/2 \text{ whenever } |x - s| < \delta_1$$

and $\delta_2 > 0$ such that

$$|g(y) - g(t)| < \epsilon/2 \text{ whenever } |y - t| < \delta_2.$$

If we set $\delta = \min(\delta_1, \delta_2)$, then $\|(x, y) - (s, t)\|_2 < \delta$ implies

$$|x - s| < \delta \leq \delta_1 \text{ and } |y - t| < \delta \leq \delta_2$$

so that

$$|f(x) - f(s)| < \epsilon/2 \text{ and } |g(y) - g(t)| < \epsilon/2$$

whence

$$\begin{aligned} \|(f(x), g(y)) - (f(s), g(t))\|_2 &\leq \|(f(x), 0) - (f(s), 0)\|_2 + \|(0, g(y)) - (0, g(t))\|_2 \\ &= |f(x) - f(s)| + |g(y) - g(t)| < \epsilon \end{aligned}$$

as required.

(ii) (You should recognise this from Analysis I.) We use the standard Euclidean metric d on $\mathbb{R}^2$. Given $\epsilon > 0$ and $(x, y) \in \mathbb{R}^2$, set

$$\delta = \frac{\min\{\epsilon, 1\}}{|x| + |y| + 2}.$$

If $d\big((x, y), (u, v)\big) < \delta$, then $|x - u|, |y - v| < \delta$, and

$$\begin{aligned}|M(x, y) - M(u, v)| &= |xy - uv| \leq |xy - xv| + |xv - uv| \\ &\leq |x||y - v| + |v||x - u| \leq |x||y - v| + \big(|y - v| + |y|\big)|x - u| \\ &\leq \delta|x| + (|y| + \delta)\delta \leq (|x| + |y| + 1)\delta < \epsilon.\end{aligned}$$

Thus M is continuous.

(iii) $m = M \circ Q$ with $Q(x, y) = (f(x), g(y))$. Since M and Q are continuous their composition m is continuous.

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Example 4.6. *(i) Let (X, d) be a metric space. If $r > 0$, then*

$$B(x, r) = \{y : d(x, y) < r\}$$

is open.

(ii) If we work in $\mathbb{R}^n$ with the Euclidean metric, then the one point set $\{\mathbf{x}\}$ is not open.

(iii) If (X, d) is a discrete metric space, then

$$\{x\} = B(x, 1/2)$$

and all subsets of X are open.

Proof. (i) If $y \in B(x, r)$, then $\delta = r - d(x, y) > 0$ and, whenever $d(z, y) < \delta$, the triangle inequality gives us

$$d(x, z) \leq d(x, y) + d(y, z) < r$$

so $z \in B(x, r)$. Thus $B(x, r)$ is open.

(ii) Choose $\mathbf{e} \in \mathbb{R}^n$ with $\|\mathbf{e}\|_2 = 1$. (We could take $\mathbf{e} = (1, 0, 0, \ldots, 0)$.) If $\delta > 0$, then, setting $\mathbf{y} = \mathbf{x} + (\delta/2)\mathbf{e}$, we have $\|\mathbf{x} - \mathbf{y}\|_2 < \delta$, yet $\mathbf{y} \notin \{\mathbf{x}\}$. Thus $\{\mathbf{x}\}$ is not open.

(iii) Observe that $d(x, x) = 0 < 1/2$ and $d(x, y) = 1 > 1/2$ for $x \neq y$. If $x \in E$, then $d(x, y) < 1/2$ implies $y = x \in E$, so E is open.

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Theorem 4.7. *If (X, d) is a metric space, then the following statements are true.*

(i) The empty set $\varnothing$ and the space X are open.

(ii) If U_α is open for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha$ is open. (In other words, the union of open sets is open.)

(iii) If U_j is open for all $1 \leq j \leq n$, then $\bigcap_{j=1}^{n} U_j$ is open.

Proof. (i) Since there are no points e in $\varnothing$, the statement

$$x \in \varnothing \text{ whenever } d(x, e) < 1$$

holds for all $e \in \varnothing$. Since every point x belongs to X, the statement

$$x \in X \text{ whenever } d(x, e) < 1$$

holds for all $e \in X$.

(ii) If $e \in \bigcup_{\alpha \in A} U_\alpha$, then we can find a particular $\alpha_1 \in A$ with $e \in U_{\alpha_1}$. Since U_{α_1} is open, we can find a $\delta > 0$ such that

$$x \in U_{\alpha_1} \text{ whenever } d(x, e) < \delta.$$

Since $U_{\alpha_1} \subseteq \bigcup_{\alpha \in A} U_\alpha$,

$$x \in \bigcup_{\alpha \in A} U_\alpha \text{whenever } d(x, e) < \delta.$$

Thus $\bigcup_{\alpha \in A} U_\alpha$ is open.

(iii) If $e \in \bigcap_{j=1}^{n} U_j$, then $e \in U_j$ for each $1 \leq j \leq n$. Since U_j is open, we can find a $\delta_j > 0$ such that

$$x \in U_j \text{ whenever } d(x, e) < \delta_j.$$

Setting $\delta = \min_{1 \leq j \leq n} \delta_j$, we have $\delta > 0$ and

$$x \in U_j \text{ whenever } d(x, e) < \delta$$

for all $1 \leq j \leq n$. Thus

$$x \in \bigcap_{j=1}^{n} U_j \text{ whenever } d(x, e) < \delta$$

and we have shown that $\bigcap_{j=1}^{n} U_j$ is open.

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Theorem 4.9. *Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X whenever U is open in Y.*

Proof. Suppose first that f is continuous and that U is open in Y. If $x \in f^{-1}(U)$, then we can find a $y \in U$ with $f(x) = y$. Since U is open in Y, we can find an $\epsilon > 0$ such that

$$z \in U \text{ whenever } \rho(y, z) < \epsilon.$$

Since f is continuous, we can find a $\delta > 0$ such that

$$\rho(y, f(w)) = \rho(f(x), f(w)) < \epsilon \text{ whenever } d(x, w) < \delta.$$

Thus

$$f(w) \in U \text{ whenever } d(x, w) < \delta.$$

In other words,

$$w \in f^{-1}(U) \text{ whenever } d(x, w) < \delta.$$

We have shown that $f^{-1}(U)$ is open.

We now seek the converse result. Suppose that $f^{-1}(U)$ is open in X whenever U is open in Y. Suppose $x \in X$ and $\epsilon > 0$. We know that the open ball

$$B(f(x), \epsilon) = \{y \in Y \,:\, \rho(f(x), y) < \epsilon\}$$

is open. Thus $x \in f^{-1}\big(B(f(x), \epsilon)\big)$ and $f^{-1}\big(B(f(x), \epsilon)\big)$ is open. It follows that there is a $\delta > 0$ such that

$$w \in f^{-1}\big(B(f(x), \epsilon)\big) \text{ whenever } d(x, w) < \delta,$$

so, in other words,

$$\rho(f(x), f(w)) < \epsilon \text{ whenever } d(x, w) < \delta.$$

Thus f is continuous.

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Example 4.10. *Let $X = \mathbb{R}$ and d be the discrete metric. Let $Y = \mathbb{R}$ and ρ be the usual (Euclidean) metric.*

(i) If we define $f : X \to Y$ by $f(x) = x$, then f is continuous but there exist open sets U in X such that $f(U)$ is not open.

(ii) If we define $g : Y \to X$ by $g(y) = y$, then g is not continuous but $g(V)$ is open in X whenever V is open in Y.

Proof. Since every set is open in X, we have $f^{-1}(V) = g(V)$ open for every V in Y and so, in particular, for every open set. Thus f is continuous.

We observe that $U = \{0\}$ is open in X and $g^{-1}(U) = f(U) = U = \{0\}$ is not open in Y. Thus g is not continuous.

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Exercise 4.11. *Consider* $\mathbb{R}^2$. *For each of the British rail express and British rail stopping metrics:*

(i) Describe the open balls. (Consider both large and small radii.)

(ii) Describe the open sets as well as you can. (There is a nice description for the British rail express metric.) Give reasons for your answers.

Solution. We start with the British rail express metric. Write

$$B_E(\delta) = \{\mathbf{x} : \|\mathbf{x}\|_2 < \delta\}$$

for the Euclidean ball centre $\mathbf{0}$ $[\delta > 0]$. If $0 < r < \|\mathbf{x}\|_2$, then

$$B(\mathbf{x}, r) = \{\mathbf{x}\}.$$

If $\|\mathbf{x}\|_2 > r > 0$, then

$$B(\mathbf{x}, r) = \{\mathbf{x}\} \cup B_E(r - \|\mathbf{x}\|_2).$$

Since open balls are open and the union of open sets is open, we deduce that every set not containing $\mathbf{0}$ and every set containing $B_E(\delta)$ for some $\delta > 0$ is open.

On the other hand, if U is open and $\mathbf{0} \in U$ then U must contain $B_E(\delta)$ for some $\delta > 0$. It follows that the collection of sets described in the last sentence of the previous paragraph constitute the open sets for the British rail express metric.

We turn now to the stopping metric. We observe that

$$B(\mathbf{0}, r) = B_E(r)$$

for $r > 0$. If $\mathbf{x} \neq \mathbf{0}$ and $0 < r < \|\mathbf{x}\|_2$, then

$$B(\mathbf{x}, r) = \left\{ \lambda \frac{\mathbf{x}}{\|\mathbf{x}\|_2} : \lambda \in (\|\mathbf{x}\|_2 - r, \|\mathbf{x}\|_2 + r) \right\}.$$

If $\mathbf{x} \neq \mathbf{0}$ and $\|\mathbf{x}\|_2 > r > 0$, then

$$B(\mathbf{x}, r) = \left\{ \lambda \frac{\mathbf{x}}{\|\mathbf{x}\|_2} : \lambda \in (0, \|\mathbf{x}\|_2 + r) \right\} \cup B_E(r - \|\mathbf{x}\|).$$

A similar argument to the previous paragraph shows that the open sets are precisely the unions of sets of the form

$$l(\mathbf{e},(a,b)) = \{\lambda\mathbf{e} : \lambda \in (a,b)\}$$

where $\mathbf{e}$ is a unit vector and $0 \leq a < b$ together with some $B_E(\delta)$ with $\delta > 0$.

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Lemma 5.2. *Consider a metric space (X,d). If a sequence x_n has a limit, then that limit is unique.*

Proof. Suppose $x_n \to x$ and $x_n \to y$. Then, given any $\epsilon > 0$, we can find N_1 and N_2 such that

$$d(x_n, x) < \epsilon/2 \text{ for all } n \geq N_1 \text{ and } d(x_n, y) < \epsilon/2 \text{ for all } n \geq N_2.$$

Taking $N = \max(N_1, N_2)$, we obtain

$$d(x,y) \leq d(x_N, x) + d(x_N, y) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since ϵ was arbitrary, $d(x,y) = 0$ and $x = y$.

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Exercise 5.3. *Consider two metric spaces (X,d) and (Y,ρ). Show that a function $f : X \to Y$ is continuous if and only if, whenever $x_n \in X$ and $x_n \to x$ as $n \to \infty$, we have $f(x_n) \to f(x)$*

Solution. Suppose that f is continuous and $x_n \to x$. Then, given $\epsilon > 0$, we can find a $\delta > 0$ such that

$$d(z,x) < \delta \Rightarrow \rho\big(f(z), f(x)\big) < \epsilon$$

and then find an N such that

$$n \geq N \Rightarrow d(x_n, x) < \delta$$

and so

$$n \geq N \Rightarrow \rho\big(f(x_n), f(x)\big) < \epsilon.$$

Thus $f(x_n) \to f(x)$.

If f is not continuous, we can find an $\epsilon > 0$ such that, given any $\delta > 0$, there exists an $z \in X$ with $d(z,x) < \delta$ and $\rho\big(f(z), f(x)\big) > \epsilon$. In particular, we can find $x_n \in X$ such that $d(x_n, x) < 1/n$ but $\rho\big(f(x_n), f(x)\big) > \epsilon$. Thus $x_n \to x$, but $f(x_n) \not\to f(x)$.

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Exercise 5.4. *In this exercise we consider the identity map between a space and itself when we equip the space with* different *metrics. We look at the three norms (and their associated metrics) defined on $C([0,1])$ in Theorem 3.10.*

Define $j_{\alpha,\beta} : (C([0,1]), \| \ \|_\alpha) \to (C([0,1]), \| \ \|_\beta)$ by $j_{\alpha,\beta}(f) = f$.

(i) Show that $j_{\infty,1}$ and $j_{\infty,2}$ are continuous, but $j_{1,\infty}$ and $j_{2,\infty}$ are not.

(ii) By using the Cauchy–Schwarz inequality $|\langle f, g\rangle| \le \|f\|_2 \|g\|_2$ with $g = 1$, or otherwise, show that $j_{2,1}$ is continuous. Show that $j_{1,2}$ is not.
[*Hint: Consider functions of the form $f_{R,K}(x) = K \max\{0, 1 - Rx\}$.*]

Solution. (i) Observe that

$$\|f\|_1 = \int_0^1 |f(t)|\,dt \le \int_0^1 \|f\|_\infty \,dt = \|f\|_\infty$$

and

$$\|f\|_2^2 = \int_0^1 |f(t)|^2\,dt \le \int_0^1 \|f\|_\infty^2 \,dt = \|f\|_\infty^2,$$

so

$$\|f\|_2 \le \|f\|_\infty.$$

Thus

$$\|f_n - f\|_\infty \to 0 \Rightarrow \|f_n - f\|_1,\ \|f_n - f\|_2 \to 0$$

and $j_{\infty,1}$ and $j_{\infty,2}$ are continuous.

However, if we put

$$f_n(t) = n^{1/3} \max\{0, 1 - nt\},$$

then

$$\|f_n - 0\|_1 = \int_0^1 f_n(t)\,dt = n^{-2/3}/2 \to 0$$

and

$$\begin{aligned}\|f_n - 0\|_2^2 &= \int_0^1 f_n(t)^2\,dt = n^{2/3}\int_0^{1/n} (1-nt)^2\,dt \\ &= n^{2/3}\left[-\frac{(1-nt)^3}{3n}\right]_0^{1/n} = \frac{n^{-1/3}}{3} \to 0\end{aligned}$$

as $n \to \infty$, so $\|f_n - 0\|_2 \to 0$, yet

$$\|f_n - 0\|_\infty = n^{1/3} \to \infty$$

so $j_{1,\infty}$ and $j_{2,\infty}$ are not continuous.

(ii) Observe that

$$\|f\|_1 = \int_0^1 |f(t)|\,dt = \langle |f|, 1\rangle \leq \|f\|_2\|1\|_2 \leq \|f\|_2.$$

Thus

$$\|f_n - f\|_2 \to 0 \Rightarrow \|f_n - f\|_1 \to 0$$

and $j_{2,1}$ is continuous.

However, if we put

$$f_n(t) = n^{2/3} \max\{0, 1 - nt\},$$

then

$$\|f_n - 0\|_1 = \int_0^1 f_n(t)\,dt = n^{-1/3}/2 \to 0,$$

yet

$$\|f_n - 0\|_2^2 = \int_0^1 f_n(t)^2\,dt = n^{4/3}\int_0^{1/n} (1-nt)^2\,dt = \frac{n^{1/3}}{3} \to \infty$$

as $n \to \infty$, so $j_{1,2}$ is not continuous.

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□

Theorem 5.8. *Let (X, d) be a metric space. A set F in X is closed if and only if its complement is open.*

Proof. *Only if* Suppose that F is closed and $E = X \setminus F$. If E is not open, we can find an $e \in E$ such that $B(e, \delta) \cap F \neq \varnothing$ for all $\delta > 0$. In particular, we can find $x_n \in F$ such that $d(x_n, e) < 1/n$ for each $n \geq 1$. Since $x_n \to e$ and F is closed, we have $e \in F$ contradicting our initial assumption that $e \in E$. Thus E is open.

If We now establish the converse. Suppose E is open and $F = X \setminus E$. Suppose $x_n \in F$ and $x_n \to x$. If $x \in E$, then, since E is open, we can find a $\delta > 0$ such that $B(x, \delta) \subseteq E$. Thus $d(x_n, x) \geq \delta$ for all n which is absurd. Thus $x \in F$ and F is closed.

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Theorem 5.9. *If (X, d) is a metric space, then the following statements are true.*

(i) The empty set $\varnothing$ and the space X are closed.

(ii) If F_α is closed for all $\alpha \in A$, then $\bigcap_{\alpha\in A} F_\alpha$ is closed. (In other words the intersection of closed sets is closed.)

(iii) If F_j is closed for all $1 \leq j \leq n$, then $\bigcup_{j=1}^n F_j$ is closed.

Proof. (i) Observe that $\varnothing = X \setminus X$ and $X = X \setminus \varnothing$.

(ii) Since F_α is closed, $X \setminus F_\alpha$ is open for all $\alpha \in A$. It follows that

$$X \setminus \bigcap_{\alpha \in A} F_\alpha = \bigcup_{\alpha \in A} (X \setminus F_\alpha)$$

is open and so $\bigcap_{\alpha \in A} F_\alpha$ is closed.

(iii) Since F_j is closed, $X \setminus F_j$ is open for all $1 \leq j \leq n$. It follows that

$$X \setminus \bigcup_{j=1}^{n} F_j = \bigcap_{j=1}^{n} (X \setminus F_j)$$

is open and so $\bigcup_{j=1}^{n} F_j$ is closed.

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Theorem 5.10. *Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed in X whenever F is closed in Y.*

Proof. *If* Suppose that f is continuous. If F is closed in Y, then $Y \setminus F$ is open, so

$$X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$$

is open. Thus $f^{-1}(F)$ is closed.

Only if Suppose $f^{-1}(F)$ is closed whenever F is. If U is open in Y, then $Y \setminus U$ is closed, so

$$X \setminus f^{-1}(U) = f^{-1}(Y \setminus U)$$

is closed. Thus $f^{-1}(U)$ is open. We have shown that f is continuous.

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Exercise 6.6. *Write $\mathcal{P}(Y)$ for the collection of subsets of Y. If X has three elements, how many elements does $\mathcal{P}\big(\mathcal{P}(X)\big)$ have?*

How many topologies are there on X?

Solution. If Y has n elements $\mathcal{P}(Y)$ has 2^n elements so $\mathcal{P}\big(\mathcal{P}(X)\big)$ has $2^{2^3} = 2^8 = 256$ elements.

Let $X = \{x, y, z\}$. We set out the types of possible topologies below.

type	number of this type
$\{\varnothing, X\}$	1
$\{\varnothing, \{x\}, X\}$	3
$\{\varnothing, \{x\}, \{y\}, \{x, y\}, X\}$	3
$\mathcal{P}(X)$	1
$\{\varnothing, \{x, y\}, X\}$	3
$\{\varnothing, \{x\}, \{x, y\}, X\}$	6
$\{\varnothing, \{z\}, \{x, y\}, X\}$	3
$\{\varnothing, \{x\}, \{z\}, \{x, y\}, \{x, z\}, X\}$	6
$\{\varnothing, \{x\}, \{x, y\}, \{x, z\}, X\}$	3

There are that 29 distinct topologies on X.

The moral of this question is that although there are far fewer topologies than simple collections of subsets and even fewer different types (non-homeomorphic topologies in later terminology) there are still quite a lot even for spaces of three points.

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Lemma 7.3. *Let (X, τ) be a topological space and A a subset of X.*

(i) $\operatorname{Int} A = \{x \in A : \exists\, U \in \tau$ *with* $x \in U \subseteq A\}$.

(ii) $\operatorname{Int} A$ *is the unique* $V \in \tau$ *such that* $V \subseteq A$ *and, if* $W \in \tau$ *and* $V \subseteq W \subseteq A$*, then* $V = W$*. (Informally,* $\operatorname{Int} A$ *is the largest open set contained in* A*.)*

Proof. (i) This is just the observation that

$$\begin{aligned}\operatorname{Int} A &= \bigcup\{U \in \tau : U \subseteq A\}\\ &= \{x \in A : \exists\, U \in \tau \text{ with } x \in U \subseteq A\}\end{aligned}$$

(ii) Since

$$\operatorname{Int} A = \bigcup\{U \in \tau : U \subseteq A\},$$

we know that $\operatorname{Int} A \subseteq A$. Since the union of open sets is open, $\operatorname{Int} A \in \tau$. If $W \in \tau$ and $W \subseteq A$, then

$$\operatorname{Int} A = \bigcup\{U \in \tau : U \subseteq A\} \supseteq W,$$

so, if $W \supseteq \operatorname{Int} A$, $W = \operatorname{Int} A$.

To prove uniqueness, suppose that V' is an open subset of A and has the property that, if U in τ and $V' \subseteq U \subseteq A$. then $V' = U$. Since V' is an open subset of A, we have $V' \subseteq \operatorname{Int} A \subseteq A$ so $V' = \operatorname{Int} A$.

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Exercise 7.4. *Consider $\mathbb{R}$ with its usual topology (i.e. the one derived from the Euclidean norm). We look at the open interval $I = (0,1)$. Show that if F is closed and $F \subseteq (0,1)$, there is a closed G with $F \subseteq G \subseteq (0,1)$ and $G \neq F$. (Thus there is no largest closed set contained in $(0,1)$.)*

Solution. Suppose that F is a closed set with $F \subseteq (0,1)$. Since $0 \notin F$ and F^c is open, we can find a $\delta_1 > 0$ such that $F \cap (-\delta_1, \delta_1) = \varnothing$. Similarly we can find a $\delta_2 > 0$ such that $F \cap (1-\delta_2, 1+\delta_2) = \varnothing$. If we set $\delta = \min\{\delta_1, \delta_2, 1\}/2$ then $[\delta, 1-\delta]$ is closed, $(0,1) \supseteq [\delta, 1-\delta] \supseteq F$, but $[\delta, 1-\delta] \neq F$.

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Lemma 7.7. *Let (X,d) be a metric space and A a subset of X. Then $\operatorname{Cl} A$ consists of all those x such that we can find $x_n \in A$ with $d(x,x_n) \to 0$. (In old fashioned terminology, the closure of A is its set of* closure points*.)*

Proof. Suppose that $x_n \in A$ with $d(x,x_n) \to 0$. Then, since $A \subseteq \operatorname{Cl} A$, $x_n \in \operatorname{Cl} A$ and so, since $\operatorname{Cl} A$ is closed, $x \in \operatorname{Cl} A$.

Suppose conversely that $x \in \operatorname{Cl} A$. Since $\operatorname{Cl} A = X \setminus \operatorname{Int} A^c$, we know that the open ball $B(x, 1/n)$ of radius $1/n$ and centre x cannot lie entirely within A^c, so there exists an $x_n \in B(x,1/n) \cap A$. We have $d(x,x_n) \to 0$, so we are done.

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Exercise 7.9. *(i) Let (X,τ) be a topological space and (Y,d) a metric space. If $f,\ g : X \to Y$ are continuous show that $f(x) = g(x)$ for all $x \in X$.then the set*

$$\{x \in X \,:\, f(x) = g(x)\}$$

is closed.

(ii) Let (X,τ) be a topological space and (Y,d) a metric space[13]. If $f,\ g : X \to Y$ are continuous and $f(x) = g(x)$ for all $x \in A$, where A is dense in X, show that $f(x) = g(x)$ for all $x \in X$.

(iii) Consider the unit interval $[0,1]$ with the Euclidean metric and $A = [0,1] \cap \mathbb{Q}$ with the inherited metric. Exhibit, with proof, a continuous map $f : A \to \mathbb{R}$ (where $\mathbb{R}$ has the standard metric) such that there does not exist a continuous map $\tilde{f} : [0,1] \to \mathbb{R}$ with $\tilde{f}(x) = f(x)$ for all $x \in [0,1]$.

Solution. (i) Let

$$E = \{x \in X \,:\, f(x) = g(x)\}.$$

We show that the complement of E is open and so E is closed.

[13] Exercise 9.7 gives an improvement of parts (i) and (ii).

Suppose $b \in X \setminus E$. Then $f(b) \neq g(b)$. We can find open sets U and V such that $f(b) \in U$, $g(b) \in V$ and $U \cap V = \varnothing$. Now $f^{-1}(U)$ is open, as is $g^{-1}(V)$, so $b \in f^{-1}(U) \cap g^{-1}(V) \in \tau$. But $f^{-1}(U) \cap g^{-1}(V) \subseteq X \setminus E$. Thus $X \setminus E$ is open and we are done.

(ii) Let E be as in (i). We have $A \subseteq E$ and E closed so $X = \operatorname{Cl} A \subseteq E = X$ and $E = X$.

(iii) We observe that $x \in A \Rightarrow x^2 \neq \frac{1}{2}$. If $x \in A$, set

$$f(x) = \begin{cases} 0 & \text{if } x^2 < \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that, if $y \in A$ and $y^2 < \frac{1}{2}$, we can find a $\delta > 0$ such that

$$|y - x| < \delta \Rightarrow x^2 < \tfrac{1}{2} \Rightarrow f(x) = f(y).$$

Similarly if $y \in A$ and $y^2 > \frac{1}{2}$ we can find a $\delta > 0$ such that

$$|y - x| < \delta \Rightarrow x^2 > \tfrac{1}{2} \Rightarrow f(x) = f(y).$$

Thus f is continuous.

Suppose that $\tilde{f} : [0, 1] \to \mathbb{R}$. is such that $\tilde{f}(x) = f(x)$ for all $x \in A$. Choose $p_n, q_n \in A$ such that $p_n^2 > \frac{1}{2} > q_n^2$ and $|p_n - 2^{-1/2}|, |q_n - 2^{-1/2}| \to 0$. Then

$$|\tilde{f}(p_n) - \tilde{f}(2^{-1/2})| + |\tilde{f}(q_n) - \tilde{f}(2^{-1/2})| \geq |\tilde{f}(p_n) - \tilde{f}(q_n)| = 1,$$

so $\tilde{f}$ cannot be continuous.

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Example 8.5. *Let $X = \mathbb{R}$ and let d be the usual metric on $\mathbb{R}$. Let $Y = (0, 1)$ (the open interval with end points 0 and 1) and let ρ be the usual metric on $(0, 1)$. Then (X, d) and (Y, ρ) are homeomorphic as topological spaces, but (X, d) is complete and (Y, ρ) is not.*

Proof. We know from first year analysis that $f(x) = \tan(\pi(y - 1/2))$ is a bijective function $f : Y \to X$ which is continuous with continuous inverse. (Recall that a strictly increasing continuous function has continuous inverse.) Thus (X, d) and (Y, ρ) are homeomorphic. We know that (X, d) is complete by the general principle of convergence.

However, $1/n$ is a Cauchy sequence in Y with no limit in Y. (If $y \in (0, 1)$, then there exists an N with $y > N^{-1}$. If $m \geq 2N$, then $|1/m - y| \geq 1/(2N)$ so $1/n \nrightarrow y$.)

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Lemma 8.6. *Let X be a space and let $\mathcal{H}$ be a collection of subsets of X. Then there exists a unique topology $\tau_{\mathcal{H}}$ such that*

(i) $\tau_{\mathcal{H}} \supseteq \mathcal{H}$, and

(ii) if τ is a topology with $\tau \supseteq \mathcal{H}$, then $\tau \supseteq \tau_{\mathcal{H}}$.

Proof. The proof follows a standard pattern, which is worth learning.
Uniqueness Suppose that σ and σ' are topologies such that

(i) $\sigma \supseteq \mathcal{H}$,

(ii) if τ is a topology with $\tau \supseteq \mathcal{H}$, then $\tau \supseteq \sigma$,

(i)$'$ $\sigma' \supseteq \mathcal{H}$,

(ii)$'$ if τ is a topology with $\tau \supseteq \mathcal{H}$, then $\tau \supseteq \sigma'$.

By (i) and (ii)$'$, we have $\sigma \supseteq \sigma'$ and by (i)$'$ and (ii), we have $\sigma' \supseteq \sigma$. Thus $\sigma = \sigma'$.

Existence Let T be the set of topologies τ with $\tau \supseteq \mathcal{H}$. Since the discrete topology contains $\mathcal{H}$, T is non-empty. Set

$$\tau_{\mathcal{H}} = \bigcap_{\tau \in T} \tau.$$

By construction, $\tau_{\mathcal{H}} \supseteq \mathcal{H}$ and $\tau \supseteq \tau_{\mathcal{H}}$ whenever $\tau \in T$. Thus we need only show that $\tau_{\mathcal{H}}$ is a topology and this we now do.

(a) $\varnothing$, $X \in \tau$ for all $\tau \in T$, so $\varnothing$, $X \in \tau_{\mathcal{H}}$.

(b) If $U_\alpha \in \tau_{\mathcal{H}}$, then $U_\alpha \in \tau$ for all $\alpha \in A$ and so $\bigcup_{\alpha \in A} U_\alpha \in \tau$ for all $\tau \in T$, whence $\bigcup_{\alpha \in A} U_\alpha \in \tau_{\mathcal{H}}$.

(c) If $U_j \in \tau_{\mathcal{H}}$, then $U_j \in \tau$ for all $1 \leq j \leq n$ and so $\bigcap_{j=1}^{n} U_j \in \tau$ for all $\tau \in T$, whence $\bigcap_{j=1}^{n} U_j \in \tau_{\mathcal{H}}$.

Thus $\tau_{\mathcal{H}}$ is a topology, as required.

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Lemma 8.9. *If (X, τ) is a topological space and $Y \subseteq X$, then the subspace topology τ_Y on Y is the collection of sets $Y \cap U$ with $U \in \tau$.*

Proof. Let $j : Y \to X$ be the inclusion map given by $j(y) = y$ for all $y \in Y$. Write

$$\sigma = \{Y \cap U \,:\, U \in \tau\}.$$

Since $Y \cap U = j^{-1}(U)$, we know that τ_Y is the smallest topology containing σ and that the result will follow if we show that σ is a topology on Y. The following observations show this and complete the proof.

(a) $\varnothing = Y \cap \varnothing$ and $Y = Y \cap X$.

(b) $\bigcup_{\alpha \in A}(Y \cap U_\alpha) = Y \cap \bigcup_{\alpha \in A} U_\alpha$.

(c) $\bigcap_{j=1}^{n}(Y \cap U_j) = Y \cap \bigcap_{j=1}^{n} U_j$.

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Lemma 8.13. *Let (X,τ) and (Y,σ) be topological spaces and λ the product topology on $X \times Y$. Then $O \in \lambda$ if and only if, given $(x,y) \in O$, we can find $U \in \tau$ and $V \in \sigma$ such that*

$$(x,y) \in U \times V \subseteq O.$$

Proof. Let μ be the collection of subsets E such that, given $(x,y) \in E$, we can find $U \in \tau$ and $V \in \sigma$ with

$$(x,y) \in U \times V \subseteq E.$$

If $U \in \tau$, then, since π_X is continuous $U \times Y = \pi_X^{-1}(U) \in \lambda$. Similarly, if $V \in \sigma$ then $X \times V \in \lambda$. Thus

$$U \times V = U \times Y \cap X \times V \in \lambda.$$

If $E \in \mu$ then, given $(x,y) \in E$, we can find $U_{(x,y)} \in \tau$ and $V_{(x,y)} \in \sigma$ such that

$$(x,y) \in U_{(x,y)} \times V_{(x,y)} \subseteq E, .$$

We observe that

$$E \subseteq \bigcup_{(x,y)\in E} U_{(x,y)} \times V_{(x,y)} \subseteq E$$

so $E = \bigcup_{(x,y)\in E} U_{(x,y)} \times V_{(x,y)}$ and, since the union of open sets is open, $E \in \lambda$. Thus $\mu \subseteq \lambda$.

It is easy to check that μ is a topology as follows.

(a) $\varnothing \in \mu$ vacuously. If $(x,y) \in X \times Y$, then $X \in \tau$, $Y \in \sigma$ and $(x,y) \in X \times Y \subseteq X \times Y$. Thus $X \times Y \in \mu$.

(b) Suppose $E_\alpha \in \mu$ for all $\alpha \in A$. If $(x,y) \in \bigcup_{\alpha\in A} E_\alpha$, then $(x,y) \in E_\beta$ for some $\beta \in A$. We can find $U \in \tau$ and $V \in \sigma$ such that

$$(x,y) \in U \times V \subseteq E_\beta$$

and so

$$(x,y) \in U \times V \subseteq \bigcup_{\alpha\in A} E_\alpha.$$

Thus $\bigcup_{\alpha\in A} E_\alpha \in \mu$.

(c) Suppose $E_j \in \mu$ for all $1 \le j \le n$. If $(x,y) \in \bigcap_{j=1}^n E_j$, then $(x,y) \in E_j$ for all $1 \le j \le n$. We can find $U_j \in \tau$ and $V_j \in \sigma$ such that

$$(x,y) \in U_j \times V_j \subseteq E_j$$

and so

$$(x,y) \in \bigcap_{j=1}^{n} U_j \times \bigcap_{j=1}^{n} V_j \subseteq \bigcap_{j=1}^{n} E_j.$$

Since $\bigcap_{j=1}^{n} U_j \in \tau$ and $\bigcap_{j=1}^{n} V_j \in \sigma$, we have shown that $\bigcap_{j=1}^{n} E_j \in \mu$.

Finally, we observe that, if $U \in \tau$, then

$$\pi_X^{-1}(U) = U \times Y$$

and $(x,y) \in U \times Y \subseteq \pi_X^{-1}(U)$ with $U \in \tau$, $Y \in \sigma$, so $\pi_X^{-1}(U) \in \mu$. Thus $\pi_X : X \times Y \to X$ is continuous if we give $X \times Y$ the topology μ. A similar result holds for π_Y so, by the minimality of λ, $\mu = \lambda$.

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Exercise 8.14. *Suppose that (X, τ) and (Y, σ) are topological spaces and we give $X \times Y$ the product topology μ. Now fix $x \in X$ and give $E = \{x\} \times Y$ the subspace topology μ_E. Show that the map $k : (Y, \sigma) \to (E, \mu_E)$ given by $k(y) = (x, y)$ is a homeomorphism.*

Solution. The proof is a direct application of Lemma 8.13. We observe that k is a bijection.

If U is open in (Y, σ), then $X \times U \in \mu$ so $k(U) = \{x\} \times U \in \mu_E$. Thus k^{-1} is continuous.

If W is open in (E, μ_E) then $W = E \cap H$ for some $H \in \mu$. If $(x, y) \in W$, then by definition, we can find $J \in \tau$, $I \in \sigma$ such that $(x, y) \in J \times I \subseteq H$. Thus $y \in I \subseteq k^{-1}(W)$ with $I \in \sigma$. We have shown that $k^{-1}(W)$ is open. Thus k is continuous.

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Lemma 8.15. *Let τ_1 and τ_2 be two topologies on the same space X.*

(i) We have $\tau_1 \subseteq \tau_2$ if and only if, given $x \in U \in \tau_1$, we can find $V \in \tau_2$ such that $x \in V \subseteq U$.

(ii) We have $\tau_1 = \tau_2$ if and only if, given $x \in U \in \tau_1$, we can find $V \in \tau_2$ such that $x \in V \subseteq U$ and, given $x \in U \in \tau_2$, we can find $V \in \tau_1$ such that $x \in V \subseteq U$.

Proof. (i) If $\tau_1 \subseteq \tau_2$ and $x \in U \in \tau_1$, then setting $V = U$ we automatically have $V \in \tau_2$ and $x \in V \subseteq U$.

Conversely, suppose that, given $x \in U \in \tau_1$, we can find $V \in \tau_2$ such that $x \in V \subseteq U$. Then, if $U \in \tau_1$ is fixed, we can find $V_x \in \tau_2$ such that $x \in V_x \subseteq U$ for each $x \in U$.

Now

$$U \subseteq \bigcup_{x \in U} V_x \subseteq U$$

so $U = \bigcup_{x \in U} V_x$ and, since the union of open sets is open, $U \in \tau_2$. Thus $\tau_1 \subseteq \tau_2$.

(ii) Observe that $\tau_1 = \tau_2$ if and only if $\tau_1 \subseteq \tau_2$ and $\tau_2 \subseteq \tau_1$.

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Exercise 9.4. *If (X, τ) is a topological space, then a subset A of X is open if and only if every point of A has an open neighbourhood $U \subseteq A$.*

Solution. If A is open, then A is an open neighbourhood of every $x \in A$.

Conversely, suppose that every $x \in A$ has an open neighbourhood U_x lying entirely within A. Then

$$A \subseteq \bigcup_{x \in A} U_x \subseteq A$$

so $A = \bigcup_{x \in A} U_x$. Thus A is the union of open sets and so open.

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Lemma 9.5. *If (X, τ) is a Hausdorff space, then the one point sets $\{x\}$ are closed.*

Proof. We must show that $A = X \setminus \{x\}$ is open. But, if $y \in A$, then $y \neq x$ so, by the Hausdorff condition, we can find $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. We see that $y \in V \subseteq A$, so every point of A has an open neighbourhood lying entirely within A. Thus A is open.

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Exercise 9.6. *Let X be infinite (we could take $X = \mathbb{Z}$ or $X = \mathbb{R}$). We say that a subset E of X lies in τ if either $E = \varnothing$ or $X \setminus E$ is finite. Show that τ is a topology and that every one point set $\{x\}$ is closed but that (X, τ) is not Hausdorff.*

What happens if X is finite?

Solution. (a) We are told that $\varnothing \in \tau$. Since $X \setminus X = \varnothing$, we have $X \in \tau$.

(b) If $U_\alpha \in \tau$ for all $\alpha \in A$, then either $U_\alpha = \varnothing$ for all $\alpha \in A$, so $\bigcup_{\alpha \in A} U_\alpha = \varnothing \in \tau$, or we can find a $\beta \in A$ such that $X \setminus U_\beta$ is finite. In the second case, we observe that

$$X \setminus \bigcup_{\alpha \in A} U_\alpha \subseteq X \setminus U_\beta,$$

so $X \setminus \bigcup_{\alpha \in A} U_\alpha$ is finite and $\bigcup_{\alpha \in A} U_\alpha \in \tau$

(c) If $U_j \in \tau$ for all $1 \leq j \leq n$, then, either $U_k = \varnothing$ for some $1 \leq k \leq n$, so $\bigcap_{j=1}^n U_j = \varnothing \in \tau$, or $X \setminus U_j$ is finite for all $1 \leq j \leq n$. In the second case, since

$$X \setminus \bigcap_{j=1}^n U_j = \bigcup_{j=1}^n (X \setminus U_j),$$

it follows that $X \setminus \bigcap_{j=1}^n U_j$ is finite and so $\bigcap_{j=1}^n U_j \in \tau$.

Thus τ is a topology.

Since $\{x\}$ is finite, $X \setminus \{x\}$ is open and so $\{x\}$ is closed.

Suppose that $x \neq y$ and $x \in U \in \tau$, $y \in V \in \tau$. Then U, $V \neq \varnothing$, so $X \setminus U$ and $X \setminus V$ is finite. It follows that

$$X \setminus U \cap V = (X \setminus U) \cup (X \setminus V)$$

is finite, and so, since X is infinite, $U \cap V \neq \varnothing$. Thus τ is not Hausdorff.

If X is finite, then τ is the discrete metric which is Hausdorff.

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Lemma 9.8. *If (X, τ) is a Hausdorff topological space and $Y \subseteq X$, then Y with the subspace topology is also Hausdorff.*

Proof. Write τ_Y for the subspace topology. If x, $y \in Y$ and $x \neq y$, then x, $y \in X$ and $x \neq y$ so we can find $U, V \in \tau$ with $x \in U$, $y \in V$ and $U \cap V = \varnothing$. Set $\tilde{U} = U \cap Y$ and $\tilde{V} = V \cap Y$. Then $\tilde{U}$, $\tilde{V} \in \tau_Y$ $x \in \tilde{U}$, $y \in \tilde{V}$ and $\tilde{U} \cap \tilde{V} = \varnothing$.

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Lemma 9.9. *If (X, τ) and (Y, σ) are Hausdorff topological spaces, then $X \times Y$ with the product topology is also Hausdorff.*

Proof. Suppose (x_1, y_1), (x_2, y_2) and $(x_1, y_1) \neq (x_2, y_2)$. Then we know that at least one of the statements $x_1 \neq x_2$ and $y_1 \neq y_2$ is true[14]. Without loss of generality, we may suppose $x_1 \neq x_2$. Since (X, τ) is Hausdorff, we can find U_1, U_2 disjoint open neighbourhoods of x_1 and x_2. We observe that $U_1 \times Y$ and $U_2 \times Y$ are disjoint open neighbourhoods of (x_1, y_1) and (x_2, y_2), so we are done.

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Exercise 10.4. *(iv) Show that the topology described in Exercise 9.6 is compact.*

(v) Let X be uncountable (we could take $X = \mathbb{R}$). We say that a subset A of X lies in τ if either $A = \varnothing$ or $X \setminus A$ is countable. Show that τ is a topology but that (X, τ) is not compact.

[14] But not necessarily both. This is the traditional silly mistake.

Solution. (iv) If $X = \varnothing$ there is nothing to prove. If not, let U_α $[\alpha \in A]$ be an open cover. Since $X \neq \varnothing$ we can choose a $\beta \in A$ such that $U_\beta \neq \varnothing$ and so $U_\beta = X \setminus F$ where F is a finite set. For each $x \in F$ we know that $x \in X = \bigcup_{\alpha \in A} U_\alpha$, so there exists an $\alpha(x) \in A$ with $x \in U_{\alpha(x)}$. We have

$$U_\beta \cup \bigcup_{x \in F} U_{\alpha(x)} = X,$$

giving us the desired open cover.

(v) I leave it the reader to show that τ is a topology. Let x_1, x_2, …, be distinct points of X. Let

$$U = X \setminus \{x_j \,:\, 1 \leq j\}$$

and $U_k = U \cup \{x_k\}$. Then $U_k \in \tau$ $[k \geq 1]$ and $\bigcup_{k \geq 1} U_k = X$. Now suppose $k(1)$, $k(2)$,…, $k(N)$ given. If $m = \max_{1 \leq r \leq N} k(r)$, then

$$x_{m+1} \notin \bigcup_{r=1}^{N} U_{k(r)}$$

so there is no finite subcover.

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Lemma 10.5. *Suppose that (X, d) is a compact metric space (that is to say, the topology induced by the metric is compact).*

(i) Given any $\delta > 0$, we can find a finite set of points E such that $X = \bigcup_{e \in E} B(e, \delta)$.

(ii) X has a countable dense subset.

Proof. (i) Observe that the open balls $B(x, \delta)$ form an open cover of X and so have a finite subcover.

(ii) For each $n \geq 1$, choose a finite subset E_n such that

$$X = \bigcup_{e \in E_n} B(e, 1/n).$$

Observe that $E = \bigcup_{n=1}^\infty E_n$ is the countable union of finite sets, so countable. If U is open and non-empty, then we can find a $u \in U$ and a $\delta > 0$ such that $U \supseteq B(u, \delta)$. Choose $N > \delta^{-1}$. We can find an $e \in E_N \subseteq E$ with $u \in B(e, 1/N)$, so

$$e \in B(u, 1/N) \subseteq B(u, \delta) \subseteq U.$$

Thus $\operatorname{Cl} E = X$ and we are done.

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Theorem 10.6. [The Heine–Borel Theorem.] *Let $\mathbb{R}$ be given its usual (Euclidean) topology. Then the closed bounded interval $[a, b]$ is compact.*

Proof. Suppose that $\mathcal{C}$ is an open cover of $[a, b]$ (i.e. the elements of $\mathcal{C}$ are open sets and $\bigcup_{U\in\mathcal{C}} U \supseteq [a, b]$). If $\mathcal{C}_1$ is a finite subcover of $[a, c]$ and $\mathcal{C}_2$ is a finite subcover of $[c, b]$, then $\mathcal{C}_1 \cup \mathcal{C}_2$ is a finite subcover of $[a, b]$.

Suppose now that $[a, b]$ has no finite subcover using $\mathcal{C}$. Set $a_0 = a$, $b_0 = b$, and $c_0 = (a_0+b_0)/2$. By the first paragraph, at least one of $[a_0, c_0]$ and $[c_0, b_0]$ has no finite subcover using $\mathcal{C}$. If $[a_0, c_0]$ has no finite subcover, set $a_1 = a_0$, $b_1 = c_0$. Otherwise, set $a_1 = c_0$, $b_1 = b_0$. In either case, we know that

(i) $a = a_0 \leq a_1 \leq b_1 \leq b_0 = b$,

(ii) If $\mathcal{F}$ is a finite subset of $\mathcal{C}$, then $\bigcup_{U\in\mathcal{F}} U \not\supseteq [a_1, b_1]$,

(iii) $b_1 - a_1 = (b - a)/2$.

Proceeding inductively, we obtain

(i)$_n$ $a \leq a_{n-1} \leq a_n \leq b_n \leq b_{n-1} \leq b$.

(ii)$_n$ If $\mathcal{F}$ is a finite subset of $\mathcal{C}$, then $\bigcup_{U\in\mathcal{F}} U \not\supseteq [a_n, b_n]$.

(iii)$_n$ $b_n - a_n = 2^{-n}(b - a)$.

The a_n form an increasing sequence bounded above by b, so, by the fundamental axiom of analysis, $a_n \to A$ for some $A \leq b$. Similarly $b_n \to B$ for some $B \geq a$. Since $b_n - a_n \to 0$, $A = B = x$, say, for some $x \in [a, b]$. Since $x \in [a, b]$ and $\bigcup_{U\in\mathcal{C}} U \supseteq [a, b]$ we can find a $V \in \mathcal{C}$ with $x \in V$. Since V is open in the Euclidean metric, we can find a $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq V$. Since a_n, $b_n \to x$ we can find an N such that $|x - a_N|$, $|x - b_N| < \delta$ and so

$$[a_N, b_N] \subseteq (x - \delta, x + \delta) \subseteq V$$

contradicting (ii)$_N$. (Just take $\mathcal{F} = \{V\}$.)

The theorem follows by reductio ad absurdum.

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Theorem 10.8. *A closed subset of a compact set is compact. [More precisely, if E is compact and F closed in a given topology, then, if $F \subseteq E$, it follows that F is compact.]*

Proof. Suppose (X, τ) is a topological space, E is a compact set in X and F is a closed subset of E. If $U_\alpha \in \tau$ $[\alpha \in A]$ and $\bigcup_{\alpha\in A} U_\alpha \supseteq F$, then $X \setminus F \in \tau$ and

$$(X \setminus F) \cup \bigcup_{\alpha\in A} U_\alpha = X \supseteq E.$$

By compactness, we can find $\alpha(j) \in A$ $[1 \leq j \leq n]$ such that

$$(X \setminus F) \cup \bigcup_{j=1}^{n} U_{\alpha(j)} \supseteq E.$$

Since $(X \setminus F) \cap F = \varnothing$ and $E \supseteq F$, it follows that

$$\bigcup_{j=1}^{n} U_{\alpha(j)} \supseteq F$$

and we are done.

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Theorem 10.9. *If (X, τ) is Hausdorff, then every compact set is closed.*

Proof. Let K be a compact set. If $x \notin K$, then, given any $k \in K$, we know that $k \neq x$ and so, since X is Hausdorff, we can find open sets U_k and V_k such that

$$x \in V_k,\ k \in U_k \text{ and } V_k \cap U_k = \varnothing.$$

Since $\bigcup_{k \in K} U_k \supseteq \bigcup_{k \in K} \{k\} = K$, we have an open cover of K. By compactness, we can find $k(1)$, $k(2)$, ..., $k(n) \in K$ such that

$$\bigcup_{j=1}^{n} U_{k(j)} \supseteq K.$$

We observe that the finite intersection $V = \bigcap_{j=1}^{n} V_{k(j)}$ is an open neighbourhood of x and that

$$V \cap K \subseteq V \cap \bigcup_{j=1}^{n} U_{k(j)} = \varnothing,$$

so $V \cap K = \varnothing$ and we have shown that every $x \in X \setminus K$ has an open neighbourhood lying entirely within $X \setminus K$. Thus $X \setminus K$ is open and K is closed.

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Example 10.11. *Give an example of a topological space and a compact set which is not closed.*

Proof. If (X, τ) has the indiscrete topology, then, if $Y \subseteq X$, $Y \neq X, \varnothing$, we have Y compact but not closed. We can take $X = \{a, b\}$ with $a \neq b$ and $Y = \{a\}$.

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Theorem 10.12. *Consider $(\mathbb{R}, \tau)$ with the standard (Euclidean) topology. A set E is compact if and only if it is closed and bounded (that is to say, there exists a M such that $|x| \leq M$ for all $x \in E$).*

Proof. If E is bounded, then $E \subseteq [-M, M]$ for some M. By the theorem of Heine–Borel, $[-M, M]$ is compact so, if E is closed, E is compact.

Since $(\mathbb{R}, \tau)$ is Hausdorff, any compact set must be closed. Finally suppose that E is compact. We have

$$E \subseteq \bigcup_{j=1}^{\infty} (-j, j).$$

By compactness, we can find $j(r)$ such that $E \subseteq \bigcup_{r=1}^{N} (-j(r), j(r))$ Taking $M = \max_{1 \leq r \leq n} j(r)$ we have $E \subseteq (-M, M)$ so E is bounded.

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Theorem 10.14. *Let (X, τ) and (Y, σ) be topological spaces and $f : X \to Y$ a continuous function. If K is a compact subset of X, then $f(K)$ is a compact subset of Y.*

Proof. Suppose that $U_\alpha \in \sigma$ for all $\alpha \in A$ and $\bigcup_{\alpha \in A} U_\alpha \supseteq f(K)$. Then

$$\bigcup_{\alpha \in A} f^{-1}(U_\alpha) = f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) \supseteq K$$

and, since f is continuous $f^{-1}(U_\alpha) \in \tau$ for all $\alpha \in A$. By compactness, we can find $\alpha(j) \in A$ $[1 \leq j \leq n]$ such that

$$\bigcup_{j=1}^{n} f^{-1}(U_{\alpha(j)}) \supseteq K$$

and so

$$\bigcup_{j=1}^{n} U_{\alpha(j)} \supseteq f\left(\bigcup_{j=1}^{n} f^{-1}(U_{\alpha(j)})\right) \supseteq f(K)$$

and we are done.

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Theorem 10.17. *Let $\mathbb{R}$ have the usual metric. If K is a closed and bounded subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous, then f is bounded and attains its bounds.*

Proof. If K is empty there is nothing to prove, so we assume $K \neq \varnothing$.

Since K is compact and f is continuous, $f(K)$ is compact. Thus $f(K)$ is a non-empty closed bounded set. Since $f(K)$ is non-empty and bounded, it has a supremum α, say. Since $f(K)$ is closed, it contains its supremum. [Observe that we can find $k_n \in K$ such that

$$\alpha - 1/n \leq f(k_n) \leq \alpha$$

and so $f(k_n) \to \alpha$. Since $f(K)$ is closed, $\alpha \in f(K)$.]

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Exercise 10.18. *Let $\mathbb{R}$ have the usual metric.*

(i) If K is a subset of $\mathbb{R}$ with the property that, whenever $f : K \to \mathbb{R}$ is continuous, f is bounded, show that that K is closed and bounded.

(ii) If K is a subset of $\mathbb{R}$ with the property that, whenever $f : K \to \mathbb{R}$ is continuous and bounded, then f attains its bounds then K is closed and bounded.

Proof. (i) If $K = \varnothing$ there is nothing to prove, so we assume $K \neq \varnothing$.

Let $f : K \to \mathbb{R}$ be defined by $f(k) = |k|$. Since f is bounded, K must be.

If $x \notin K$, then the function $f : K \to \mathbb{R}$ given by $f(k) = |k - x|^{-1}$ is continuous and so bounded. Thus we can find an $M > 0$ such that $|f(k)| < M$ for all $k \in K$. It follows that $|x - k| > M^{-1}$ for all $k \in K$ and the open ball $B(x, M^{-1})$ lies entirely in the complement of K. Thus K is closed.

(ii) If K is unbounded, then, setting $f(x) = \tan^{-1} x$, we see that f is bounded on K, but does not attain its bounds. If K is not closed, then we can find $a \in \mathrm{Cl}(K)$ with $a \notin K$. If we set

$$f(x) = \tan^{-1} \frac{1}{a - x}$$

then f is bounded on K but does not attain its bounds.

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Theorem 10.20. *Let (X, τ) be a compact and (Y, σ) a Hausdorff topological space. If $f : X \to Y$ is a continuous bijection, then it is a homeomorphism.*

Proof. Since f is a bijection, $g = f^{-1}$ is a well defined function. If K is closed in X, then (since a closed subset of a compact space is compact) K is compact so $f(K)$ is compact. But a compact subset of a Hausdorff space is closed so $g^{-1}(K) = f(K)$ is closed. Thus g is continuous and we are done. (If U is open in X then $X \setminus U$ is closed so $Y \setminus g^{-1}(U) = g^{-1}(X \setminus U)$ is closed and $g^{-1}(U)$ is open.)

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Theorem 10.22. *Let τ_1 and τ_2 be topologies on the same space X.*

(i) If $\tau_1 \supseteq \tau_2$ and τ_1 is compact, then so is τ_2.

(ii) If $\tau_1 \supseteq \tau_2$ and τ_2 is Hausdorff, then so is τ_1.

(iii) If $\tau_1 \supseteq \tau_2$, τ_1 is compact and τ_2 is Hausdorff, then $\tau_1 = \tau_2$.

Proof. (i) The map $\iota : (X, \tau_1) \to (X, \tau_2)$ is continuous and so takes compact sets to compact sets. In particular, since X is compact, in τ_1, $X = \iota X$ is compact in τ_2.

(ii) If $x \neq y$ we can find $x \in U \in \tau_2$ and $y \in V \in \tau_2$ with $U \cap V = \varnothing$. Automatically $x \in U \in \tau_1$ and $y \in V \in \tau_1$ so we are done.

(iii) The map $\iota : (X, \tau_1) \to (X, \tau_2)$ is a continuous bijection and so a homeomorphism.

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Exercise 10.23. *(i) Give an example of a Hausdorff space (X, τ) and a compact Hausdorff space (Y, σ) together with a continuous bijection $f : X \to Y$ which is not a homeomorphism.*

(ii) Give an example of a compact Hausdorff space (X, τ) and a compact space (Y, σ) together with a continuous bijection $f : X \to Y$ which is not a homeomorphism.

Solution. Let τ_1 be the indiscrete topology on $[0, 1]$, τ_2 the usual (Euclidean) topology on $[0, 1]$ and τ_3 the discrete topology on $[0, 1]$. Then $([0, 1], \tau_1)$ is compact (but not Hausdorff), $([0, 1], \tau_2)$ is compact and Hausdorff, and $([0, 1], \tau_3)$ is Hausdorff (but not compact). The identity maps $\iota : ([0, 1], \tau_2) \to ([0, 1], \tau_1)$ and $\iota : ([0, 1], \tau_3) \to ([0, 1], \tau_3)$ are continuous bijections but not homeomorphisms.

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Theorem 11.1. *The product of two compact spaces is compact. (More formally, if (X, τ) and (Y, σ) are compact topological spaces and λ is the product topology, then $(X \times Y, \lambda)$ is compact.)*

Proof. Let $O_\alpha \in \lambda$ $[\alpha \in A]$ and

$$\bigcup_{\alpha \in A} O_\alpha = X \times Y.$$

Then, given $(x, y) \in X \times Y$, we can find $U_{x,y} \in \tau$, $V_{x,y} \in \sigma$ and $\alpha(x, y) \in A$ such that

$$(x, y) \in U_{x,y} \times V_{x,y} \subseteq O_{\alpha(x,y)}.$$

In particular, we have

$$\bigcup_{y\in Y}\{x\}\times V_{x,y}=\{(x,y)\,:\,y\in Y\}$$

for each $x\in X$ and so

$$\bigcup_{y\in Y}V_{x,y}=Y.$$

By compactness, we can find a positive integer $n(x)$ and $y(x,j)\in Y$ $[1\leq j\leq n(x)]$ such that

$$\bigcup_{j=1}^{n(x)}V_{x,y(x,j)}=Y.$$

Now $U_x=\bigcap_{j=1}^{n(x)}U_{x,y(x,j)}$ is the finite intersection of open sets in X and so open. Further $x\in U_x$ and so

$$\bigcup_{x\in X}U_x=X.$$

By compactness, we can find x_1, x_2, ..., x_m such that

$$\bigcup_{r=1}^{m}U_{x_r}=X.$$

It follows that

$$\begin{aligned}\bigcup_{r=1}^{m}\bigcup_{j=1}^{n(x_r)}O_{x_r,y(x_r,j)}&\supseteq\bigcup_{r=1}^{m}\bigcup_{j=1}^{n(x_r)}U_{x_r,y(x_r,j)}\times V_{x_r,y(x_r,j)}\\&\supseteq\bigcup_{r=1}^{m}\bigcup_{j=1}^{n(x_r)}U_{x_r}\times V_{x_r,y(x_r,j)}\\&\supseteq\bigcup_{r=1}^{m}U_{x_r}\times Y\\&\supseteq X\times Y\end{aligned}$$

and we are done.

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Exercise 11.7. *Consider the complex plane with its usual metric. Let*

$$\partial D=\{z\in\mathbb{C}\,:\,|z|=1\}$$

and give ∂D the subspace topology τ. Give $\mathbb{R}$ its usual topology and define an equivalence relation $\sim$ by $x \sim y$ if $x - y \in \mathbb{Z}$. We write $\mathbb{R}/\sim = \mathbb{T}$ and give $\mathbb{T}$ the quotient topology. The object of this exercise is to show that ∂D and $\mathbb{T}$ are homeomorphic.

(i) Verify that $\sim$ is indeed an equivalence relation.

(ii) Show that, if we define $f : \mathbb{R} \to \partial D$ by $f(x) = \exp(2\pi i x)$, then $f(U)$ is open whenever U is open.

(iii) If $q : \mathbb{R} \to \mathbb{T}$ is the quotient map $q(x) = [x]$ show that $q(x) = q(y)$ if and only if $f(x) = f(y)$. Deduce that $q\big(f^{-1}(\{\exp(2\pi i x)\})\big) = [x]$ and that the equation $F(\exp(2\pi i x)) = [x]$ gives a well defined bijection $F : \partial D \to \mathbb{T}$.

(iv) Show that $F^{-1}(V) = f\big(q^{-1}(V)\big)$ and deduce that F is continuous.

(v) Show that $\mathbb{T}$ is Hausdorff and explain why ∂D is compact. Deduce that F is a homeomorphism.

Solution. (i) Observe that $x - x = 0 \in \mathbb{Z}$, so $x \sim x$.

Observe that $x \sim y$ implies $x - y \in \mathbb{Z}$, so $y - x = -(x - y) \in \mathbb{Z}$ and $y \sim x$.

Observe that, if $x \sim y$ and $y \sim z$, then $x - y$, $y - z \in \mathbb{Z}$, so

$$x - z = (x - y) + (y - z) = x - z \in \mathbb{Z}$$

and $x \sim z$.

(ii) If $x \in U$ an open set, then we can find a $1 > \delta > 0$ such that $|x-y| < \delta$ implies $y \in U$.

By simple geometry, any $z \in \mathbb{C}$ with $|z| = 1$ and $|\exp(2\pi i x) - z| < \delta/100$ can be written as $z = \exp(2\pi i y)$ with $|y - x| < \delta$. Thus

$$\partial D \cap \{z \in \mathbb{C} \,:\, |z - \exp(2\pi i x)| < \delta/100\} \subseteq f(U).$$

We have shown that $f(U)$ is open.

(iii) We have

$$\begin{aligned} q(x) = q(y) &\Leftrightarrow y \in [x] \Leftrightarrow x - y \in \mathbb{Z} \Leftrightarrow \exp(2\pi i(x - y)) = 1 \\ &\Leftrightarrow \exp(2\pi i x) = \exp(2\pi i y) \Leftrightarrow f(x) = f(y). \end{aligned}$$

It follows that the equation $F(\exp(2\pi i x)) = [x]$ gives a well defined bijection $F : \partial D \to \mathbb{T}$.

(iv) Observe that

$$F^{-1}([x]) = \{\exp(2\pi i t) \,:\, \exp(2\pi i t) = \exp(2\pi i x)\} = f\big(q^{-1}([x])\big)$$

and so $F^{-1}(V) = f\big(q^{-1}(V)\big)$. If V is open, then, since q is continuous, $q^{-1}(V)$ is open so, by (ii), $F^{-1}(V)$ is open. Thus F is continuous.

(v) If $[x] \neq [y]$, then we know that $x - y \notin \mathbb{Z}$ and the set

$$\{|t| \,:\, t - (x-y) \in \mathbb{Z},\ |t| < 1\}$$

is finite and non-empty. Thus there exists a $\delta > 0$ such that

$$\{|t| \,:\, t - (x-y) \in \mathbb{Z},\ |t| < \delta\} = \varnothing.$$

Let

$$U_x = \bigcup_{j=-\infty}^{\infty} (j + x - \delta/4, j + x + \delta/4) \text{ and } U_y = \bigcup_{j=-\infty}^{\infty} (j + y - \delta/4, j + y + \delta/4).$$

Observe that U_x and U_y are open in $\mathbb{R}$ and $q^{-1}\big(q(U_x)\big) = U_x$, $q^{-1}\big(q(U_y)\big) = U_y$, and so $q(U_x)$ and $q(U_y)$ are open in the quotient topology. Since $[x] \in q(U_x)$, $[y] \in q(U_y)$ and $q(U_x) \cap q(U_y) = \varnothing$, we have shown that the quotient topology is Hausdorff.

Since ∂D is closed and bounded in $\mathbb{C}$ and we can identify $\mathbb{C}$ with $\mathbb{R}^2$ as a metric space, ∂D is compact.

Since a continuous bijection from a compact to a Hausdorff space is a homeomorphism, F is a homeomorphism.

[Remark. It is just as simple to show that the natural map from $\mathbb{T}$ (which we know to be compact, why?) to ∂D (which we know to be Hausdorff, why?) is a bijective continuous map. Or we could show continuity in both directions and not use the result on continuous bijections.]

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Example 12.1. *Give an example of a metric space (X, d) which is bounded (in the sense that there exists an M with $d(x, y) \leq M$ for all x, $y \in X$) but for which there exist sequences with no convergent subsequence.*

Solution. Consider the discrete metric on $\mathbb{Z}$. If $x_n = n$ and $x \in \mathbb{Z}$, then $d(x, x_n) = 1$ for all n with at most one exception. Thus the sequence x_n can have no convergent subsequence.

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Theorem 12.4. *If the metric space (X, d) is compact, it is sequentially compact.*

Proof. Let x_n be a sequence in X. If it has no convergent subsequence, then, for each $x \in X$ we can find a $\delta(x) > 0$ and an $N(x)$ such that $x_n \notin B(x, \delta(x))$ for all $n \geq N(x)$. Since

$$X = \bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} B(x, \delta(x)) \subseteq X,$$

the $B(x, \delta(x))$ form an open cover and, by compactness, have a finite subcover. In other words, we can find an M and $y_j \in X$ $[1 \leq j \leq M]$ such that

$$X = \bigcup_{j=1}^{M} B\big(y_j, \delta(y_j)\big).$$

Now set $N = \max_{1 \leq j \leq M} N(y_j)$. Since $N \geq N(y_j)$, we have $x_N \notin B\big(y_j, \delta(y_j)\big)$ for all $1 \leq j \leq M$. Thus $x_N \notin \bigcup_{j=1}^{M} B\big(y_j, \delta(y_j)\big) = X$ which is absurd.

The result follows by reductio ad absurdum.

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Theorem 12.5. *If the metric space (X, d) is compact, then d is complete.*

Proof. Let (x_n) be a Cauchy sequence. By sequential compactness, the sequence has a convergent subsequence and so is convergent.

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Theorem 12.6. *Let (X, d) be a metric space. If a subsequence of a Cauchy sequence converges, then the series converges.*

Proof. Suppose that (x_n) is Cauchy, $n(j) \to \infty$ and $x_{n(j)} \to a$. Given $\epsilon > 0$, we can find an N such that $d(x_n, x_m) < \epsilon/2$ for n, $m \geq N$ and a k such that $n(k) \geq N$ and $d(x_{n(k)}, a) < \epsilon/2$. Thus, if $n \geq N$,

$$d(x_n, a) \leq d(x_n, x_{n(k)}) + d(x_{n(k)}, a) < \epsilon$$

and we are done.

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Lemma 12.7. *Suppose that (X, d) is a sequentially compact metric space and that the collection U_α with $\alpha \in A$ is an open cover of X. Then there exists a $\delta > 0$ such that, given any $x \in X$, there exists an $\alpha(x) \in A$ such that the open ball $B(x, \delta) \subseteq U_{\alpha(x)}$.*

Proof. Suppose the first sentence is true and the second sentence false. Then, for each $n \geq 1$, we can find an x_n such that the open ball $B(x_n, 1/n) \nsubseteq U_\alpha$ for all $\alpha \in A$. By sequential compactness, we can find $y \in X$ and $n(j) \to \infty$ such that $x_{n(j)} \to y$.

Since $y \in X$, we must have $y \in U_\beta$ for some $\beta \in A$. Since U_β is open, we can find an ϵ such that $B(y, \epsilon) \subseteq U_\beta$. Now choose J sufficiently large that $n(J) > 2\epsilon^{-1}$ and $d(x_{n(J)}, y) < \epsilon/2$. We now have, using the triangle inequality, that

$$B(x_{n(J)}, 1/n(J)) \subseteq B(x_{n(J)}, \epsilon/2) \subseteq B(y, \epsilon) \subseteq U_\beta,$$

contradicting the definition of $x_{n(J)}$.

The result follows by reductio ad absurdum.

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Theorem 12.8. *If the metric space (X, d) is sequentially compact, it is compact.*

Proof. Let $(U_\alpha)_{\alpha \in A}$ be an open cover and let δ be defined as in Lemma 12.7. The $B(x, \delta)$ form a cover of X. If they have no finite subcover, then given x_1, x_2, $\dots x_n$ we can find an $x_{n+1} \notin \bigcup_{j=1}^{n} B(x_j, \delta)$. Consider the sequence x_j thus obtained. We have $d(x_{n+1}, x_k) > \delta$ whenever $n \geq k \geq 1$ and so $d(x_r, x_s) > \delta$ for all $r \neq s$. It follows that, if $x \in X$, $d(x_n, x) > \delta/2$ for all n with at most one exception. Thus the sequence of x_n has no convergent subsequence.

It thus follows, by reductio ad absurdum, that the $B(x, \delta)$ have a finite subcover. In other words, we can find an M and $y_j \in X$ $[1 \leq j \leq M]$ such that

$$X = \bigcup_{j=1}^{M} B(y_j, \delta).$$

We thus have

$$X = \bigcup_{j=1}^{M} B(y_j, \delta) \subseteq \bigcup_{j=1}^{M} U_{\alpha(y_j)} \subseteq X$$

so $X = \bigcup_{j=1}^{M} U_{\alpha(y_j)}$ and we have found a finite subcover.

Thus X is compact.

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Theorem 13.5. *If A contains at least two points, then a topological space (X, τ) is connected if and only if every locally constant function $f : X \to A$ is constant.*

Proof. Suppose first that (X, τ) is connected and $f : (X, \tau) \to (A, \Delta)$ is continuous (where Δ is the discrete topology). Choose $t \in X$. Since every subset of A is open in the discrete topology $\{f(t)\}$ and $A \setminus \{f(t)\}$ are open so

$$U = \{x \in X \,:\, f(x) = f(t)\} = f^{-1}(\{f(t)\})$$

and

$$V = \{x \in X \,:\, f(x) \neq f(t)\} = f^{-1}(A \setminus \{f(t)\})$$

are open. Since $U \cap V = \varnothing$, $U \cup V = X$, U is non-empty and X is connected, we have $V = \varnothing$ and f constant.

Conversely, if (X, τ) is not connected, we can find U, $V \in \tau$ such that $U \cap V = \varnothing$, $U \cup V = X$, U, $V \neq \varnothing$. Choosing a, $b \in A$ with $a \neq b$ and setting $f(x) = a$ for $x \in U$, $f(x) = b$ for $x \in V$ we obtain a locally constant non-constant f.

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Exercise 13.8. *Prove the following results.*

(i) If (X, τ) and (Y, σ) are topological spaces, E is a connected subset of X and $g : E \to Y$ is continuous, then $g(E)$ is connected. (More briefly, the continuous image of a connected set is connected.)

(ii) If (X, τ) is a connected topological space and $\sim$ is an equivalence relation on X, then $X/\sim$ with the quotient topology is connected.

(iii) If (X, τ) and (Y, σ) are connected topological spaces, then $X \times Y$ with the product topology is connected.

(iv) If (X, τ) is a connected topological space and E is a subset of X, it does not follow that E with the subspace topology is connected.

Proof. (i) If $g(E)$ is not connected we can find a non-constant continuous $f : g(E) \to \mathbb{R}$ taking only the values 0 and 1. Setting $F = f \circ g$ (the composition of f and g), we know that $F : E \to \mathbb{R}$ is non-constant, continuous and only takes the values 0 and 1. Thus E is not connected.

(ii) $X/\sim$ is the continuous image of X under the quotient map which we know to be continuous.

(iii) Suppose $X \times Y$ with the product topology is not connected. Then we can find a non-constant continuous function $f : X \times Y \to \mathbb{R}$ taking only the values 0 and 1. Take $(x, y), (u, v) \in X \times Y$ with $f(x, y) \neq f(u, v)$. Then, if $f(x, v) = f(x, y)$, it follows that $f(x, v) \neq f(u, v)$. Without loss of generality, suppose that $f(x, v) \neq f(x, y)$. Then we know that the function $\theta : Y \to X \times Y$ given by $\theta(z) = (x, z)$ is continuous. (Use Exercise 8.14 or argue directly as follows. If Ω is open in $X \times Y$ and $z \in \theta^{-1}(\Omega)$, then $(x, z) \in \Omega$, so we can find U open in X and V open in Y such that $(x, z) \in U \times V \subseteq \Omega$. Thus $z \in V \subseteq \theta^{-1}(\Omega)$ and we have shown $\theta^{-1}(\Omega)$ open.) If we set $F = f \circ \theta$, then $F : Y \to \mathbb{R}$ is non-constant, continuous and only takes the values 0 and 1. Thus Y is not connected.

(iv) $\mathbb{R}$ is connected with the usual topology, but $E = (-2, -1) \cup (1, 2)$ is not.

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Lemma 13.9. *Let E be a subset of a topological space (X, τ). If E is connected so is* $\operatorname{Cl} E$.

Proof. Suppose that U and V are open sets with

$$\operatorname{Cl} E \subseteq U \cap V \text{ and } \operatorname{Cl} E \cap U \cap V = \varnothing.$$

Then (since $\operatorname{Cl} E \supseteq E$) we have $E \subseteq U \cap V$ and $E \cap U \cap V = \varnothing$. Since E is connected we know that either $E \cap U = \varnothing$ or $E \cap V = \varnothing$.

Without loss of generality, suppose $E \cap V = \varnothing$. Then $V \subseteq E^c$ so

$$V \subseteq \operatorname{Int} E^c = (ClE)^c$$

and $\operatorname{Cl} E \cap V = \varnothing$.

Thus $\operatorname{Cl} E$ is connected.

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Lemma 13.10. *We work in a topological space (X, τ).*

(i) Let $x_0 \in X$. If $x_0 \in E_\alpha$ and E_α is connected for all $\alpha \in A$, then $\bigcup_{\alpha \in A} E_\alpha$ is connected.

(ii) Write $x \sim y$ if there exists a connected set E with x, $y \in E$. Then $\sim$ is an equivalence relation.

(iii) The equivalence classes $[x]$ are connected.

(iv) If F is connected and $F \supseteq [x]$, then $F = [x]$.

Proof. (i) Let U and V be open sets such that

$$U \cup V \supseteq \bigcup_{\alpha \in A} E_\alpha \text{ and } U \cap V \cap \bigcup_{\alpha \in A} E_\alpha = \varnothing.$$

Without loss of generality, let $x_0 \in U$. Then

$$U \cup V \supseteq E_\alpha \text{ and } U \cap V \cap E_\alpha = \varnothing$$

for each $\alpha \in A$. But $x_0 \in U \cap E_\alpha$ so $U \cap E_\alpha \neq \varnothing$, and so, by the connectedness of E_α, we have

$$U \supseteq E_\alpha$$

for all $\alpha \in A$. Thus $U \supseteq \bigcup_{\alpha \in A} E_\alpha$. We have shown that $\bigcup_{\alpha \in A} E_\alpha$ is connected.

(ii) Observe that if U and V are sets (open or not) such that

$$U \cup V \supseteq \{x\}, \text{ and } U \cap V \cap \{x\} = \varnothing.$$

then either $x \notin U$ and $U \cap \{x\} = \varnothing$ or $x \in U$ so $U \supseteq \{x\}$. Thus the one point set $\{x\}$ is connected and $x \sim x$.

The symmetry of the definition tells us that, if $x \sim y$, then $y \sim x$.

If $x \sim y$ and $y \sim z$, then x, $y \in E$ and y, $z \in F$ for some connected sets E and F. By part (i), $E \cup F$ is connected (observe that $y \in E$, F) so, since x, $z \in E \cup F$, $x \sim z$.

We have shown that $\sim$ is an equivalence relation.

(iii) If $y \in [x]$, then there exists a connected set E_y with $x, y \in E_y$. By definition $[x] \supseteq E_y$ so

$$[x] = \bigcup_{y \in [x]} \{y\} \subseteq \bigcup_{y \in [x]} E_y \subseteq [x]$$

whence

$$[x] = \bigcup_{y \in [x]} E_y$$

and, by part (i), $[x]$ is connected.

(iv) If F is connected and $[x] \subseteq F$, then $x \in F$ and, by definition of $\sim$, $[x] \supseteq F$. It follows that $F = [x]$.

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Lemma 13.13. *If (X, τ) is a topological space and we write $x \sim y$ if x is path-connected to y, then $\sim$ is an equivalence relation.*

Proof. If $x \in X$, then the map $\gamma : [0,1] \to X$ defined by $\gamma(t) = x$ for all t is continuous. (Observe that, if F is a closed set in X, then $\gamma^{-1}(F)$ takes the value $\varnothing$ or $[0,1]$ both of which are closed.) Thus $x \sim x$.

If $x \sim y$, then we can find a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. The map $T : [0,1] \to [0,1]$ given by $T(t) = 1 - t$ is continuous so the composition $\tilde{\gamma} = \gamma \circ T$ is. Observe that $\tilde{\gamma}(0) = y$ and $\tilde{\gamma}(1) = x$ so $y \sim x$.

If $x \sim y$ and $y \sim z$, then we can find continuous maps $\gamma_j : [0,1] \to X$ with $\gamma_1(0) = x$, $\gamma_1(1) = y$, $\gamma_2(0) = y$ and $\gamma_2(1) = z$. Define $\gamma : [0,1] \to X$ by

$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } t \in [0, 1/2] \\ \gamma_2(2t - 1) & \text{if } t \in (1/2, 1]. \end{cases}$$

If U is open in X, then

$$\gamma^{-1}(U) = \{t/2 \,:\, t \in \gamma_1^{-1}(U)\} \cup \{(1+t)/2 \,:\, t \in \gamma_2^{-1}(U)\}$$

is open.

(If more detail is required we argue as follows. Suppose $s \in \gamma^{-1}(U)$. If $s \in (0, 1/2)$, then $2s \in \gamma_1^{-1}(U)$ so, since $\gamma_1^{-1}(U)$ is open we can find a $\delta > 0$ with $s > \delta$ such that $(2s - \delta, 2s + \delta) \subseteq \gamma_1^{-1}(U)$. Thus $(s - \delta/2, s + \delta/2) \subseteq \gamma^{-1}(U)$. If $s = 0$ then $0 \in \gamma_1^{-1}(U)$ so, since $\gamma_1^{-1}(U)$ is open we can find a $\delta > 0$ with $1 > \delta$ such that $[0, \delta) \subseteq \gamma_1^{-1}(U)$. Thus $[s, \delta/2) = [0, \delta/2) \subseteq \gamma^{-1}(U)$. The cases $s \in (1/2, 1]$ are dealt with similarly. This leaves the case $s = 1/2$. Arguing as before, we can find $\delta_1, \delta_2 > 0$ with $1 > \delta_1, \delta_2$ such that

$$(1 - \delta_1, 1] \subseteq \gamma_1^{-1}(U) \text{ and } [0, \delta_2) \subseteq \gamma_2^{-1}(U).$$

Setting $\delta = \min(\delta_1, \delta_2)$ we have

$$(1/2 - \delta/2, 1/2 + \delta/2) \subseteq \gamma^{-1}(U).$$

We see that the case $s = 1/2$ is really the only one which requires care.)

Thus γ is continuous and, since $\gamma(0) = x$, $\gamma(1) = z$, $x \sim z$.

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Theorem 13.14. *If a topological space is path-connected, then it is connected.*

Proof. Suppose that (X, τ) is path-connected and that U and V are open sets with $U \cap V = \varnothing$ and $U \cup V = X$. If $U \neq \varnothing$, choose $x \in U$. If $y \in X$, we can find $f : [0, 1] \to X$ continuous with $f(0) = x$ and $f(1) = y$. Now the continuous image of a connected set is connected and $[0, 1]$ is connected, so $f([0, 1])$ is connected. Since

$$U \cap V \cap f([0, 1]) = \varnothing,\ U \cup V \supseteq f([0, 1]) \text{ and } U \cap f([0, 1]) \neq \varnothing,$$

we know that $U \supseteq f([0, 1])$ so $y \in U$. Thus $U = X$. We have shown that X is connected.

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Exercise 13.15. *Show that the non-empty bounded connected subsets of* $\mathbb{R}$ *(with the usual topology) are the intervals. (By intervals we mean sets of the form* $[a, b]$, $[a, b)$, $(a, b]$ *and* (a, b) *with* $a \leq b$. *Note that* $[a, a] = \{a\}$, $(a, a) = \varnothing$.*)*

Describe, without proof, all the connected subsets of $\mathbb{R}$.

Solution. Since $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) are path connected, they are connected.

Suppose, conversely, that E is bounded and contains at least two points. Since E is bounded $\alpha = \inf E$ and $\beta = \sup E$ exist. Further $\alpha < \beta$. If $c \in (\alpha, \beta) \setminus E$ we can find $x, y \in E$ such that $\alpha < x \leq c$ and $c \leq y < \beta$. If $c \notin E$, then $U = (-\infty, c)$ and $V = (c, \infty)$ are open $U \cap V = \varnothing$, $U \cup V \supseteq E$ but $x \in U \cap E$, $y \in V \cap E$ so $U \cap E$, $V \cap E \neq \varnothing$ and E is not connected.

Thus, if E is connected, $E \supseteq (\alpha, \beta)$ and E is one of $[\alpha, \beta]$, (α, β) $(\alpha, \beta]$ or $[\alpha, \beta)$.

The same kind of argument shows that the connected subsets of $\mathbb{R}$ are precisely the sets of the form $[a, b]$, $[a, b)$, $(a, b]$, (a, b) $(-\infty, b]$, $(-\infty, b)$, $[a, \infty)$, (a, ∞) and $\mathbb{R}$ $[a \leq b]$.

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Theorem 13.16. *If we give $\mathbb{R}^n$ the usual topology, then any open set Ω which is connected is path-connected.*

Proof. If $\Omega = \varnothing$, there is nothing to prove, so we assume Ω non-empty.

Pick $\mathbf{x} \in \Omega$ and let U be the set of all points in Ω which are path-connected to $\mathbf{x}$ and let V be the set of all points in Ω which are not. We shall prove that U and V are open.

Suppose first that $\mathbf{u} \in U$. Since Ω is open, we can find an open ball $B(\mathbf{u}, \delta)$ centre $\mathbf{u}$, radius $\delta > 0$ lying entirely within Ω. If $\mathbf{y} \in B(\mathbf{u}, \delta)$, then $\mathbf{u}$ is path-connected to $\mathbf{y}$ in $B(\mathbf{u}, \delta)$ and so in U. (Consider $\gamma : [0, 1] \to \Omega$ given by $\gamma(t) = t\mathbf{u} + (1 - t)\mathbf{y}$.) Since $\mathbf{x}$ is path-connected to $\mathbf{u}$ and $\mathbf{u}$ is path-connected to $\mathbf{y}$, it follows that $\mathbf{x}$ is path-connected to $\mathbf{y}$ in Ω so $\mathbf{y} \in U$.

Now suppose that $\mathbf{v} \in V$. Since Ω is open, we can find an open ball $B(\mathbf{v}, \delta)$ centre $\mathbf{v}$, radius $\delta > 0$ lying entirely within Ω. If $\mathbf{y} \in B(\mathbf{v}, \delta)$, then $\mathbf{v}$ is path-connected to $\mathbf{y}$ in $B(\mathbf{v}, \delta)$ and so in V. It follows that, if $\mathbf{y}$ is path-connected to $\mathbf{x}$, then so is $\mathbf{v}$. But $\mathbf{v} \in V$, so $\mathbf{y}$ is not path-connected to $\mathbf{x}$. Thus $\mathbf{y} \in V$.

Since $U \cup V = \Omega$ and $U \cap V = \varnothing$, the connectedness of Ω shows that $U = \Omega$ and Ω is path-connected.

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Example 13.17. *We work in $\mathbb{R}^2$ with the usual topology. Let*

$$E_1 = \{(0, y) : |y| \le 1\} \text{ and } E_2 = \{(x, \sin 1/x) : 0 < x \le 2/\pi\}$$

and set $E = E_1 \cup E_2$.

(i) Sketch E.

(ii) Explain why E_1 and E_2 are path-connected and show that E is closed and connected.

(iii) Suppose, if possible, that $\mathbf{x} : [0, 1] \to E$ is continuous and $\mathbf{x}(0) = (1, \sin 1)$, $\mathbf{x}(1) = (0, 0)$. Explain why we can find $0 < t_1 < t_2 < t_3 < \dots$ such that $x(t_j) = \big((j + \frac{1}{2})\pi\big)^{-1}$. By considering the behaviour of t_j and $y(t_j)$, obtain a contradiction.

(iv) Deduce that E is not path-connected.

Solution. Part (i) is left to the reader.

(ii) If y_1, $y_2 \in [-1, 1]$, the function $\mathbf{f} : [0, 1] \to E_1$ given by

$$\mathbf{f}(t) = \big(0, (1 - t)y_1 + ty_2\big)$$

is continuous and $\mathbf{f}(0) = (0, y_1)$ and $\mathbf{f}(1) = (0, y_2)$, so E_1 is path-connected.

If (x_1, y_1), $(x_2, y_2) \in E_2$, then $y_j = \sin 1/x_j$ and setting

$$\mathbf{g}(t) = \Big((1-t)x_1 + tx_2, \sin\big(1/((1-t)x_1 + tx_2)\big)\Big)$$

we see that $\mathbf{g}$ is continuous and $\mathbf{g}(0) = (x_1, y_1)$ and $\mathbf{g}(1) = (x_2, y_2)$, so E_2 is path-connected.

We next show that E is closed. Suppose that $(x_r, y_r) \in E$ and $(x_r, y_r) \to (x, y)$. If $x = 0$, then we note that, since $|y_r| \leq 1$ for all r and $y_r \to y$, we have $|y| \leq 1$ and $(x, y) \in E_1 \subseteq E$. If $x \neq 0$, then $1 \geq x > 0$ (since $x_r \geq 0$ for all r). We can find an N such that $|x - x_r| < x/2$ and so $x_r > x/2$ for all $r \geq N$. Thus, by continuity,

$$(x_r, y_r) = (x_r, \sin 1/x_r) \to (x, \sin 1/x) \in E_2 \subseteq E.$$

Thus E is closed.

Now suppose, if possible, that E is disconnected. Then we can find U and V open such that

$$U \cap E \neq \varnothing,\ V \cap E \neq \varnothing,\ U \cup V \supseteq E \text{ and } U \cap V \cap E = \varnothing.$$

Then

$$U \cup V \supseteq E_j \text{ and } U \cap V \cap E_j = \varnothing.$$

and so, since E_j is path-connected, so connected, we have $U \cap E_j = \varnothing$ or $V \cap E_j = \varnothing$ $[j = 1, 2]$. Without loss of generality, assume $V \cap E_1 = \varnothing$ so $U \supseteq E_1$. Since $(0, 0) \in E_1$, we have $(0, 0) \in U$. Since U is open, we can find a $\delta > 0$ such that $(x, y) \in U$ whenever $\|(x, y)\|_2 < \delta$. If n is large,

$$((n\pi)^{-1}, 0) \in U \cap E_2 = U \cap V \cap E,$$

contradicting our initial assumptions. By reductio ad absurdum, E is connected.

(iii) Write $\mathbf{x}(t) = (x(t), y(t))$. Since $\mathbf{x}$ is continuous, so is x. Since $x(0) = 1$ and $x(1) = 0$, the intermediate value theorem tells us that we can find t_1 with $0 < t_1 < 1$ and $x(t_1) = (\frac{3}{2}\pi)^{-1}$. Applying the intermediate value theorem again, we can find t_2 with $0 < t_2 < t_1$ and $x(t_2) = (\frac{5}{2}\pi)^{-1}$. We continue inductively.

Since the t_j form a decreasing sequence bounded below by 0, we have $t_j \to T$ for some $T \in [0, 1]$. Since y is continuous

$$(-1)^j = \sin\big(1/x(t_j)\big) = y(t_j) \to y(T)$$

which is absurd.

(iv) Part (iii) tells us that there is no path joining $(0, 0)$ and $(1, 0)$ in E, so E is not path-connected.

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Lemma 14.2. *Let (X, τ) be a topological space. Then $U \in \tau$ if and only if, given $x \in U$, we can find a neighbourhood N of x with $N \subseteq U$.*

Proof. If $U \in \tau$ then U is a neighbourhood of x for all $x \in U$.

Conversely, if given any $x \in U$, we can find a neighbourhood N_x of x with $N_x \subseteq U$, then we can find an open neighbourhood U_x of x with $U_x \subseteq N_x$. Since

$$U \subseteq \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} U_x \subseteq \bigcup_{x \in U} N_x \subseteq \bigcup_{x \in U} U = U,$$

we have $U = \bigcup_{x \in U} U_x \in \tau$.

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Lemma 14.3. *Let (X, τ) and (Y, σ) be topological spaces. Then $f : X \to Y$ is continuous if and only if, given $x \in X$ and M a neighbourhood of $f(x)$ in Y, we can find a neighbourhood N of x with $f(N) \subseteq M$.*

Proof. Only if If $f : X \to Y$ is continuous, $x \in X$ and M is a neighbourhood of $f(x)$, then we can find a $V \in \sigma$ with $f(x) \in V \subseteq M$. Since f is continuous $f^{-1}(V) \in \tau$. Thus, since $x \in f^{-1}(V)$, we have that $f^{-1}(V)$ is an open neighbourhood and so a neighbourhood of x. Setting $N = f^{-1}(V)$, we have $f(N) = V \subseteq M$ as required.

If Suppose that, given $x \in X$ and M a neighbourhood of $f(x)$ in Y, we can find a neighbourhood N of x with $f(N) \subseteq M$. Let V be open in Y. If $x \in X$ and $f(x) \in V$, then V is a neighbourhood of $f(x)$ so there exists a neighbourhood N_x of x with $f(N_x) \subseteq V$. We now choose U_x an open neighbourhood of x with $U_x \subseteq N_x$. We have

$$f(U_x) \subseteq V$$

and so $U_x \subseteq f^{-1}(V)$ for all $x \in f^{-1}(V)$. Thus

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} \{x\} \subseteq \bigcup_{x \in f^{-1}(V)} U_x \subseteq \bigcup_{x \in f^{-1}(V)} f^{-1}(V) = f^{-1}(V).$$

It follows that $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x \in \tau$. We have shown that f is continuous.

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Lemma 14.6. *Let X be a set and $\mathcal{B}$ a collection of subsets of X. Let $\tau_{\mathcal{B}}$ be the collection of sets U such that, whenever $x \in U$, we can find a $B \in \mathcal{B}$ such that $x \in B \subset U$.*

Then $\tau_{\mathcal{B}}$ is a topology if and only if $\mathcal{B}$ is a basis.

Proof. We first prove necessity. If τ_B is a topology, then $X \in \tau_{\mathcal{B}}$ and so for each $x \in X$ we can find a $B_x \in \mathcal{B}$ with $x \in B_x$. Thus

$$\bigcup_{B \in \mathcal{B}} B \supseteq \bigcup_{x \in X} B_x \supseteq \bigcup_{x \in X} \{x\} = X$$

so $\bigcup_{B \in \mathcal{B}} B = X$.

Next we observe that, by definition, $\mathcal{B} \subseteq \tau_{\mathcal{B}}$. Thus if B_1, $B_2 \in \mathcal{B}$ we must have $B_1 \cap B_2 \in \tau_{\mathcal{B}}$ and, by definition, if $x \in B_1 \cap B_2$ we can find a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Thus $\mathcal{B}$ is a basis.

We now prove sufficiency. Suppose that $\mathcal{B}$ is a basis. We observe that, using the definition, $\mathcal{B} \subseteq \tau_{\mathcal{B}}$ and whenever $\mathcal{A} \subseteq \tau_{\mathcal{B}}$ we have $\bigcup_{A \in \mathcal{A}} A \in \tau_{\mathcal{B}}$.

We have $\varnothing \in \tau_{\mathcal{B}}$ vacuously and, by the definition of a basis, $X = \bigcup_{B \in \mathcal{B}} B \in \tau_{\mathcal{B}}$.

Finally, if U_1, $U_2 \in \tau_B$ then whenever $x \in U_1 \cap U_2$ we can find B_1, $B_2 \in \mathcal{B}$ with $x \in B_1 \subset U_1$, $x \in B_2 \subset U_2$. By the definition of a basis, we can find $B_3 \in \mathcal{B}$ with

$$x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2.$$

Thus $U_1 \cap U_2 \in \tau_{\mathcal{B}}$. Thus $\tau_{\mathcal{B}}$ is a topology.

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20 Executive summary

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