Professor Yotov has pointed out to me that Theorem 3.4.8 is false. I can only plead a temporary insanity. The appropriate corrections (I hope) follow starting on the next page. (Thanks to Greg Price for a correction to the correction.)

Note also the correction from Dr. Andrej Radovic. In Exercise 1.2.3 (i) $x$ should have a subscript to give

$$a_i x_i = y_i \quad [1 \leq i \leq r]$$

Correction from Greg Price. Last line of Exercise 11.5.25:

Show that $\tilde{T} \in s'$ and $\tilde{T} \neq 0$, but $\tilde{T} b = 0$ for all $b \in c_{00}$. Deduce that $\Theta_{c_{00}}$ is not surjective.
Theorem 3.4.8 is false as may be seen by looking at the $2 \times 2$ matrix
\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \]

We therefore need to rewrite the rest of Section 3.4 as follows.

Theorem 1.3.2 may be interpreted in terms of elementary matrices.

Lemma 3.4.9 Given any $n \times n$ matrix $A$, we can find elementary matrices $F_1, F_2, \ldots, F_p$ and $G_1, G_2, \ldots, G_q$ together with a diagonal matrix $D$ such that
\[ F_pF_{p-1}\ldots F_1AG_1G_2\ldots G_q = D. \]

A simple modification now gives the central theorem of this section.

Theorem 3.4.10 Given any $n \times n$ matrix $A$, we can find elementary matrices $L_1, L_2, \ldots, L_p$ and $M_1, M_2, \ldots, M_q$ together with a diagonal matrix $D$ such that
\[ A = L_1L_2\ldots L_pDM_1M_2\ldots M_q. \]

Proof. By Lemma 3.4.9, we can find elementary matrices $F_r$ and $G_s$ together with a diagonal matrix $D$ such that
\[ F_pF_{p-1}\ldots F_1AG_1G_2\ldots G_q = D. \]

Since elementary matrices are invertible and their inverses are elementary (see Lemma 3.4.6), we can take $L_r = F_r^{-1}$, $M_s = G_s^{-1}$ and obtain
\[ L_1L_2\ldots L_pDM_1, M_2, \ldots, M_q \]
\[ = F_1^{-1}F_2^{-1}\ldots F_p^{-1}F_{p-1}\ldots F_2F_1AG_1G_2\ldots G_qG_q^{-1}G_{q-1}\ldots G_1^{-1} = A \]
as required. \hfill \square

There is an obvious connection with the problem of deciding when there is an inverse matrix.

Lemma 3.4.11 Let $D = (d_{ij})$ be a diagonal matrix.

(i) If all the diagonal entries $d_{ii}$ of $D$ are non-zero, $D$ is invertible and the inverse $D^{-1} = E$ where $E = (e_{ij})$ is given by $e_{ii} = d_{ii}^{-1}$ and $e_{ij} = 0$ for $i \neq j$.

(ii) If some of the diagonal entries of $D$ are zero, then $D$ is not invertible.
Proof. (i) If all the diagonal entries of $D$ are non-zero, then, taking $E$ as proposed, we have

$$DE = ED = I$$

by direct calculation.

(ii) If $d_{rr} = 0$ for some $r$, then, if $B = (b_{ij})$ is any $n \times n$ matrix, we have

$$\sum_{j=1}^{n} d_{rj}c_{jk} = 0$$

so $DC$ has all entries in the $r$th row equal to zero. and, in particular $DC \neq I$. \hfill \qed

Lemma 3.4.12 Let $L_1, L_2, \ldots, L_p, M_1, M_2, \ldots, M_q$ be elementary $n \times n$ matrices and let $D$ be an $n \times n$ diagonal matrix. Suppose that

$$A = L_1L_2\ldots L_pDM_1M_2\ldots M_q.$$

(i) If all the diagonal entries $d_{ii}$ of $D$ are non-zero, then $A$ is invertible.

(ii) If some of the diagonal entries of $D$ are zero, then $A$ is not invertible.

Proof. Since elementary matrices are invertible (Lemma 3.4.6 (v) and (vi)) and the product of invertible matrices is invertible (Lemma 3.4.3), we have $A = LDM$ where $L$ and $M$ are invertible.

If all the diagonal entries $d_{ii}$ of $D$ are non-zero, then $D$ is invertible and so, by Lemma 3.4.3, $A = LDM$ is invertible.

If $A$ is invertible, then $D = M^{-1}AL^{-1}$ is the product of invertible matrices, so invertible. Thus none of the diagonal entries of $D$ can be zero. \hfill \qed

As a corollary we obtain a result promised at the beginning of this section.

Lemma 3.4.13 If $A$ and $B$ are $n \times n$ matrices such that $AB = I$, then $A$ and $B$ are invertible with $A^{-1} = B$ and $B^{-1} = A$.

Proof. Combine the results of Theorem 3.4.10 with those of Lemma 3.4.12. \hfill \qed

Later we shall see how a more abstract treatment gives a simpler and more transparent proof of this fact.

We are now in a position to provide the complementary result to Lemma 3.4.4.

Lemma 3.4.14 If $A$ is an $n \times n$ square matrix such that the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = y_i \quad [1 \leq i \leq n]$$

has a unique solution for each choice of $y_i$, then $A$ has an inverse.
Proof. If we fix $k$, then our hypothesis tells us that the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = \delta_{ik} \quad [1 \leq i \leq n]$$

has solution. Thus, for each $k$ with $1 \leq k \leq n$, we can find $x_{jk}$ with $1 \leq j \leq n$ such that

$$\sum_{j=1}^{n} a_{ij}x_{jk} = \delta_{ik} \quad [1 \leq i \leq n].$$

If we write $X = (x_{jk})$ we obtain $AX = I$ so $A$ is invertible. \qed

We also need to rewrite the proof of theorem 4.3.6 as follows.

We can now exploit Theorem 3.4.10 which tells us that, given any $3 \times 3$ matrix $A$, we can find elementary matrices $L_1, L_2, \ldots, L_p$ and $M_1, M_2, \ldots, M_q$ together with a diagonal matrix $D$ such that

$$A = L_1L_2\ldots L_pDM_1M_2\ldots M_q.$$

**Theorem 4.3.6** If $A$ and $B$ are $3 \times 3$ matrices then $\det BA = \det B \det A$.

**Proof.** We know that we can write $A$ in the form given in the paragraph above so

$$\det BA = \det(BL_1L_2\ldots L_pDM_1M_2\ldots M_q)$$

$$= \det(BL_1L_2\ldots L_pDM_1M_2\ldots M_q - 1) \det M_q$$

$$= \det(BL_1L_2\ldots L_pDM_1M_2\ldots M_q - 1) \det M_q$$

$$= \det(BL_1L_2\ldots L_pD) \det M_1 \det M_2 \ldots \det M_q$$

$$= \det((BL_1L_2\ldots L_p)D) \det M_1 \det M_2 \ldots \det M_q$$

$$= \det(BL_1L_2\ldots L_p) \det D \det M_1 \det M_2 \ldots \det M_q$$

$$= \det B \det L_1 \det L_2 \ldots \det L_p \det D \det M_1 \det M_2 \ldots \det M_q$$

Looking at the special case $B = I$, we see that

$$\det A = \det L_1 \det L_2 \ldots \det L_p \det D \det M_1 \det M_2 \ldots \det M_q,$$

and so $\det BA = \det B \det A$. \qed