Prfessor Yotov has pointed out to me that Theorem 3.4.8 is false. I can only plead a temporary insanity. The appropriate corrections (I hope) follow.

Theorem 3.4.8 is false as may be seen by looking at the 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

We therefore need to rewrite the rest of Section 3.4 as follows.

Theorem 1.3.2 may be interpreted in terms of elementary matrices.

Lemma 3.4.9 *Given any $n \times n$ matrix A, we can find elementary matrices $F_1, F_2, \dots, F_p$ and $G_1, G_2, \dots, G_q$ together with a diagonal matrix D such that*

$$F_p F_{p-1} \dots F_1 A G_1 G_2 \dots G_q = D.$$

A simple modification now gives the central theorem of this section.

Theorem 3.4.10 *Given any $n \times n$ matrix A, we can find elementary matrices $L_1, L_2, \dots, L_p$ and $M_1, M_2, \dots, M_q$ together with a diagonal matrix D such that*

$$A = L_1 L_2 \dots L_p D M_1 M_2 \dots M_q.$$

Proof. By Lemma 3.4.9, we can find elementary matrices F_r and G_s together with a diagonal matrix D such that

$$F_p F_{p-1} \dots F_1 A G_1 G_2 \dots G_q = D.$$

Since elementary matrices are invertible and their inverses are elementary (see Lemma 3.4.6), we can take $L_r = F_r^{-1}$, $M_s = G_s^{-1}$ and obtain

$$\begin{aligned} &L_1 L_2 \dots L_p D M_1,\ M_2,\ \dots,\ M_q \\ &\qquad = F_1^{-1} F_2^{-1} \dots F_p^{-1} F_p \dots F_2 F_1 A G_1 G_2 \dots G_q G_q^{-1} G_{q-1}^{-1} \dots{}_1^{-1} = A \end{aligned}$$

as required. □

There is an obvious connection with the problem of deciding when there is an inverse matrix.

Lemma 3.4.11 *Let $D = (d_{ij})$ be a diagonal matrix.*

(i) If all the diagonal entries d_{ii} of D are non-zero, D is invertible and the inverse $D^{-1} = E$ where $E = (e_{ij})$ is given by $e_{ii} = d_{ii}^{-1}$ and $e_{ij} = 0$ for $i \neq j$.

(ii) If some of the diagonal entries of D are zero, then $BDC \neq I$ for all $n \times n$ matrices B and C.

Proof. (i) If all the diagonal entries of D are non-zero, then, taking E as proposed, we have

$$DE = ED = I$$

by direct calculation.

(ii) If $d_{rr} = 0$ for some r, then, if $B = (b_{ij})$ is any $n \times n$ matrix, we have

$$\sum_{k=1}^{n} \left(\sum_{j=1}^{n} b_{rj} d_{jk} \right) c_{kl} = \sum_{k=1}^{n} 0 \times c_{kl} = 0$$

so BDC has all entries in the rth row equal to zero. In particular $BDC \neq I$. □

Lemma 3.4.12 *Let L_1, L_2, …, L_p, M_1, M_2, …, M_q be elementary $n \times n$ matrices and let D be an $n \times n$ diagonal matrix. Suppose that*

$$A = L_1 L_2 \ldots L_p D M_1 M_2 \ldots M_q.$$

(i) If all the diagonal entries d_{ii} of D are non-zero, then A is invertible.

(ii) If some of the diagonal entries of D are zero, then A is not invertible.

Proof. Since elementary matrices are invertible (Lemma 3.4.6 (v) and (vi)) and the product of invertible matrices is invertible (Lemma 3.4.3), we have $A = LDM$ where L and M are invertible.

If all the diagonal entries d_{ii} of D are non-zero, then D is invertible and so, by Lemma 3.4.3, $A = LDM$ is invertible.

If A is invertible, then $D = M^{-1} A L^{-1}$ is the product of inertible matrices, so invertible. Thus none of the diagonal entries of D can be zero. □

As a corollary we obtain a result promised at the beginning of this section.

Lemma 3.4.13 *If A and B are $n \times n$ matrices such that $AB = I$, then A and B are invertible with $A^{-1} = B$ and $B^{-1} = A$.*

Proof. Combine the results of Theorem 3.4.10 with those of Lemma 3.4.12. □

Later we shall see how a more abstract treatment gives a simpler and more transparent proof of this fact.

We are now in a position to provide the complementary result to Lemma 3.4.4.

Lemma 3.4.14 *If A is an $n \times n$ square matrix such that the system of equations*

$$\sum_{j=1}^{n} a_{ij} x_j = y_i \qquad [1 \leq i \leq n]$$

has a unique solution for each choice of y_i, then A has an inverse.

Proof. If we fix k, then our hypothesis tells us that the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = \delta_{ik} \qquad [1 \leq i \leq n]$$

has solution. Thus, for each k with $1 \leq k \leq n$, we can find x_{jk} with $1 \leq j \leq n$ such that

$$\sum_{j=1}^{n} a_{ij}x_{jk} = \delta_{ik} \qquad [1 \leq i \leq n].$$

If we write $X = (x_{jk})$ we obtain $AX = I$ so A is invertible. $\square$

We also need to rewrite the proof of theorem 4.3.6 as follows.

We can now exploit Theorem 3.4.10 which tells us that, given any 3×3 matrix A, we can find elementary matrices $L_1,\ L_2,\ \dots,\ L_p$ and $M_1,\ M_2,\ \dots,\ M_q$ together with a diagonal matrix D such that

$$A = L_1 L_2 \dots L_p D M_1 M_2 \dots M_q.$$

Theorem 4.3.6 *If A and B are 3×3 matrices then* $\det BA = \det B \det A$.

Proof. We know that we can write A in the form given in the paragraph above so

$$\begin{aligned}
\det BA &= \det(BL_1L_2\dots L_pDM_1M_2\dots M_q)\\
&= \det\big(BL_1L_2\dots L_pDM_1M_2\dots M_{q-1})M_q\big)\\
&= \det\big(BL_1L_2\dots L_pDM_1M_2\dots M_{q-1}\big)\det M_q\\
&\ \vdots\\
&= \det(BL_1L_2\dots L_pD)\det M_1\det M_2\dots\det M_q\\
&= \det\big((BL_1L_2\dots L_p)D\big)\det M_1\det M_2\dots\det M_q\\
&= \det(BL_1L_2\dots L_p)\det D\det M_1\det M_2\dots\det M_q\\
&\ \vdots\\
&= \det B\det L_1\det L_2\dots\det L_p\det D\det M_1\det M_2\dots\det M_q
\end{aligned}$$

Looking at the special case $B = I$, we see that

$$\det A = \det L_1\det L_2\dots\det L_p\det D\det M_1\det M_2\dots\det M_q,$$

and so $\det BA = \det B\det A$. $\square$