Linear Analysis

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Small print The syllabus for the course is defined by the Faculty Board Schedules (which are minimal for lecturing and maximal for examining). Several of the results are called Exercises. I will do some as part of the lectures but others will be left to the reader. In general these are simple verifications or recall results from earlier courses. If you find that you cannot do one of these, consult your supervisor.

I have sketched solutions for supervisors to the exercises in the five example sheets at the end. These should be available from the departmental secretaries or in tex, ps, pdf and dvi format by e-mail.

I should **very much** appreciate being told of any corrections or possible improvements and might even part with a small reward to the first finder of particular errors. These notes are written in $\LaTeX\,2_\varepsilon$ and should be available in tex, ps, pdf and dvi format from my home page

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1 Some inequalities

Inequalities lie at the heart of analysis. In this section we prove some inequalities which lie at the heart of this course.

We start with some observations from Part 1A.

Exercise 1.1. *Suppose $f : (0, \infty) \to \mathbb{R}$ is twice differentiable with $f''(x) < 0$ for all $x \in \mathbb{R}$. Then if $0 < t < s$ and $0 < \lambda < 1$, it follows that*

$$\lambda f(t) + (1 - \lambda) f(s) < f\big(\lambda t + (1 - \lambda) s\big).$$

In other words, if $f''(x) < 0$, then f is strictly concave. Applying the result with $f = \log$, we obtain the following result.

Lemma 1.2. *Suppose p, q are real and positive with*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, if a, $b > 0$, we have

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q}$$

with equality if and only if $a = b$.

Our result remains true if a, $b \geq 0$.

We can now obtain our first version of the Hölder inequality[1]. (Here and elsewhere we will write $\mathbb{F}$ to mean either $\mathbb{R}$ or $\mathbb{C}$. This reflects the fact that many theorems of Linear Analysis apply both to real and complex vector spaces. However, just as in other branches of analysis and algebra, there are important theorems which apply only to complex or only to real spaces.)

Theorem 1.3. *Suppose p, q are real and positive with*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If a_j, $b_j \in \mathbb{F}$, then

$$\sum_{j=1}^{n} |a_j b_j| \leq \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |b_j|^q \right)^{1/q}$$

with equality if and only if we can find A, $B \in \mathbb{R}$, not both zero, with $A|b_j|^q = B|a_j|^p$ for all $1 \leq j \leq n$.

I will probably leave out the examination of the case of equality in Theorems 1.3 and 1.5 but once you have mastered the main proof it a very instructive exercise to look closely the cases of equality.

If we set $p = 2$ in Hölder's inequality we recover the Cauchy-Schwarz inequality.

Our Hölder inequality is complemented by a 'reverse Hölder inequality'.

[1] Mathematical nomenclature should not be confused with historic truth. Hölder's paper makes it clear that he is discussing an inequality previously proved by L. G. Rogers (1862–1933). Rogers was Professor of Mathematics at Yorkshire College (now the University of Leeds) and a fine mathematician 'with little ambition or desire for recognition'. In 1913, Ramanujan conjectured some remarkable identities which no-one could prove. In 1917, whilst looking through old journals, Ramanujan came across a 1894 paper of Rogers in which a more general form of the identities were proved. Rogers' election to the Royal Society followed.

Theorem 1.4. *(i) Suppose $p,\ q$ are real and positive with*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $a_j \in \mathbb{F}$ and

$$\sum_{j=1}^{n} |a_j b_j| \leq A \left(\sum_{j=1}^{n} |b_j|^q \right)^{1/q}$$

for all choices of $b_j \in \mathbb{F}$ then

$$\left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} \leq A.$$

(ii) (Stronger version) Suppose $p,\ q$ are real and positive with

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $a_j \in \mathbb{F}$ and

$$\left| \sum_{j=1}^{n} a_j b_j \right| \leq A \left(\sum_{j=1}^{n} |b_j|^q \right)^{1/q}$$

for all choices of $b_j \in \mathbb{F}$ then

$$\left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} \leq A.$$

Putting Hölder's inequality and the reverse Hölder inequality together, we get Minkowski's inequality.

Theorem 1.5. *Suppose $p > 1$. Then*

$$\left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} + \left(\sum_{j=1}^{n} |b_j|^p \right)^{1/p} \geq \left(\sum_{j=1}^{n} |a_j + b_j|^p \right)^{1/p}$$

with equality only if we can find $\alpha,\ \beta \in \mathbb{F}$, not both zero, such that $\beta a_j = \alpha b_j$ for all $1 \leq j \leq n$.

(Exercise 22.2 contains an apparently simpler proof, but the explicit use of the reverse Hölder inequality in the proof above may help fix it in the reader's mind.)

For us, as for Minkowski, this has a clear geometrical interpretation.

Theorem 1.6. *Suppose $p > 1$. Then*

$$\|\mathbf{a}\|_p = \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p}$$

defines a norm on $\mathbb{F}^n$.

We can extend most of our results (but not all) to the cases $p = 1$ and $p = \infty$.

Exercise 1.7. *(i) Show that*

$$\|\mathbf{a}\|_1 = \sum_{j=1}^{n} |a_j|$$

defines a norm on $\mathbb{F}^n$.

(ii) Show that

$$\|\mathbf{a}\|_\infty = \max_{1 \leq j \leq n} |a_j|$$

defines a norm on $\mathbb{F}^n$.

(iii) Show that, if $\mathbf{a}, \mathbf{b} \in \mathbb{F}^n$, then

$$\sum_{j=1}^{n} |a_j||b_j| \leq \|\mathbf{a}\|_\infty \|\mathbf{b}\|_1.$$

(iv) State and prove the two *'reverse Hölder' inequalities corresponding to (iii).*

(v) If $\|\mathbf{a} + \mathbf{b}\|_1 = \|\mathbf{a}\|_1 + \|\mathbf{b}\|_1$, does it follow that we can find $\alpha, \beta \in \mathbb{F}$, not both zero, such that $\beta a_j = \alpha b_j$ for all $1 \leq j \leq n$? Give reasons.

(vi) If $\|\mathbf{a}+\mathbf{b}\|_\infty = \|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty$, does it follow that we can find $\alpha, \beta \in \mathbb{F}$, not both zero, such that $\beta a_j = \alpha b_j$ for all $1 \leq j \leq n$? Give reasons.

2 In finite dimension, norms are equivalent

We have produced a variety of norms on $\mathbb{F}^n$, but the following theorem (which the reader may have seen in Part 1B) shows that they are all equivalent in a rather strong sense (called Lipschitz equivalence).

Theorem 2.1. *Suppose E is a finite dimensional vector space over $\mathbb{F}$. If $\|.\|_1$ and $\|.\|_2$ are two norms on E, then we can find a constant $K > 0$ such that*

$$K\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2 \geq K^{-1}\|\mathbf{x}\|_1$$

for all $\mathbf{x} \in E$.

Note that the proof depends ultimately on the fundamental axiom of analysis.

However, the unit balls of the different norms $\|.\|_p$ look more and more different as the dimension of the space increases.

Exercise 2.2. *(i) If $\infty > s > r \geq 1$ show, by applying Hölder's inequality with $a_j = x_j^s$, $b_j = 1$ and $p = s/r$, that, if $x_j \geq 0$, we have*

$$\left(\sum_{j=1}^{n} x_j^r\right)^{1/r} \leq \left(\sum_{j=1}^{n} x_j^s\right)^{1/s} n^{r^{-1}-s^{-1}}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Thus, if we work in $\mathbb{F}^n$,

$$\|\mathbf{x}\|_r \leq n^{r^{-1}-s^{-1}} \|\mathbf{x}\|_s$$

and we cannot improve the inequality.

(ii) If $\infty > s > r \geq 1$, show, by setting $t = x_j/\|\mathbf{x}\|_s$ in the inequality $t^r \geq t^s$ $[0 \leq t \leq 1]$, or otherwise, that, if $x_j \geq 0$,

$$\left(\sum_{j=1}^{n} x_j^s\right)^{1/s} \leq \left(\sum_{j=1}^{n} x_j^r\right)^{1/r}$$

and identify the cases of equality.

Conclude that, if we work in $\mathbb{F}^n$,

$$\|\mathbf{x}\|_s \leq \|\mathbf{x}\|_r$$

and we cannot improve the inequality.

(iii) State and prove (the proofs are easy) the corresponding results when we allow $s = \infty$.

Once we reach the simplest infinite dimensional space Theorem 2.1 fails.

Definition 2.3. *We write c_{00} for the space of sequences*

$$\mathbf{a} = (a_1, a_2, \dots)$$

with $a_n \in \mathbb{F}$ and only finitely many a_j non-zero. [Warning: The notation c_{00} is not universal.]

Exercise 2.4. *(i) Check that* c_{00} *is a vector space.*

(ii) Check that

$$\|\mathbf{a}\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{1/p}$$

defines a norm on c_{00} *for* $\infty > p \geq 1$.

(iii) Define an appropriate norm $\|.\|_\infty$ *and show that is a norm.*

(iv) Show that, given any $\infty \geq s > r \geq 1$ *and any* $K > 0$*, we can find a non-zero* $\mathbf{a} \in c_{00}$ *such that*

$$\frac{\|\mathbf{a}\|_r}{\|\mathbf{a}\|_s} > K.$$

3 Banach spaces

It is generally thought that the algebraic properties of large objects are rather dull and this is certainly the case for normed spaces. The remedy is to introduce extra analytic structure, in this case completeness.

Definition 3.1. *A Banach space is a complete normed space.*

We observe that the norms we have placed on c_{00} are not complete.

Exercise 3.2. *Verify this statement.*

Later, in Lemma 6.8, we shall show that c_{00} cannot be given a complete norm. Thus c_{00} is good for nothing except the production of counterexamples. (But it is very good for this purpose.)

However, the work we have already done enables us to produce some nice infinite dimensional Banach spaces.

Theorem 3.3. *Let* $1 \leq p < \infty$. *If* l^p *is the collection of sequences*

$$\mathbf{a} = (a_1, a_2, \dots)$$

with $a_n \in \mathbb{F}$ *and*

$$\sum_{j=1}^{\infty} |a_j|^p \text{ convergent,}$$

then l^p *is a vector space with the usual coordinatewise definition of addition and multiplication by scalars. If we set*

$$\|\mathbf{a}\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{1/p},$$

then $\|.\|_p$ *is a complete norm on* l^p.

Lemma 3.4. *If l^∞ is the collection of bounded sequences*

$$\mathbf{a} = (a_1, a_2, \dots)$$

with $a_n \in \mathbb{F}$, then l^∞ is a vector space with the usual coordinatewise definition of addition and multiplication by scalars. If we set

$$\|\mathbf{a}\|_\infty = \sup_{1 \leq j} |a_j|$$

then $\|.\|_\infty$ is a complete norm on l^∞.

At some point, it became fashionable to write l_p instead of l^p. The reader must be prepared for both notations.

Exercise 3.5. *(i) Show that, if $\infty \geq s > r \geq 1$, then $l^r \subseteq l^s$, $l^r \neq l^s$.*

(ii) If $\infty \geq s > r \geq 1$ show that, if $\mathbf{x} \in l^r$, then $\mathbf{x} \in l^s$ and $\|\mathbf{x}\|_s \leq \|\mathbf{x}\|_r$. Give an explicit example of a sequence $\mathbf{x}(k) \in l^r$ with $\|\mathbf{x}(k)\|_s = 1$ but $\|\mathbf{x}(k)\|_r \to \infty$ as $k \to \infty$.

Although we seek to study infinite dimensional Banach spaces by using geometrical intuition, they differ in important respects from finite dimensional spaces. Here is one example.

Theorem 3.6. *The closed unit ball*

$$\bar{B} = \{\mathbf{x} \in E \,:\, \|\mathbf{x}\| \leq 1\}$$

of an infinite dimensional Banach space $(E, \|.\|)$ is not compact.

Our proof depends on two results which are of independent interest.

Lemma 3.7. *Any finite dimensional subspace of a Banach space is closed.*

Theorem 3.8. [Lemma of F. Riesz] *If F is a closed subspace of a Banach space $(E, \|.\|)$ and $F \neq E$, then, given any $\epsilon > 0$, we can find an $\mathbf{e} \in E$ with $\|\mathbf{e}\| = 1$ such that*

$$\|\mathbf{e} - \mathbf{f}\| > 1 - \epsilon$$

for all $\mathbf{f} \in F$.

The technique that we use to prove Theorem 3.8 is very useful in a wide variety of circumstances.

4 Continuous linear functions

When we studied linear maps on finite dimensional normed spaces we made repeated use of the fact that they were continuous.

Exercise 4.1. *Suppose $(U, \|.\|_U)$ and and $(V, \|.\|_V)$ are finite dimensional normed spaces. Show that every linear map $\alpha : U \to V$ is continuous.*

We cannot make this assumption when we study linear maps on infinite dimensional spaces.

Exercise 4.2. *Consider the space c_{00} of sequences of complex numbers only finitely many of which are non-zero, equipped with the norm*

$$\|\mathbf{a}\|_1 = \sum_{j=1}^{\infty} |a_j|.$$

Give an example of a linear map $\alpha : c_{00} \to \mathbb{F}$ which is not continuous.

The treatment of continuous linear maps on infinite dimensional vector spaces is much aided by some simple observations.

Lemma 4.3. *Suppose $(U, \|.\|_U)$ and and $(V, \|.\|_V)$ are normed spaces and $\alpha : U \to V$ is linear. The following statements are equivalent.*
(i) α is continuous everywhere.
(ii) α is continuous at $\mathbf{0}$.
(iii) There exists a C such that

$$\|\alpha\mathbf{u}\|_V \leq C\|\mathbf{u}\|_U$$

for all $\mathbf{u} \in U$.

For this reason continuous linear maps are sometimes called bounded linear maps.

We can define the *operator norm* and derive its elementary properties in exactly the same way as in the finite dimension case.

Definition 4.4. *Suppose $(U, \|.\|_U)$ and $(V, \|.\|_V)$ are normed spaces and $\alpha : U \to V$ is continuous and linear. We define the operator norm $\|\alpha\|$ of α as follows.*

$$\|\alpha\| = \sup_{\|\mathbf{u}\|_U \leq 1} \|\alpha\mathbf{u}\|_V.$$

Exercise 4.5. *(i) Suppose* $(U, \|.\|_U)$ *and* $(V, \|.\|_V)$ *are normed spaces. The collection* $\mathcal{L}(U, V)$ *of continuous linear maps* $\alpha : U \to V$ *forms a vector space under pointwise addition and multiplication by scalars. The operator norm is indeed a norm on* $\mathcal{L}(U, V)$*.*

(ii) Suppose $(U, \|.\|_U)$ *is a normed space. Then the identity map* $\iota \in \mathcal{L}(U, U)$ *and* $\|\iota\| = 1$*.*

(iii) Suppose $(U, \|.\|_U)$*,* $(V, \|.\|_V)$ *and* $(W, \|.\|_W)$ *are normed spaces. If* $\alpha \in \mathcal{L}(U, V)$ *and* $\beta \in \mathcal{L}(V, W)$*, then* $\beta\alpha \in \mathcal{L}(U, W)$ *and*

$$\|\beta\alpha\| \leq \|\beta\|\|\alpha\|.$$

Theorem 4.6. *If* $(U, \|.\|_U)$ *and and* $(V, \|.\|_V)$ *are Banach spaces, it follows that* $\big(\mathcal{L}(U, V), \|.\|\big)$ *is a Banach space.*

(For an improvement see Exercise 18.8.) A particularly important example of the space $(\mathcal{L}(U, V), \|.\|)$ occurs when $V = \mathbb{F}$.

Definition 4.7. *If* $(U, \|.\|_U)$ *is a normed space over* $\mathbb{F}$ *we say that a continuous linear map* $T : U \to \mathbb{F}$ *is a* bounded linear functional*. The space* $U' = \mathcal{L}(U, \mathbb{F}, \|.\|)$ *is called the* dual *space of* U*.*

Theorem 4.8. *If* $\infty > p \geq 1$ *and* $p^{-1}+q^{-1} = 1$*, then* $(l^p)' = l^q$ *(More exactly, there is a natural identification of* $(l^p)'$ *and* l^q*. When we have introduced more definitions, you may prefer the statement that there is a natural isometric isomorphism between* $(l^p)'$ *and* l^q*.)*

The following important exercise illustrates both our method of proving Theorem 4.8 and its limitations.

Exercise 4.9. *(i) Show that* c*, the set of all sequences* $\mathbf{a}$ *such that* a_j *tends to a limit as* $j \to \infty$*, is a closed subspace of* l^∞ *with its usual norm. Show that* c_0*, the set of all sequences* $\mathbf{a}$ *such that* $a_j \to 0$ *as* $j \to \infty$*, is a closed subspace of* l^∞ *with its usual norm.*

Because of these results we may consider c *and* c_0 *as Banach spaces under the supremum norm (that is to say the* l^∞ *norm).*

(ii) Show that c_0' *can be identified in a natural manner with* l^1*.*

(iii) Identify the elements of c' *in a reasonably natural manner.*

It is an important fact that the method we use to prove Theorem 4.8 fails when we try to find the dual of l^∞. In some sense l^∞ is too large for us to deal with. Lemma 4.12 make it clear what is going on.

Definition 4.10. *We say that a subset* E *of a metric space* (X, d) *is dense in* X*, if given any* $x \in X$*, we can find* $x_n \in E$ *with* $x_n \to x$*.*

Definition 4.11. *A metric space (X, d) is separable if we can find a countable dense subset of X.*

Lemma 4.12. *The space l^p is separable if $\infty > p \geq 1$ but l^∞ is not separable.*

5 Second duals

Students find it hard to understand the treatment of second duals in 1B Algebra but the subject actually becomes easier to understand when treated in a more general setting.

Lemma 5.1. *Let U be a normed vector space. Then the map $J : U \to U''$ given by*

$$(Ju)(T) = Tu$$

for $u \in U$ and $T \in U'$ is a well defined continuous linear map.

In 1B Algebra, when U is a finite dimensional space it is easy to show that J is injective. In the infinite dimensional case we need a further idea.

Definition 5.2. *[Warning. This is non-standard.] We say that a dual U' of a normed space U is* sufficiently rich *if, whenever $u \in U$,*

$$\sup_{T \in U', \|T\| \leq 1} \|Tu\| = \|u\|.$$

It is easy to check that all the spaces we have met so far are sufficiently rich.

Exercise 5.3. *(i) Show that the duals of the l^p spaces are sufficiently rich $[1 \leq p \leq \infty]$. (Note that, although we cannot fully identify the dual of l^∞ we can, nonetheless, show that it is sufficiently rich.)*

(ii) Show that if U is a Banach space with sufficiently rich dual then $\mathcal{L}(U, U)$ has sufficiently rich dual. (We need this later in the proof of Lemma 13.8 but the reader should not loose any sleep over this.)

The reader will find it easy to check (and should carry out the check) that any specific space has a sufficiently rich dual.

If we use the axiom of choice, then it can be shown that all duals are sufficiently rich. Any text that the reader consults will use the axiom of choice and will therefore omit the condition that the dual is sufficiently rich.

Lemma 5.4. *Let U be a normed vector space with a sufficiently rich dual. Then the map $J : U \to U''$ introduced in Lemma 5.1 is isometric (that is to say $\|Ju\|_{U''} = \|u\|_U$) and so injective.*

(Here and everywhere else, we use the operator norm on U' and U''.)

If U' is sufficiently rich, it is reasonable to use J to give natural identification of U with JU and write $U \subseteq U''$.

Lemma 5.5. *If U is a Banach space with sufficiently rich dual then (with the natural identification) U is a closed subspace of U''.*

In 1B Algebra, when U is a finite dimensional space, a dimensional argument shows that J is surjective (and so, with the natural identification, $U = U''$). If $1 < p < \infty$ then the work we have already done shows that $(l^p)'' = l^p$.

However, the following important example shows that U may be a proper subspace of its second dual U''.

Exercise 5.6. *(Part (i) and most of part (iii) were done in Exercise 4.9.) Let c_0 be the the subset of l^∞ consisting of those sequences* **a** *such that that $a_j \to 0$ as $j \to \infty$.*

(i) Show that c_0 is a closed subspace of l^∞ and so $(c_0, \|.\|_\infty)$ is a Banach space.

(ii) Show that c_0 is separable.

(iii) Show that c_0' can be identified in a natural manner with l^1. Show also that the dual of c_0 is sufficiently rich.

(iv) Deduce that c_0'' can identified in a natural manner with l^∞ and so the mapping $J : c_0 \to c_0''$ introduced in Lemma 5.1 cannot be surjective.

There are two ways of looking at Banach spaces. One is to study each space in its own right using the the language and insights of linear analysis as tools. Each of the spaces l^1, l^2 and l^∞ could be, and has been, the object of a lifetime's study. This is the point of view of the present course or, at least, the present lecturer[2].

The second way of looking at Banach spaces is to study them as general structures. In this case we do not study individual spaces but isomorphism classes or isometric isomorphism classes of Banach spaces.

Definition 5.7. *(i) The normed spaces $(U, \|.\|_U)$ and $(V, \|.\|_V)$ are* isomorphic *if there exists a vector space isomorphism $T : U \to V$ such that both T and T^{-1} are continuous.*

(ii) The normed spaces $(U, \|.\|_U)$ and $(V, \|.\|_V)$ are isometrically isomorphic *if there exists a vector space isomorphism $T : U \to V$ such that*

$$\|Tu\|_V = \|u\|_U$$

for all $u \in U$.

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[2] Who has spent most of his mathematical life studying a particular representation of l^1.

From this point of view, all we know so far about the l^p spaces is that l^∞ is not isomorphic to l^p for $1 \leq p < \infty$. (If the reader is interested, Exercise 18.9 shows that l^1 is not isomorphic to l^p for $p \neq 1$ and Exercise 22.5 shows that l^2 is not isomorphic to l^p for $p \neq 2$. It has been shown that, in fact, spaces l^p with distinct p are not isomorphic.) We give a very simple example of Banach space isomorphism in Exercise 6.15.

For the avoidance of doubt, the reader is instructed that, both in the notes and in examination questions, statements about l^p and similar spaces refer to their concrete realisations and not to their isomorphism classes. If isomorphism is to be considered, this will be stated explicitly.

6 Baire category

The Baire category is a profound triviality which condenses the folk wisdom of a generation of ingenious mathematicians into a single statement.

Theorem 6.1. *If (X, d) is a complete metric space and U_j is an open dense subset of X for each $j \geq 1$, then $\bigcap_{j=1}^\infty U_j \neq \varnothing$.*

(The same ideas are used to prove a similar result for compact topological spaces in Exercise 19.5.)

In some sense,the property of belonging to U_j is stable (since U_j is open, small perturbations leave us within U_j) and the property of not belonging to U_j unstable (since U_j is dense, we can move to U_j by arbitrarily small perturbations).

Exercise 6.2. *Consider the space of $n \times n$ complex matrices with the operator norm. Recall that given any matrix A we can find a non-singular matrix B such that BAB^{-1} is upper triangular. By using this result, or otherwise, show that the set of of matrices with n distinct eigenvalues is open and dense.*

For historical reasons Baire's theorem is associated with some rather unhelpful nomenclature.

Definition 6.3. *Consider a metric space (X, d). If E_j is closed with dense complement and $E \subseteq \bigcup_{j=1}^\infty E_j$, then E is said to be of* first category*.*

Baire's theorem can be restated as follows.

Theorem 6.4. [Baire's category theorem] *If (X, d) is a complete metric space, then X is not of first category.*

The following remark is very useful.

Lemma 6.5. *Consider a metric space (X, d). The countable union of sets of the first category is of first category.*

The next remark that may already have occurred to the reader.

Exercise 6.6. *If (X, d) is a complete metric space and E is a subset of first category, then $X \setminus E$ is dense in E.*

Recall that a point x in a metric space (X, d) is called *isolated* if we can find a $\delta > 0$ such that $d(x, y) < \delta$ implies $y = x$.

Lemma 6.7. *A complete metric space without isolated points is uncountable.*

Observe that this gives us a new proof that $\mathbb{R}$ is uncountable (and so transcendental numbers exist) which does not depend on establishing decimal representation.

Lemma 6.8. *(i) If E is an infinite dimensional Banach space over $\mathbb{F}$, then E cannot have a countable spanning set. In other words, we cannot find a sequence e_1, e_2, ... in E such that every $u \in E$ can be written*

$$u = \sum_{j=1}^{N} \lambda_j e_j$$

for some $\lambda_j \in \mathbb{F}$ and some $N \geq 1$.

(ii) The space c_{00} cannot be given a complete norm.

Exercise 6.9. *Consider the Banach space $(l^p, \|.\|_p)$. We know that* considered as a set*, l^r is a subset of l^p whenever $p \geq r \geq 1$. With this convention $\bigcup_{p>r} l^r$ is of first category in $(l^p, \|.\|_p)$.*

Banach and Steinhauss used the Baire category theorem to isolate another piece of folk wisdom.

Theorem 6.10. [Principle of uniform boundedness] *Suppose $(U, \|.\|_U)$ and $(V, \|.\|)$ are Banach spaces. If we have a family $\mathcal{T}$ of continuous linear maps $T : U \to V$ such that $\sup_{T\in\mathcal{T}} \|Tu\|_V < \infty$ for each $u \in U$, then $\sup_{T\in\mathcal{T}} \|T\| < \infty$.*

Here is a typical use of the principle. We work on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ (but if the reader prefers she may work on $[-\pi, \pi]$). To see why this result may be interestiong recall applied lecturers writing down the following 'aspirational prose'.

We have $g_n \to \delta$, that is to say the continuous function g_n tends to the delta function, and so

$$\int g_n(t) f(t)\, dt \to \int g_n(t)\delta(t)\, dt = f(0).$$

Exercise 6.11. *Suppose that* $g_n : \mathbb{T} \to \mathbb{R}$ *is continuous and*

$$\frac{1}{2\pi}\int_{\mathbb{T}} g_n(t)f(t)\,dt \to f(0)$$

as $n \to \infty$ *for all continuous functions* $f : \mathbb{T} \to \mathbb{R}$*. Then the following must be true.*

(i) There exists a constant K *such that*

$$\frac{1}{2\pi}\int_{\mathbb{T}} |g_n(t)|\,dt \leq K$$

for all $n \geq 1$*.*

(ii) If $\delta > 0$ *and* f *is a continuous function with* $f(t) = 0$ *for* $|t| < \delta$*, then*

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(t)g_n(t)\,dt \to 0$$

as $n \to \infty$*.*

(iii) We have

$$\frac{1}{2\pi}\int_{\mathbb{T}} g_n(t)\,dt \to 1$$

as $n \to \infty$*.*
[In Exercise 22.7 we establish that these necessary conditions are also sufficient. In Exercise 22.9 we use Exercise 6.11 to establish that the Fourier series of a continuous function need not converge pointwise to that function.]

We use the Baire category theorem to prove the following series of rather more subtle results.

Theorem 6.12. [Open mapping theorem] *Suppose* $(U, \|.\|_U)$ *and* $(V, \|.\|)$ *are Banach spaces. If* $T \in \mathcal{L}(U, V)$ *is surjective, then* T *maps open sets in* U *to open sets in* V*.*

Exercise 6.13. *It is easy to see that linearity is essential for results like these. Give an example of a continuous surjective map* $f : \mathbb{R} \to \mathbb{R}$ *which is not open.*

We give an example of the use of the open mapping theorem in Exercise 18.11 The reader may recall a very useful 'open mapping theorem' in complex variable theory. The following is an immediate consequence of Theorem 6.12.

Theorem 6.14. [Inverse mapping theorem][3] *Suppose that* $(U, \|.\|_U)$ *and* $(V, \|.\|_V)$ *are Banach spaces. If* $T \in \mathcal{L}(U, V)$ *is bijective, then* T^{-1} *is continuous (so* T *is an isomorphism).*

[3] Called the *inversion theorem* in the syllabus.

(For a variation on this theme, see Exercise 21.4.)

Here is a simple example of the use of Theorem 6.14.

Exercise 6.15. *The space c of sequences with limits and the space c_0 of sequences with limit zero (both equipped with the supremum norm) are Banach space isomorphic.*

We introduce the last of this group of theorems with an exercise.

Exercise 6.16. *(i) Let (X,d) be a metric space. If $f : X \to X$ is continuous, then the graph*

$$\big\{\big(x, f(x)\big) \,:\, x \in X\big\}$$

is closed with respect to the product metric.

(ii) If $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = x^{-2}$ for $x \neq 0$ and $g(0) = 0$, then the graph

$$\big\{\big(x, g(x)\big) \,:\, x \in X\big\}$$

is closed in the usual metric but g is not continuous.

Theorem 6.17. [Closed graph theorem] *Suppose $(U, \|.\|_U)$ is Banach space and $T : U \to U$ is a linear function. If the graph*

$$\big\{\big(u, T(u)\big) \,:\, u \in U\big\}$$

is closed with respect to the product norm, then T is continuous.

To see how such a theorem can be used, we recall some definitions and results from from 1B algebra. The reader can check that they apply without change in the infinite dimensional case.

Exercise 6.18. *If U is a vector space, we say that a linear map $P : U \to U$ is a* projection *if $P^2 = P$. Show that, for such a P,*

$$(I - P)^{-1}(0) = P(U) \text{ and } P^{-1}(0) = (I - P)(U).$$

Show further that every $u \in U$ can be written uniquely in the form $u = v + w$ with $v \in P(U)$ and $w \in P^{-1}(0)$.

Theorem 6.19. *Suppose $(U, \|.\|_U)$ is Banach space and $P : U \to U$ is a projection. Then P is continuous if and only if the kernel $P^{-1}(0)$ and image $P(U)$ are closed.*

7 Continuous functions

We recall the discussion of continuous functions in the Topological and Metric Spaces course.

Exercise 7.1. *Let (X, τ) and (Y, σ) be topological spaces. The following two statements about a function $f : X \to Y$ are equivalent.*

(i) If $U \in \sigma$, then $f^{-1}(U) \in \tau$.

(ii) Given $x \in X$ and $V \in \sigma$ with $f(x) \in V$, we can find a $W \in \tau$ with $x \in W$ and $F(W) \subseteq V$.

Any f satisfying the conditions of Exercise 7.1 is called continuous. We shall be interested in continuous functions $F : X \to \mathbb{F}$ where (X, τ) is a topological space[4] and $\mathbb{F}$ has its usual topology.

Even if the reader has not seen the next three exercises before, she should have no difficulty with them.

Exercise 7.2. *(i) Let (X, τ) be a topological space and let $f_n : X \to \mathbb{F}$ be continuous. Suppose that $f : X \to \mathbb{F}$ is such that we can find $\epsilon_n \to 0$ with*

$$|f_n(x) - f(x)| < \epsilon_n \text{ for all } x \in X.$$

Show that f is continuous. (In other words, the uniform limit of continuous functions is continuous.)

(ii) Let $C_0(X)$ be the space of bounded continuous functions $f : X \to \mathbb{F}$. Show that

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

defines a complete norm on $C_0(X)$.

Note that the completeness of l^∞ is a special case where $X = \mathbb{N}$ and τ is the discrete topology.

Exercise 7.3. *If (X, τ) is compact, show that every continuous function $f : X \to \mathbb{R}$ is bounded.*

Exercise 7.4. *Show that, if E is subset of $\mathbb{F}^n$ with the usual topology, then every continuous function $f : E \to \mathbb{F}^n$ is bounded if and only if E is compact.*

These results strongly suggest that we should study the space $C(X) = C_{\mathbb{F}}(X)$ of continuous functions $f : X \to \mathbb{F}$ with the uniform norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$ in the case when X is compact.

However, if we simply demand that X is compact, the *space* $C(X)$ may not have much to do with the *set* X.

[4]As usual, we shall sometimes merely refer to X with the topology τ being understood.

Exercise 7.5. *If X has the indiscrete topology $\tau = \{X, \varnothing\}$, then $C(X)$ consists of the constant functions.*

The following simple observation puts us on a profitable path.

Exercise 7.6. *If $C(X)$ is such that, given $x \neq y$, we can find an $f \in C(X)$ with $f(x) \neq f(y)$ (informally, if $C(X)$* separates *the points of X), then X is Hausdorff.*

In this section we prove the remarkable fact that the converse also holds for compact spaces. Thus it is natural to study $C(X)$ when X is compact and Hausdorff[5].

We need to recall a couple of elementary topological results.

Exercise 7.7. *(i) In a compact space, every closed set is compact.*
(ii) In a Hausdorff space, singleton sets $\{a\}$ are closed.

We now start our theorem sequence.

Theorem 7.8. *If (X, τ) is compact and Hausdorff, then, given A and B non-empty disjoint closed sets, we can find disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

(A space satisfying the conclusions of Theorem 7.8 is called normal. See Exercise 22.11 for more on this topic.)

Theorem 7.9. [Urysohn's lemma] *If (X, τ) is compact and Hausdorff, then, given A and B non-empty disjoint closed sets, we can find an $f \in C_{\mathbb{R}}(X)$ such that $0 \leq f(x) \leq 1$ for all $x \in X$ and*

$$f(a) = 1 \text{ when } a \in A$$
$$f(b) = 0 \text{ when } b \in B.$$

Exercise 7.7 now tells us that $C(X)$ separates points whenever X is compact and Hausdorff. It is, perhaps, worth remarking that Urysohn's lemma has a much simpler proof if τ is derived from a metric.

The following simple remark comes in useful in our proof of Urysohn's lemma.

Exercise 7.10. *Let (X, τ) be a topological space and let $\mathbb{R}$ have its usual topology. A function $f : X \to \mathbb{R}$ is continuous if and only if $f^{-1}\big((-\infty, a)\big)$ is open and $f^{-1}\big((-\infty, a]\big)$ is closed for all $a \in \mathbb{R}$.*

[5] This result so impressed a retired French general that he proposed using the word 'compact' to mean 'compact and Hausdorff'. The innovation was not popular but the reader should be aware of this possible source of confusion.

In fact we can prove an apparently stronger result than Urysohn's lemma.

Theorem 7.11. [Tietze's extension theorem] *If Y is closed subset of a compact Hausdorff space (X,τ), then, given any $f \in C_{\mathbb{R}}(Y)$ (where Y has the subspace topology), we can find an $F \in C_{\mathbb{R}}(X)$ such that $F(y) = f(y)$ for all $y \in Y$.*

To see that Tietze's extension theorem is non-trivial consider the following example.

Exercise 7.12. *Consider the closed interval $X = [-4,4]$ with the usual topology and the open interval $Y = (0,1)$. Show that if $f : Y \to \mathbb{R}$ is defined by $f(y) = \sin(1/y)$ then f is continuous but there does not exist an $F \in C(X)$ such that $F(y) = f(y)$ for all $y \in Y$.*

We strengthen Theorem 7.11 in two steps.

Corollary 7.13. *If Y is closed subset of a compact Hausdorff space (X,τ), then, given any $f \in C_{\mathbb{F}}(Y)$, we can find an $F \in C_{\mathbb{F}}(X)$ such that $F(y) = f(y)$ for all $y \in Y$.*

Corollary 7.14. *If Y is closed subset of a compact Hausdorff space (X,τ), then, given any $f \in C_{\mathbb{F}}(Y)$, we can find an $F \in C_{\mathbb{F}}(X)$ such that $F(y) = f(y)$ for all $y \in Y$ and $\|F\|_\infty = \|f\|_\infty$.*

8 The Stone–Weierstrass theorem

Unless the reader has lead a very sheltered life she will have done the following important exercise many times before. (If not, she should do it at once.)

Exercise 8.1. [Cauchy's example] *Let $E(x) = \exp(-1/x^2)$ for $x \neq 0$ and $E(0) = 0$.*

(i) Show that E is infinitely differentiable on $\mathbb{R} \setminus \{0\}$ with

$$E^{(n)}(x) = P_n(1/x)E(x)$$

for some polynomial P_n.

(ii) Show that E is infinitely differentiable everywhere with $E^{(n)}(0) = 0$ for all n.

(iii) Use the fact that a power series is infinitely differentiable term by term to show that we cannot find $a_j \in \mathbb{R}$ with $E(x) = \sum_{j=-\infty}^{\infty} a_j x^j$.

(Exercise 19.7, which uses the Baire category theorem from a later section, provides an even stronger result.) Weierstrass must, therefore, have been delighted to prove the following result.

Theorem 8.2. *The set of real polynomials is uniformly dense in $C_{\mathbb{R}}([a,b])$.*

In other words, given any continuous real function $f : [a,b] \to \mathbb{R}$ and any $\epsilon > 0$, we can find a real polynomial with

$$|P(t) - f(t)| < \epsilon$$

for all $t \in [a,b]$.

When Stone was asked to contribute an article to the first issue of the *American Mathematical Monthly*, he produced the following far reaching extension of Weierstrass's theorem.

Theorem 8.3. [The Stone–Weierstrass theorem] *Consider a compact Hausdorff space X. Suppose that A is a subspace of $C_{\mathbb{R}}(X)$ with the following properties.*
(i) If f, $g \in A$ then $f \times g \in A$.
(ii) $1 \in A$.
(iii) If x, $y \in X$ then we can find an $f \in A$ such that $f(x) \neq f(y)$.
Then A is dense in $(C_{\mathbb{R}}, \|.\|_\infty)$.

(If A is a subspace of $C(X)$ satisfying (i), we sometimes say that A is a *subalgebra* of $C(X)$.)

Our proof of the Stone–Weierstrass theorem makes use of the following fact.

Lemma 8.4. *We can find $a_j \in \mathbb{R}$ such that*

$$(1-x)^{1/2} = \sum_{j=0}^{\infty} a_j x^j$$

for all real x with $|x| < 1$.

Our version of the Stone–Weierstrass theorem deals with *real valued* functions. The following example shows that it will not apply in the complex case without modification.

Example 8.5. *We work in the complex plane $\mathbb{C}$. Let*

$$\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\} \text{ and } D = \{z \in \mathbb{C} : |z| < 1\}.$$

We write $A(\bar{D})$ for the set of $f \in C(\bar{D})$ such that f is analytic on D. Then $A(\bar{D})$ is a subspace of $C_{\mathbb{C}}(\bar{D})$ with the following properties.
(i) If f, $g \in A(\bar{D})$ then $f \times g \in A$.
(ii) $1 \in A(\bar{D})$.
(iii) If z, $w \in \bar{D}$ then we can find an $f \in A(\bar{D})$ such that $f(z) \neq f(w)$.
However, $A(\bar{D})$ is not uniformly dense in $C(\bar{D})$.

Instead we produce the following variation.

Theorem 8.6. [The complex Stone–Weierstrass theorem] *Consider a compact Hausdorff space X. Suppose that A is a subspace of $C_{\mathbb{C}}(X)$ with the following properties.*

(i) If $f, g \in A$, then $f \times g \in A$.
(ii) $1 \in A$.
(iii) If $x, y \in X$, then we can find an $f \in A$ such that $f(x) \neq f(y)$.
(iv) If $f \in A$, then its complex conjugate $f^* \in A$.
Then A is dense in $(C_{\mathbb{C}}(X), \|.\|_\infty)$.

The following exercise gives a typical application and clears up matters left vague in the 1B methods course.

Exercise 8.7. *We work on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. If $f : \mathbb{T} \to \mathbb{C}$ is continuous we write*

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \exp(-int)\, dt.$$

(i) The collection of trigonometric polynomials $\sum_{j=-n}^{n} a_j \exp(ijt)$ is uniformly dense in $C_{\mathbb{C}}(\mathbb{T})$.

(ii) (Uniqueness of Fourier series.) If $f, g \in C_{\mathbb{C}}(\mathbb{T})$ and $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{Z}$, then $f = g$.

(iii) If $f \in C_{\mathbb{C}}(\mathbb{T})$ and $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ converges, then

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \exp(int)$$

for all $t \in \mathbb{T}$.

Exercise 19.3 gives another example of Stone–Weierstrass in action.

9 Ascoli–Arzelà

It is frequently possible to show that a problem can be solved ‘apart from an error which can be made as small as we like’. Under these circumstances an appeal to compactness, if available, will often show that the problem has an exact solution.

The Ascoli–Arzelà theorem enables us to characterise the compact subsets of $C(X)$ when X is a compact metric space.

Definition 9.1. *Let (X, τ) be a topological space and (Y, ρ) a metric space. We say that a collection $\mathcal{F}$ of functions $f : X \to Y$ is* equicontinuous *at x if given $\epsilon > 0$ we can find a $U \in \tau$ with $x \in U$ such that*

$$y \in U \text{ implies } \rho(f(x), f(y)) < \epsilon \text{ for all } f \in \mathcal{F}.$$

If $\mathcal{F}$ is equicontinuous at all points of X we say that $\mathcal{F}$ is equicontinuous.

Exercise 9.2. *If (X, d) and (Y, ρ) are metric spaces, write out the definition of equicontinuity in ϵ, δ form.*

Theorem 9.3. [Ascoli–Arzelà] *Let (X, τ) be a compact Hausdorff space. Then a subset $\mathcal{F}$ of $C(X)$ is compact under the the uniform norm if and only if both the following conditions hold.*

(i) $\mathcal{F}$ is closed and bounded in the uniform norm.

(ii) $\mathcal{F}$ is equicontinuous.

We shall prove the Ascoli–Arzelà theorem by a direct attack. A cleaner proof depending on results from the Metric and Topological course is given in Exercise 19.11 but the basic ideas of the two proofs are the same.

A typical example of the use of these ideas appears in the proof of the following nice result.

Theorem 9.4. *If $\eta > 0$ and $f : [x_0 - \eta, x_0 + \eta] \times [y_0 - \eta, y_0 + \eta] \to \mathbb{R}$ is continuous, then we can find a δ with $\eta \geq \delta > 0$ and a differentiable function*

$$\phi : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$$

such that $\phi(x_0) = y_0$ and

$$\phi'(t) = f(t, \phi(t))$$

for all $t \in (x_0 - \delta, x_0 + \delta)$.

In Part 1B we used the contraction mapping theorem (another idea from 'abstract analysis') to prove the following theorem.

Theorem 9.5. *If $\eta > 0$, $K > 0$ and $f : [x_0 - \eta, x_0 + \eta] \times [y_0 - \eta, y_0 + \eta] \to \mathbb{R}$ satisfies the Lipschitz condition.*

$$|f(x, y) - f(x', y)| \leq K|x - x'|$$

for all x, $x' \in [x_0 - \eta, x_0 + \eta]$ and all $y \in [y_0 - \eta, y_0 + \eta]$, then we can find a δ with $\eta \geq \delta > 0$ and a unique differentiable function

$$\phi : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$$

such that $\phi(x_0) = y_0$ and

$$\phi'(t) = f(t, \phi(t))$$

for all $t \in (x_0 - \delta, x_0 + \delta)$.

Exercise 9.6. *By using the mean value theorem, establish that if*

$$f : [x_0 - \eta, x_0 + \eta] \times [y_0 - \eta, y_0 + \eta] \to \mathbb{R}$$

has continuous first partial derivative $\partial f(x, y)/\partial x$, then it satisfies a Lipschitz condition.

Our new theorem establishes existence under much more general conditions that those of Theorem 9.5, but the solution need not be unique.

Exercise 9.7. *The differential equation $x'(t) = 3x(t)^{2/3}$ has more than one solution with $x_0 = 0$.*

It is helpful, when considering the form of our proof for Theorem 9.4, to observe that if there are different solutions of the equations then a series of 'approximate solutions' may switch between approximating one solution and another.

Even in the Lipschitz case we cannot hope to prove more than the existence of *local* solutions since no *global* solution may exist.

Exercise 9.8. *Find all the solutions of $x'(t) = (1 + x(t)^2)$. Observe that there is no solution which is valid over an interval of length greater than π.*

10 Inner product spaces

Since the reader's first arrival in Cambridge she has been bombarded with inner product spaces. In this section we recall some of the results she already knows. She should check where appropriate that the results hold in infinite dimensional spaces.

Definition 10.1. *Let V be a vector space over $\mathbb{C}$. Suppose that exists a map $p : V^2 \to \mathbb{C}$ such that, writing $\langle u, v\rangle = p(u, v)$ we have*

(i) $\langle \lambda_1 u_1 + \lambda_2 u_2, v\rangle = \lambda_1\langle u_1, v\rangle + \lambda_2\langle u_2, v\rangle$ for all λ_1, $\lambda_2 \in \mathbb{C}$, u_1, u_2, $v \in V$.

(ii) $\langle u, v\rangle = \langle v, u\rangle^*$ for all u, $v \in V$.

(iii) $\langle u, u\rangle \geq 0$ for all $u \in V$.

(iv) $\langle u, u\rangle = 0$ implies $u = 0$.

Then we say that (V, p) is an inner product space. We call p an inner product.

A similar definition applies with $\mathbb{C}$ replaced by $\mathbb{R}$ except that the complex conjugation in condition (ii) is superfluous.

Exercise 10.2. *(i) (Cauchy–Schwarz) If V is an inner product space then*

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle$$

with equality if and only if u and v are linearly dependent.

(ii) If V is an inner product space then

$$\|u\|_2^2 = \langle u, u\rangle, \ \|u\|_2 \geq 0$$

defines a norm on V.

(iii) (Parallelogram law) With the notation of (ii)

$$\|u+v\|_2^2 + \|u-v\|_2^2 = 2\big(\|u\|_2^2 + \|v\|_2^2\big).$$

We derived the norm from the inner product but the process can be reversed and we can recover the inner product from the norm.

Exercise 10.3. [The polarisation identity] *With the notation and assumptions of Exercise 10.2,*

$$4\langle u, v\rangle = \|u+v\|_2^2 - \|u-v\|_2^2 + i(\|u+iv\|_2^2 - \|u-iv\|_2^2)$$

for all u $v \in V$.

(For an interesting sidelight see Exercise 22.12.)

Definition 10.4. *Let V be an inner product space.*

(i) If u, $v \in V$ and $\langle u, v\rangle = 0$ we say that u and v are orthogonal and write $u \perp v$.

(ii) A collection E of vectors is said to be orthonormal if, whenever e, $f \in E$

$$\langle e, f\rangle = \begin{cases} 0 & \text{if } e \neq f, \\ 1 & \text{if } e = f. \end{cases}$$

We have the following extensions of Pythagoras's theorem.

Exercise 10.5. *Consider an inner product space V. Suppose e_1, e_2, $\ldots$, e_n are orthonormal vectors in V and $f \in V$. Then*

$$\|f - \sum_{j=1}^{n} \lambda_j e_j\|_2^2 \geq \|f\|_2^2 - \sum_{j=1}^{n} |\langle f, e_j\rangle|^2$$

with equality if and only if $\lambda_j = \langle f, e_j\rangle$.

Theorem 10.6. **[Bessel's inequality]** *Consider an inner product space V. Suppose $e_1, e_2, \dots$ is an orthonormal sequence of vectors in V and $f \in V$. Then*

$$\sum_{j=1}^{\infty} |\langle f, e_j\rangle|^2 \leq \|f\|_2^2$$

with equality if and only if

$$\left\| f - \sum_{j=1}^{N} \langle f, e_j\rangle e_j \right\|_2 \to 0$$

as $n \to \infty$.

We illustrate these familiar general results with a familiar special case. Note that Exercise 10.7 (iii) resolves a problem left open by the 1B mathematical methods course.

Exercise 10.7. *We work on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.*

(i) Show that if $f \in C_{\mathbb{R}}(\mathbb{T})$, $f(t) \geq 0$ for all t and

$$\frac{1}{2\pi} \int_{\mathbb{T}} f(t)\, dt = 0,$$

then $f(t) = 0$ for all t.

(ii) Show that the formula

$$\langle f, g\rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) g(t)^*\, dt = 0$$

defines an inner product on $C_{\mathbb{C}}(\mathbb{T})$. From now on we consider $C_{\mathbb{C}}(\mathbb{T})$ with this inner product.

(iii) Show that, if we write $e_j(t) = \exp ijt$, then the e_j are orthonormal. By using the fact that the trigonometric polynomials are dense in $(C_{\mathbb{C}}(\mathbb{T}), \|.\|_\infty)$ (Exercise 8.7), show that the trigonometric polynomials are dense in $(C_{\mathbb{C}}(\mathbb{T}), \|.\|_2)$. Hence show that

$$\left\| f - \sum_{j=-M}^{N} \langle f, e_j\rangle e_j \right\|_2 \to 0$$

as $M, N \to \infty$.

(iv) (Parseval's formula) Use (iii) to show that, if we write $\hat{f}(j) = \langle f, e_j\rangle$, then

$$\sum_{j=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|^2\, dt$$

for all $f \in C_{\mathbb{C}}(\mathbb{T})$. Show also that

$$\sum_{j=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)^* = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(t)^*\,dt.$$

However, we note the following important fact.

Exercise 10.8. $(C(\mathbb{T}), \|.\|_2)$ *is not complete.*

This result needs careful proof. We need to show, not that Cauchy sequence does not converge to the *obvious* answer, but that it does not converge to *any* continuous function. The next exercise illustrates this remark.

Exercise 10.9. *Write*

$$\Delta_n(t) = \begin{cases} 1 - 2^n|t| & \text{for } |t| \leq 2^{-n}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that,if we define $f_n \in C(\mathbb{T})$ by

$$f_n = \sum_{j=1}^{n} n\Delta_n(t - 2\pi r/n),$$

then $\|f\|_2 \to 0$.

11 Hilbert space

The work of this section depends on the following key result.

Theorem 11.1. *Let V be an infinite inner product space. The following statements are equivalent.*
(i) V is separable.
(ii) There exists an orthonormal sequence e_j such that

$$\left\| f - \sum_{j=1}^{n} \langle f, e_j\rangle e_j \right\|_2 \to 0$$

as $n \to \infty$ for all $f \in V$.

Our proof calls on an old friend, the Gramm–Schmidt orthogonalisation process.

Exercise 11.2. *Suppose V is an inner product space. If $e_1, e_2, \ldots, e_n$ are orthonormal and $f \in V$ then either*

(i) $f = \sum_j^n \langle f, e_j \rangle e_j$ and $f \in \operatorname{span}\{e_1, e_2, \ldots, e_n\}$, or

(ii) $f \neq \sum_j^n \langle f, e_j \rangle e_j$ in which case $f \notin \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. In this case, setting

$$u = f - \sum_j^n \langle f, e_j \rangle e_j$$

and $e_{n+1} = \|u\|_2^{-1} u$, we have $e_1, e_2, \ldots, e_{n+1}$ orthonormal and

$$\operatorname{span}\{e_1, e_2, \ldots, e_{n+1}\} = \operatorname{span}\{e_1, e_2, \ldots, f\}.$$

From now on, if

$$\left\| f - \sum_{j=1}^n f_j \right\|_2 \to 0,$$

we feel free to write

$$f = \sum_{j=1}^\infty f_j.$$

Exercise 11.3. *(Uniqueness) Let V be an infinite inner product space. If we have an orthonormal sequence e_j, then, if $\lambda_j \in \mathbb{F}$,*

$$\sum_{j=1}^\infty \lambda_j e_j = 0$$

implies $\lambda_j = 0$ for all j.
[Note this is result of analysis and not of algebra since it involves limits.]

Definition 11.4. *If U is an inner product space, we say that an orthonormal sequence e_j in U is a basis*[6] *(or more exactly an orthonormal basis) for U if*

$$x = \sum_{j=1}^\infty \langle x, e_j \rangle e_j$$

for all $x \in U$.

We immediately obtain the following remarkable result.

[6]NB This is not an *algebraic* basis.

Theorem 11.5. **[Riesz–Fisher]** *All separable complete infinite dimensional inner product spaces are inner product isomorphic. More precisely, if U and V are separable complete infinite dimensional inner product spaces with inner products p_U and p_V, then there exists a linear map $T : U \to V$ such that*

$$p_V(Tx, Ty) = p_U(x, y)$$

for all $x, y \in U$. We note that T is automatically an isometric Banach space isomorphism.

Since all separable complete infinite dimensional inner product spaces are isomorphic, we simply talk about the Hilbert[7] space H. Sometimes people talk about complete inner product spaces which are not separable and are then careful to talk about 'non-separable Hilbert spaces' but the study of such large spaces has not yet been very profitable. (If you want to see such a space, consult Exercise 22.17.)

Our arguments also give the following results more or less for free.

Exercise 11.6. *Consider l^2. If $\mathbf{a}, \mathbf{b} \in l^2$ then $\sum_{j=1}^{\infty} a_j b_j^*$ is absolutely convergent. Further*

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^{\infty} a_j b_j^*$$

defines an inner product which induces the norm $\|.\|_2$. With this inner product, l^2 is (inner product isomorphic to) Hilbert space.

Exercise 11.7. *Let U be a separable infinite dimensional inner product space. Then there exists an inner product preserving linear map $J : U \to H$ of U into the Hilbert space H such that $J(U)$ is dense in H.*

If the reader knows about such things, she will be able to restate Exercise 11.7 as the observation that the completion of a separable infinite dimensional inner product space is (inner product isomorphic to) Hilbert space.

Lemma 11.8. *If U is an inner product space, with basis e_j, then U is complete if and only if*

$$\sum_{j=1}^{\infty} x_j e_j$$

converges whenever $\sum_{j=1}^{\infty} |x_j|^2$ converges.

[7] Hilbert developed the theory H in a non-abstract way for particular purposes. There is a, no doubt apocryphal, story of his asking 'What is this Hilbert space which the young people are talking about?'.

12 The dual of Hilbert space

We already know that Hilbert space is isometrically isomorphic to l^2 and we know that l^2 has dual space isometrically isomorphic to itself. Thus the dual space of Hilbert space is isometrically isomorphic to itself.

However, Hilbert space is the infinite dimensional space in which our geometrical intuition has freest play and it is instructive to follow a geometric path to a closely related result. Not only does this avoid the inelegant use of specific bases, but it provides additional insight into the structure of Hilbert space[8]..

Theorem 12.1. *Let U be a complete inner product space. If F is closed subspace and $a \in U$, then we can find a unique $f_0 \in F$ such that*

$$\|a - f_0\|_2 \leq \|a - f\|_2$$

for all $f \in F$.

(See also Exercises 20.4 and 20.5.)

Lemma 12.2. *With the hypotheses and notation of Theorem 12.1, $f_0 \in F$ is the unique element of F such that $a - f_0$ is orthogonal to every element of F.*

We immediately deduce the following pleasing result.

Theorem 12.3. [Riesz representation] *If U is a complete inner product space and $T \in U'$ (that is to say, $T : U \to \mathbb{F}$ is a continuous linear map), then there is a unique $w \in U$ with*

$$Tu = \langle u, w \rangle$$

for all $u \in U$.

Exercise 12.4. *If U is a complete inner product space and we define by*

$$J(v)u = \langle u, v \rangle$$

for all $u, v \in U$, then $J(v) \in U'$ for all $v \in U$ and J has the following properties.

(i) $J(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1^* J(v_1) + \lambda_2^* J(v_2)$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and all $v_1, v_2 \in V$. (We say that J is anti-linear.*)*

(ii) $\|J(v)\| = \|v\|_2$ for all $v \in V$.

(iii) J is surjective.

[8] Since we do not use bases, our results will also apply to non-seperable Hilbert spaces but the reader may ignore this.

Thus (using the polarisation identity of Exercise 10.3) $J : U \to U'$ is an inner product *anti-isomorphism* and U' is naturally anti-isomorphic to U. (If the reader is interested, but only if she is interested, she may glance at Exercise 22.15.)

Theorem 12.1 and Lemma 12.2 also give us information on orthogonal complements which will be used later.

Lemma 12.5. *If F is a closed subspace of a Hilbert space H, then*

$$F^{\perp} = \{g \in H : \langle g, f\rangle = 0 \text{ for all } f \in F\}$$

is a closed subspace of H. Every $u \in H$ can be written in one and only one way as

$$u = f + g$$

with $f \in F$ and $g \in F^{\perp}$.

13 The spectrum

When we studied linear maps $\alpha : \mathbb{C}^n \to \mathbb{C}^n$, we were particularly interested in those $\lambda \in \mathbb{C}$ such that $\alpha - \lambda\iota$ was not invertible. This interest carries over to infinite dimensional spaces[9]. The elementary theory is no harder for general Banach spaces than for Hilbert spaces[10] so we shall work in the general context.

Definition 13.1. *If U is a Banach space over $\mathbb{C}$ and $T : U \to U$ is a continuous linear map, we define the spectrum $\sigma(T)$ of T by*

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not invertible}\}.$$

The inverse mapping theorem shows that, if $\lambda \notin \sigma(T)$, then $(T - \lambda I)^{-1}$ is a continuous linear map. The structure of the spectrum can be exceedingly intricate but some useful general results can be obtained by applying the following simple 'master theorem'.

Theorem 13.2. *If U is a Banach space over $\mathbb{C}$ and $T : U \to U$ is a continuous linear map with $\|T\| < 1$, then $\sum_{j=0}^{\infty} T^j$ converges in the uniform norm and $I - T$ is invertible with*

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j.$$

[9] Even in the finite dimensional case, the study of such things in real vector spaces turned out to be less interesting, so we shall stick to complex Banach spaces.

[10] However is only true for the elementary theory.

Lemma 13.3. *If U is a Banach space over $\mathbb{C}$ and $T : U \to U$ is a continuous linear map, then $\sigma(T)$ is bounded.*

Lemma 13.4. *If U is a Banach space over $\mathbb{C}$ and $T : U \to U$ is a continuous linear map, then $\sigma(T)$ is closed.*

Definition 13.5. *If U is a Banach space over $\mathbb{C}$ and $T : U \to U$ is a continuous linear map, we say that λ is an* eigenvalue *of T if*

$$\ker(T - \lambda I) \neq \{0\}.$$

If u is a non-zero element of $\ker(T - \lambda I)$, we call u an eigenvector *with associated eigenvalue λ.*

Exercise 13.6. *With the notation just introduced, every eigenvalue of T lies in $\sigma(T)$.*

Example 13.7. *(i) If K is a non-empty closed bounded set in $\mathbb{C}$, then we can find a continuous linear map $T : l^2 \to l^2$ with $\sigma(T) = K$.*

(ii) We can find a continuous linear map $T : l^2 \to l^2$ such that $\sigma(T) = \{0\}$ but 0 is not an eigenvalue.

Recall that every Banach space we have studied has a sufficiently rich dual in the sense of Definition 5.2.

Lemma 13.8. *If U is a Banach space over $\mathbb{C}$ with sufficiently rich dual and $T : U \to U$ is a continuous linear map, then $\sigma(T)$ is non-empty.*

If we were prepared to develop complex analysis for $\mathcal{L}(U, U)$ valued functions from scratch, we could replace Lemma 13.8 by a stronger and simpler result.

Lemma 13.9. *If U is a Banach space over $\mathbb{C}$ and $T : U \to U$ is a continuous linear map, then $\sigma(T)$ is non-empty.*

14 Self-adjoint compact operators on Hilbert space

In the previous section we developed the elementary theory of the spectrum for general Banach spaces. From now on we are only interested in Hilbert space.

The reader will recall the very pretty theory of diagonalisation for self-adjoint (that is to say, Hermitian) maps $\alpha : V \to V$ on finite dimensional inner product spaces. We conclude this course by developing a parallel theory for Hilbert space. We need two definitions of which only the second is really new.

Definition 14.1. *Let H be a Hilbert space. A continuous linear map $T : H \to H$ is called* self-adjoint *(or* Hermitian*) if*

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

for all x, $y \in H$.

Exercise 14.2. *The eigenvalues of a self-adjoint continuous linear map are real.*

Definition 14.3. *Let H be a Hilbert space. A continuous linear map $T : H \to H$ is called* compact *if $\mathrm{Cl}(T(B))$, the closure of the image under T of the unit ball $B = \{x : \|x\| \leq 1\}$, is compact.*

Exercise 14.4. *Let H be a Hilbert space. Show that a continuous linear map $T : H \to H$ is compact, if given any $x_n \in H$ with $\|x_n\| \leq 1$, we can find $n(j) \to \infty$ and a $y \in H$ such that*

$$\|Tx_{n(j)} - y\|_2 \to 0.$$

Exercise 21.7 gives some insight into what the compact operators[11] look like.

We state the theorem which we wish to prove.

Theorem 14.5. [The spectral theorem] *Let H be a Hilbert space. If $T : H \to H$ is a continuous linear compact self-adjoint map, we can find an orthonormal basis e_n of eigenvectors whose associated eigenvalues λ_n are real and satisfy the condition $\lambda_n \to 0$ as $n \to \infty$.*

Exercise 14.6. *Show that the following is an equivalent statement of Theorem 14.5.*

Let H be a Hilbert space. If $T : H \to H$ is a continuous linear compact self-adjoint map we can find an orthonormal basis e_n and a sequence λ_n of real numbers with $\lambda_n \to 0$ such that

$$Tu = \sum_{j=1}^{\infty} \lambda_j \langle u, e_j\rangle e_j$$

for all $u \in H$.

We give yet another equivalent form in Exercise 21.9.

[11] A continuous linear map $T : H \to H$ is called an operator.

Exercise 14.7. *Let H be a Hilbert space. with orthonormal basis e_n. If λ_n is a sequence of real numbers with $\lambda_n \to 0$, then the equation*

$$Tu = \sum_{j=1}^{\infty} \lambda_j \langle u, e_j \rangle e_j$$

defines a continuous linear compact self-adjoint map $T : H \to H$.

The proof of Theorem 14.5 parallels its finite dimensional analogue but additional work is required. For the rest of the section we work in a Hilbert space H and 'T is an operator' will mean that $T : H \to H$ is a continuous linear map.

Lemma 14.8. *If T is a self-adjoint operator, then*

$$\sup_{\|x\|_2=1} |\langle x, Tx \rangle| = \|T\|.$$

Lemma 14.9. *If T is a compact self-adjoint operator, then at least one of $\|T\|$ or $-\|T\|$ is an eigenvalue.*

The next result recalls 1B Linear Algebra.

Exercise 14.10. *If T is a self-adjoint operator and e is an eigenvector for T, then, writing*

$$e^\perp = \{f \in H \,:\, \langle e, f \rangle = 0\},$$

we know that $T(e^\perp) \subseteq e^\perp$. The map $T|_{e^\perp} : e^\perp \to e^\perp$ is self-adjoint and, if T is compact, so is $T|_{e^\perp}$.

Lemma 14.11. *If T is a compact operator then, given any $\epsilon > 0$, T has only finitely many orthonormal eigenvectors with associated eigenvalues having absolute values greater than ϵ.*

Putting these these results together we obtain the spectral theorem for compact self-adjoint operators.

Exercise 14.12. *Suppose that T is a compact self-adjoint (ie Hermitian) operator. Consider the following properties which T may or may not have.*

(A) $T^{-1}(0) = \{0\}$.

(B) $T^{-1}(0)$ has dimension r for some $r \geq 1$.

(C) $T^{-1}(0)$ has infinite dimension.

(a) T has infinitely many eigenvalues.

(b) T has s eigenvalues for some $s \geq 1$.

Which of the pairs (X,y) can be true of T and which cannot? Give reasons or examples.

This completes the course, but I have added two extra sections. The first is an extended exercise on the use of the spectral theorem which is strongly recommended to the reader.

15 Using the spectral theorem

In mathematical methods you studied Sturm–Liouville equations

$$\frac{d}{dt}\Big(p(t)y'(t)\Big) + q(t)y(t) = f(t)$$

on an interval $[a, b]$ subject to conditions

$$A_1 y(a) + A_2 y'(a) = 0,\ \ B_1 y(b) + B_2 y'(b) = 0$$

with $(A_1, A_2) \neq (0, 0)$, $(B_1, B_2) \neq (0, 0)$, p continuously differentiable, f, q continuous and $p(t) > 0$ for all $t \in [a, b]$. You showed that it is generally possible to find a continuous *Green's function* $G : [a, b]^2 \to \mathbb{R}$ with $G(s, t) = G(t, s)$ such that

$$y(t) = \int_a^b G(s, t) f(s)\, ds$$

solves the given problem.

We shall not into the details here. (They are in [3] §19 and in [5].) The next exercise gives a particular case.

Exercise 15.1. *(i) If $G : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $g(t) = G(t, t)$, write down $g'(t)$.*

(ii) By using the fundamental theorem of the calculus and differentiation under the integral show, that under conditions on F that you should specify,

$$\frac{d}{dt} \int_a^t F(s, t)\, ds = F(t, t) + \int_a^t \frac{\partial F}{\partial t}(s, t)\, ds.$$

(iii) Show that, if

$$G(s, t) = \begin{cases} (1 - s)t & \text{if } 1 \geq t \geq s \geq 0, \\ s(1 - t) & \text{if } 1 \geq s > t \geq 0, \end{cases}$$

then, if $f : [0, 1] \to \mathbb{R}$ is continuous,

$$y(t) = \int_0^1 f(s) G(s, t)\, ds$$

defines a twice differentiable function with $y(0) = y(1) = 0$ and

$$y''(t) = f(t)$$

for $t \in [0, 1]$.

We now investigate the equation

$$y(t) = \int_a^b G(s,t)f(s)\,ds$$

using the methods of linear analysis.

Exercise 15.2. *Suppose that $G : [a,b]^2 \to \mathbb{R}$ is continuous. Show that, if $f : [a,b] \to \mathbb{C}$ is continuous, then $Lf : [a,b] \to \mathbb{C}$ given by*

$$Lf(t) = \int_a^b G(s,t)f(s)\,ds$$

is continuous.

Exercise 15.3. *(This is a reprise of parts Exercises 10.7 and 10.8.) Show that, if we set*

$$\langle f, g\rangle = \int_a^b f(t)g(t)^*\,dt,$$

we obtain $C([a,b])$ as an inner product space. Show that $C([a,b])$ is an infinite dimensional separable inner product space but is not complete.

Exercise 15.4. *We consider $C([a,b])$ both with the uniform norm $\|.\|_\infty$ and the inner product derived norm $\|.\|_2$. We shall use the Cauchy–Schwarz inequality for integrals repeatedly.*

(i) Show that

$$L : (C([a,b]), \|.\|_2) \to (C([a,b]), \|.\|_\infty)$$

is a continuous linear map.

(ii) Show that the collection of Lf such that $f \in C([a,b])$ and $\|f\|_2 \leq 1$ is equicontinuous.

(iii) Show (Exercise 19.3 is relevant) that, if $G(s,t) = G(t,s)$ for all t, $s \in [a,b]$, then

$$\langle Lf, g\rangle = \langle f, Lg\rangle$$

for all f, $g \in (C([a,b])$.

We know that $C([a,b])$ is not a complete inner product space, so we cannot apply the spectral theorem directly. However, Exercise 11.7 tells us that there exists an inner product preserving linear map $J : C([a,b]) \to H$ of U into the Hilbert space H such that $J(C([a,b]))$ is dense in H.

Exercise 15.5. *The results of this exercise are not hard but the reader should not sleep walk through them.*

(i) Show that, if $u \in H$, $u_n \in C([a,b])$ and $\|Ju_n - u\|_2 \to 0$, then Lu_n converges uniformly in $C([a,b])$ to a continuous function g say. Show that if $v_n \in C([a,b])$ and $\|Jv_n - u\|_2 \to 0$, then

$$\|Lv_n - g\|_\infty \to 0.$$

Thus we can write $\tilde{L}u = g$.

(ii) Show that $\tilde{L}$ is a well defined function $\tilde{L} : H \to C([a,b])$.

(iii) Show that

$$\tilde{L} : H \to (C([a,b]), \|.\|_\infty)$$

is a continuous linear map.

(iv) Show that the collection of $\tilde{L}f$ such that $f \in H$ and $\|f\|_2 \leq 1$ is equicontinuous.

Exercise 15.6. *We now define $\breve{L} = J\tilde{L}$.*

(i) Show that

$$\breve{L} : H \to H.$$

is a continuous linear map.

(ii) Show that $\breve{L}$ is compact.

(iii) From now on we suppose $G(s,t) = G(t,s)$ for all s, $t \in [a,b]$. Show that $\breve{L}$ is self-adjoint.

(iv) Deduce that we can find an orthonormal basis w_n and a sequence λ_n of real numbers with $\lambda_n \to 0$ such that

$$\breve{L}u = \sum_{j=1}^{\infty} \lambda_j \langle u, w_j \rangle w_j$$

for all $u \in H$.

The result of the previous exercise tells us something about $\breve{L}$, which is an operator on H, and we are interested in L, which is an operator on $C([a,b])$. However, this is soon remedied.

Exercise 15.7. *(i) If $\lambda_j \neq 0$, use the fact that $\lambda_j w_j = \breve{L}w_j$ to show that $w_j = Je_j$ for some $e_j \in C([a,b])$.*

(ii) Conclude that, if $G : [a,b]^2 \to \mathbb{R}$ is continuous and $G(s,t) = G(t,s)$, then either *we can find an orthonormal sequence v_j in $C([a,b])$ and a sequence ζ_j of non-zero real numbers with $\zeta_j \to 0$ having the property*

$$\left\| \int_a^b f(s)G(s,.)\,ds - \sum_{j=1}^{N} \zeta_j \langle f, v_j \rangle v_j \right\|_2 \to 0$$

as $N \to \infty$*,* or *we can find a finite orthonormal collection* v_j *in* $C([a,b])$ *and* ζ_j *non-zero real numbers with*

$$\int_a^b f(s)G(s,t)\,ds = \sum \zeta_j \langle f, v_j\rangle v_j(t).$$

(iii) Show that $\langle f, v_j\rangle = 0$ *for all* j *implies* $f = 0$ *whenever* $f \in C([a,b])$ *if and only if*

$$\int_a^b f(t)G(s,t)\,dt = 0$$

for all s *implies* $f = 0$ *whenever* $f \in C([a,b])$*.*

Exercise 15.8. *Briefly identify the 'eigenfunctions'* v_j *associated with non-zero eigenvalues in the case of Exercise 15.1.*

16 Where next?

In this section which will neither be examined nor lectured, I look at the different ways in which the ideas of this course can be developed.

Measure theory Measure theory interacts with linear analysis in many ways.

(1) We have seen that $C([a,b])$ with the usual inner product can be identified with a dense subset of of a complete inner product space. It is a surprising fact that we can realise this complete inner product space as a space of functions $L^2([a,b])$ on $[a,b]$ by using Lebesgue integration.

In much the same way, it can be shown that, if we write

$$\|f\|_p = \left(\int_a^b |f(x)|^p\,dx\right)^{1/p},$$

then $\|.\|_p$ is a norm on $C([a,b])$ (see Exercise 18.1) and the completion gives rise in a natural manner to a space of $L^p([a,b])$ on $[a,b]$. These spaces are natural subjects for linear analysis.

(2) Although we have studied the space $C([a,b])$ with the the uniform norm, we did not try to identify its dual (for some members of the dual see Exercise 18.7). It is not hard to show that that the dual space can be identified with the space of *Borel measures.*

(3) The theory of compact self-adjoint operators that we have developed on Hilbert space corresponds to the theory of Fourier sums

$$f(t) \sim \sum_{j=-\infty}^{\infty} \hat{f}(j)\exp(ijt).$$

If we are to get something like the theory of Fourier transforms with the putative inversion formula

$$f(t) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x) \exp(ixt)\, dx$$

we need to extend our notions of integration. (In fact, we only need to extend the ideas of Riemann integration, but, nonetheless, the extension is quite subtle.)

Examples Spaces like l^p and $C([a,b]), \|.\|_\infty)$ are good examples of Banach spaces to start with because they have a great deal of structure. For the same reason, they are inadequate if we wish to understand what a general Banach space might look like.

As a result of seventy years of hard work (including that of our own Professor Gowers) we know that the good behaviour of l^p, $C([a,b]), \|.\|_\infty)$ and similar spaces is not typical of Banach spaces in general. The study of Banach spaces (like the study of most general mathematical objects) requires a plentiful stock of examples.

The axiom of choice The reader will be aware of a principle called the *axiom of choice*[12]. This asserts that, given a non-empty collection $\mathcal{A}$ of non-empty sets, we can find a function

$$f : \mathcal{A} \to \bigcup_{A \in \mathcal{A}} A$$

such that $f(A) \in A$. (That is to say, f *chooses* an element $f(A)$ from each $A \in \mathcal{A}$.) Mathematical logicians have shown that, if the ordinary axioms for set theory are consistent, then they remain consistent if we add the axiom of choice but that the axiom of choice is not implied by the ordinary axioms.

It turns out that the general study of Banach spaces takes a more elegant form if we assume the axiom of choice and, for this reason, it is customary to assume it. Here are some consequences of this assumption.

Theorem 16.1. *Assuming the axiom of choice, every vector space V over $\mathbb{F}$ has an algebraic basis (that is to say a subset E such that any $v \in V$ may be written uniquely as a finite sum*

$$v = \sum_{g \in G} \lambda_g g$$

with G a finite subset of E and $\lambda_g \in \mathbb{F}$).

[12] For historical reasons this axiom has acquired an air of glamour and mystery which it it hardly deserves.

Using this theorem, it is easy to prove the following result which reinforces the lesson of Exercise 4.2.

Theorem 16.2. *Assuming the axiom of choice, if U is any infinite dimensional normed vector space over $\mathbb{F}$, then there exists a linear map $\alpha : U \to \mathbb{F}$ which is not continuous.*

We can also prove the following supplement to Exercise 2.4.

Theorem 16.3. *Assuming the axiom of choice, we can find an infinite dimensional vector space U and two complete norms $\|.\|_A$ $\|.\|_B$ on U such that*

$$\sup_{u \neq 0} \frac{\|u\|_A}{\|u\|_B} = \sup_{u \neq 0} \frac{\|u\|_B}{\|u\|_A} = \infty.$$

(For example we can set up an algebraic isomorphism between l^2 and l^∞.)

The axiom of choice also enables us to prove a beautiful result of Hahn–Banach. We shall not discuss this but here are some of its consequence. The first result sheds light on the paragraph following Theorem 4.8.

Lemma 16.4. *We work in l^∞ and define $\mathbf{e}_n \in l^\infty$ by*

$$e_{nj} = \begin{cases} 1 & \text{if } j = n, \\ 0 & \text{otherwise.} \end{cases}$$

Assuming the axiom of choice, there there exists a non-zero continuous linear functional $T : l^\infty \to \mathbb{C}$ such that

$$T\mathbf{e}_n = 0$$

for all n.

The second consequence was already stated in the discussion of Definition 5.2.

Theorem 16.5. *Assuming the axiom of choice, every Banach space has a sufficiently rich dual.*

The strengths and weaknesses of linear analysis using the axiom of choice are well illustrated by Lemma 16.4. On the one hand, it asserts the existence of an object T without giving any clue as to what it looks like. On the other hand, if we did not know the result of Lemma 16.4, we could waste an awful lot of time trying to show that no such object exists.

17 Books

There are many excellent introductions to linear analysis. The book of Bollobás [1] has the advantage of being based on this course and a subsequent Part III course. I think that [3] and [2] are nice and reasonably simple.

If you wish to learn more about Hilbert space then [5] is an excellent introduction and, if you simply want to learn more analysis in a non-exam driven way, then Rudin's *Real and Complex Analysis* is a masterpiece.

References

[1] B. Bollobás, *Linear Analysis*, CUP, 1991.

[2] C. Gofman and G. Pedrick, *A First Course in Functional Analysis*, Prentice Hall, 1965. (This is now reissued by AMS, Chelsea)

[3] J. D. Pryce *Basic Methods of Linear Functional Analysis*, Hutchinson, 1973. (Out of print but should be in college libraries.)

[4] W.Rudin *Real and Complex Analysis*, McGraw Hill, 2nd Ed, 1974.

[5] N. Young *An Introduction to Hilbert Space*, CUP, 1988.

18 First example sheet

Students who are unsure of their ground should check that they can do the exercises in the main text. Strong students should at least glance at the supplementary example sheet. The order of the exercises roughly follows the order of the lectures.

Exercise 18.1. *In this exercise,* $\infty > p > 1$ *and* $p^{-1} + q^{-1} = 1$. *We work with the space* $C([a, b])$ *of continuous functions on* $[a, b]$.

(i) Prove Hölder's integral inequality

$$\int_a^b |f(t)g(t)|\,dt \leq \left(\int_a^b |f(t)|^p\,dt\right)^{1/p} \left(\int_a^b |g(t)|^q\,dt\right)^{1/q}$$

for all $f,\ g \in C([a, b])$.

(ii) State and prove an appropriate reverse form of Hölder's integral inequality.

(iii) Show that

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p}$$

defines a norm on $C([a,b])$.

(iv) Show that $(C([a,b]), \|.\|_p)$ *is not complete. (We shall consider the particular case* $p = 2$ *in Exercise 10.8.)*

(v) By applying Hölder's integral inequality with $g = 1$, $p = v/u$, *or otherwise, show that*

$$\|F\|_u \leq (b-a)^{(u^{-1}-v^{-1})} \|F\|_v$$

when $\infty > v > u > 1$.

(vi) Show that, if $\infty > v > u > 1$, *then, given any* $K > 0$, *we can find an* $f \in C([a,b])$ *such that*

$$\|f\|_v > K\|f\|_u.$$

[Note that the inequalities in (v) and (vi) run in the **opposite** *way to the* l^p *case.]*

(vii) [Optional extra] Show that, if $\infty > v > u > 1$, *then given* $K > 0$, *we can find continuous functions* f *and* g *which are zero outside some interval such that*

$$\left(\int_{-\infty}^{\infty} |f(x)|^u \, dx \right)^{1/u} > K \left(\int_{-\infty}^{\infty} |f(x)|^v \, dx \right)^{1/v},$$
$$\left(\int_{-\infty}^{\infty} |g(x)|^v \, dx \right)^{1/v} > K \left(\int_{-\infty}^{\infty} |g(x)|^u \, dx \right)^{1/u}.$$

Exercise 18.2. *Suppose* $1 > p > 0$.

(a) Find (x_1, x_2), $(y_1, y_2) \in \mathbb{R}^2$ *such that*

$$\big((x_1+y_1)^p + (x_2+y_2)^p) \big)^{1/p} > (x_1^p + x_2^p)^{1/p} + (y_1^p + y_2^p)^{1/p}.$$

(b) Show, by considering the behaviour of $1 + t^p - (1+t)^p$, *or otherwise, that, if* a, $b \geq 0$, *then*

$$a^p + b^p \geq (a+b)^p.$$

(c) Show that, if we write l^p *for the space of complex sequences* $\mathbf{a}$ *with*

$$\sum_{j=1}^{\infty} |a_j|^p < \infty,$$

then l^p *can be made into a vector space in the standard way.*

Show that, if we set

$$d(\mathbf{a},\mathbf{b}) = \sum_{j=1}^{\infty} |a_j - b_j|^p,$$

then d is a complete metric on l^p.

Exercise 18.3. *Show that we can find a constant A_n such that*

$$\sup_{t\in[0,1]} |p'(t)| \leq A_n \sup_{t\in[0,1]} |p(t)|$$

for every real polynomial of degree n or less.

Exercise 18.4. *Let E and F be normed spaces. Let A be a dense subset of E, and let $T_n : E \to F$ be a continuous linear map for each $n \geq 1$. Show that if*

(a) there exists a K with $\|T_n\| \leq K$ for all n, and

(b) $T_n(a) \to 0$ for all $a \in A$,

then $T_n(e) \to 0$ for all $e \in E$.

Is the result true if condition (a) is dropped? Give a proof or a counterexample.

If (a) and (b) hold, does it follow that $\|T_n\| \to 0$ as $n \to \infty$? Give a proof or a counterexample.

Exercise 18.5. *(A useful fact.) Let $(V, \|.\|)$ be a normed space. Show that it is a Banach space if and only if $\sum_{j=1}^{\infty} x_j$ converges whenever $\sum_{j=1}^{\infty} \|x_j\|$ converges.*

In the special case when $V = \mathbb{C}$ and $\|z\| = |z|$ deduce that absolute convergence implies convergence.

Exercise 18.6. *(i) Consider $C([0,1])$ with the uniform norm. Show that*

$$E = \{f \in C([0,1]) \,:\, f(0) = 0\}$$

is a closed subspace of $C([0,1])$ and explain why this means that E is a Banach space under the uniform norm.

Show that

$$F = \left\{ f \in E \,:\, \int_0^1 f(t)\,dt = 0 \right\}$$

is a closed subspace of E. Show that there does not exist a $g \in E$ such that $\|g\|_\infty = 1$ and

$$\|g - f\|_\infty \geq 1$$

for all $f \in F$.

Thus Theorem 3.8 cannot be improved in general.

(ii) Show, however, that, if F is a subspace of a finite dimensional normed space space $(E, \|.\|)$ and $F \neq E$, then we can find an $\mathbf{e} \in E$ with $\|\mathbf{e}\| = 1$ such that

$$\|\mathbf{e} - \mathbf{f}\| \geq 1.$$

for all $\mathbf{f} \in F$.

Exercise 18.7. *In this question we work with real valued continuous functions although similar results hold for the complex valued case.*

(i) Consider the space $(C([a,b]), \|.\|_\infty)$. Show that, if $s \in [a,b]$ and

$$\delta_s(g) = g(s),$$

then $\delta_s \in C([a,b])'$. What is $\|\delta_s\|$ and why? Can you find a $g \in C([a,b])$ with $\|g\|_\infty = 1$ and $\delta_s(g) = \|\delta_s\|$? Give reasons. Show that $C([a,b])$ has a sufficiently rich dual in the sense of Definition 5.2.

(ii) Consider the space $(C([a,b]), \|.\|_\infty)$. If $F \in C([a,b])$ set

$$T_F(g) = \int_a^b F(t)g(t)\,dt.$$

Show that $T_F \in C([a,b])'$. What is $\|T_F\|$ and why? Can you always find a $g \in C([a,b])$ with $\|g\|_\infty = 1$ and $T_F(g) = \|T_F\|$? Give reasons.

(iii) Consider the space $(C([a,b]), \|.\|_1)$ where, as usual,

$$\|g\|_1 = \int_a^b |g(t)|\,dt.$$

If δ_s and T_F are defined as before, show that δ_s is not continuous, but T_F is.

(iv) [Optional extra] Continuing with the ideas of (iii), find $\|T_F\|$ and prove your answer.

Exercise 18.8. *(i) Show that if $(U, \|.\|_U)$ is a normed space and $(V, \|.\|_V)$ is a Banach space, then $(\mathcal{L}(U,V), \|.\|)$ is a Banach space.*

(ii) Consider c_{00} (the space of sequences with all but finitely many terms zero) with the norm

$$\|\mathbf{a}\|_* = \sum_{j=1}^{\infty} |a_j|$$

and the space l^1 with its usual norm. Let $\mathcal{L}(l^1, C_{00})$ be defined as in Theorem 4.6.

If we set

$$T_n(\mathbf{a}) = (a_1, 2^{-1}a_2, 3^{-1}a_3, \dots, n^{-1}a_n, 0, 0, \dots),$$

show that $T_n \in \mathcal{L}(l^1, c_{00})$. *Show that the* T_n *form a Cauchy sequence in* $\mathcal{L}(l^1, c_{00})$ *with no limit point.*

Thus Theorem 4.6 may fail if $(V, \|.\|)$ *is not complete.*

Exercise 18.9. *If* $T : U \to V$ *is an isomorphism between the Banach spaces* U *and* V *(that is to say, a linear bijection such that* T *and* T^{-1} *are continuous), show that the map* $T' : V' \to U'$ *between the dual spaces given by*

$$T'(v')u = v'(Tu)$$

for all $v' \in V'$ *and* $u \in U$ *is a well defined isomorphism between* V' *and* U'. *(Observe, that, on general grounds, the verification must consist of routine and rather easy steps.)*

Deduce that l^1 *cannot be isomorphic to* l^p *for any* $p > 1$.

Exercise 18.10. *Suppose that* X, Y *and* Z *are Banach spaces. Suppose that* $F : X \times Y \to Z$ *is linear and continuous in each variable separately, that is to say that, if* y *is fixed,*

$$F(., y)) : X \to Z$$

is a continuous linear map and, if x *is fixed,*

$$F(x, .) : Y \to Z$$

is a continuous linear map. Show, by using the principle of uniform boundedness, that there exists an M *such that*

$$\|F(x, y)\|_Z \leq M\|x\|_X\|y\|_Y$$

for all $x \in X$, $y \in Y$. *Deduce that* F *is continuous.*

Exercise 18.11. *Suppose that* U *is a vector space with two complete norms* $\|.\|_A$ *and* $\|.\|_B$. *By applying the open mapping theorem to an appropriate linear map, show that if there exists a* K *such that*

$$K\|u\|_A \geq \|u\|_B$$

for all $u \in U$, *then there exists a* K' *such that*

$$K'\|u\|_B \geq \|u\|_A$$

for all $u \in U$. *Thus comparable complete norms are equivalent.*
[We could also use the inverse mapping theorem but this comes to much the same thing.]

Exercise 18.12. *(i) (Dini's theorem) Let (X, d) be a compact metric space. Suppose $f_n : X \to \mathbb{R}$ is a sequence of continuous functions such that, for each fixed $x \in X$, $f_n(x)$ is a decreasing sequence with $f_n(x) \to 0$ as $n \to \infty$. By considering*

$$B_n = \{x : f_n(x) < \epsilon\}$$

for any fixed $\epsilon > 0$ show that $f_n \to 0$ uniformly on X.

(ii) Show, by means of an example, that the condition (X, d) compact cannot be dropped. Show, by means of an example, that the condition f_n decreasing cannot be dropped. Show, by means of an example, that the condition f_n continuous cannot be dropped.

(iii) Set $p_0 = 0$ and $p_{n+1}(x) = \frac{1}{2}x^2 + p_n(x) - \frac{1}{2}p_n(x)^2$. Explain why p_n is a polynomial. Show that

$$p_n(x) \leq p_{n+1}(x) \leq |x|$$

and all $n \geq 0$ for all $x \in [0, 1]$. Hence deduce that $p_n(x) \to |x|$ as $n \to \infty$ for all $x \in [0, 1]$. Now use Dini's theorem to show that the convergence is uniform.

Explain how to use this result as a replacement for Lemma 8.4 in the proof of the Stone–Weierstrass theorem.

19 Second example sheet

Students who are unsure of their ground should check that they can do the exercises in the main text. Strong students should at least glance at the supplementary example sheet. The order of the exercises roughly follows the order of the lectures.

Exercise 19.1. *(i) Here is a typical use of the Stone–Weierstrass theorem. If $f \in C[0, 1]$, we say that f has nth moment*

$$E_n(f) = \int_0^1 f(t)t^n \, dt.$$

Show that, if all the moments of f vanish, then

$$\int_0^1 f(t)P(t) \, dt = 0$$

for all polynomials. Use the Stone–Weierstrass theorem to deduce that

$$\int_0^1 f(t)g(t) \, dt = 0$$

for all $g \in C[0,1]$. *Deduce that* $f = 0$.

(ii) (Optional) Let $\omega = \exp(i\pi/4)$. *Show that*

$$\int_0^\infty y^n e^{-\omega y}\, dy = n!\omega^{-n-1}$$

and deduce that

$$\int_0^\infty y^{4n+3} \exp(-2^{-1/2}y) \sin(2^{-1/2}y)\, dy = 0.$$

By making the substitution $x = y^4/4$, *show that*

$$\int_0^\infty x^n \exp(-x^{1/4}) \sin(x^{1/4})\, dy = 0$$

for all n *although* $x \mapsto \exp(-x^{1/4}) \sin(x^{1/4})$ *is a well behaved non-zero continuous function. Why does the argument of part (i) fail?*
[Both parts have obvious relevance to the question of what we can say about a random variable X *from knowledge of its moments.]*

Exercise 19.2. *(The Riemann–Lebesgue lemma) (i) The Riemann–Lebesgue lemma tells us that, if* $f \in C(\mathbb{T})$, *then* $\hat{f}(n) \to 0$ *as* $|n| \to \infty$. *There are many ways of proving this but you are asked to prove it by finding a dense subalgebra of* $C(\mathbb{T})$ *for which the result is true 'for obvious reasons' and then using a density argument to extend the result to all of* $C(\mathbb{T})$.

(ii) (Optional) Suppose that $\phi(n) > 0$ *and* $\phi(n) \to 0$. *Show that we can find* $0 < n_1 < n_2 < \dots$ *such that* $\sum_{j=1}^\infty \phi(n_j)$ *converges. By considering* $\sum_{j=1}^\infty \phi(n_j) \cos n_j x$, *or otherwise, show that there exists an* $f \in C(\mathbb{T})$ *such that* $\phi(n)^{-1}\hat{f}(n) \nrightarrow 0$. *Thus part (i) cannot be improved.*

Exercise 19.3. *In this question you may use results about Riemann integration in one dimension but not in higher dimensions.*

(i) Suppose that $f : [0,1]^2 \to \mathbb{R}$ *is continuous. Explain why the function* $g : [0,1] \to \mathbb{R}$ *defined by*

$$g(x) = \int_0^1 f(x,y)\, dy$$

is continuous and so

$$\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx$$

exists.

(ii) Show that, if u, $v : [0,1] \to \mathbb{R}$ are continuous,

$$\int_0^1 \left(\int_0^1 u(x)v(y)\,dy \right)\,dx = \int_0^1 \left(\int_0^1 u(x)v(y)\,dx \right)\,dy.$$

(iii) Use the theorem of Stone–Weierstrass to show that the collection of functions h of the form

$$h(x,y) = \sum_{j=1}^{n} u_j(x)v_j(y)$$

with u_j, $v_j : [0,1] \to \mathbb{R}$ continuous, is uniformly dense in $C([0,1]^2)$.

(iv) Deduce that

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right)\,dx = \int_0^1 \left(\int_0^1 f(x,y)\,dx \right)\,dy$$

for all $f \in C([0,1]^2)$.

Exercise 19.4. *(i) Show that we cannot find a sequence P_n of polynomials such that $P_n(x) \to \exp(x)$ uniformly on $[0,\infty)$ as $n \to \infty$. Show that we cannot find a sequence Q_n of polynomials such that $Q_n(x) \to \exp(-x)$ uniformly on $[0,\infty)$ as $n \to \infty$.*

Why do these results not contradict the Stone–Weierstrass theorem?

(ii) Let A_0 be the subset of $C_{\mathbb{R}}([0,1])$ consisting of polynomials of the form $\sum_{j=1}^{n} a_j x^j$. Show that A_0 is not uniformly dense in $C_{\mathbb{R}}([0,1])$. Why does this not contradict the Stone–Weierstrass theorem? Identify the uniform closure of A_0 and prove your statement.

(iii) Let A_1 be the subset of $C_{\mathbb{R}}([0,1])$ consisting of polynomials of the form $\sum_{j=0}^{n} a_j x^{2j}$. Show that A_1 is uniformly dense in $C_{\mathbb{R}}([0,1])$.

(iii) Let A_2 be the subset of $C_{\mathbb{R}}([-1,1])$ consisting of polynomials of the form $\sum_{j=0}^{n} a_j x^{2j}$. Show that A_2 is not uniformly dense in $C_{\mathbb{R}}([-1,1])$. Why does this not contradict the Stone–Weierstrass theorem? Identify the uniform closure of A_2 and prove your statement.

Exercise 19.5. *(i) Show that a topological space (X,τ) is compact if and only if it satisfies the following condition:–*

If $\mathcal{F}$ is a non-empty collection of closed sets such that, whenever

$$F_1,\ F_2,\ \dots,\ F_n \in \mathcal{F},$$

we have $\bigcap_{j=1}^{n} F_j \neq \varnothing$, then $\bigcap_{F\in\mathcal{F}} F \neq \varnothing$.

(ii) By quoting the appropriate theorem or (if you need the practice, reproving the result which is used in obtaining that theorem) show that if (X, τ) is compact and Hausdorff, then, given $V \in \tau$ and $x \in V$, we can find a closed set F and an open set U with

$$x \in U \subseteq F \subseteq V.$$

(iii) (Baire's theorem for compact Hausdorff spaces.) If (X, τ) is a compact Hausdorff topological space and U_j is an open subset of X with the property that $U_j \cap V \neq \varnothing$ for all non-empty open V $[j \geq 1]$, show that $\bigcap_{j=1}^{\infty} U_j \neq \varnothing$.

Exercise 19.6. *(i) If A is a set of first category in a complete metric space (X, d) without isolated points show that $X \setminus A$ is uncountable.*

(ii) If A is a set of first category in a complete metric space (X, d) without isolated points, show, by considering the subspace

$$\mathrm{Cl}\{x : d(x, y) < \delta\},$$

or otherwise, that

$$\{x : d(x, y) \leq \delta\} \setminus A$$

is uncountable for all $y \in X$ and $\delta > 0$.
[Be careful. A subspace may have isolated points even if the original space does not.]

Exercise 19.7. *Consider the space $C_{\mathbb{R}}^{\infty}([0,1])$ of infinitely differentiable real functions on $[0,1]$ (with appropriate conventions about left and right derivatives at end points).*

(i) Show that, if we define

$$d(f, g) = \sum_{j=0}^{\infty} \min(2^{-j}, \|f^{(j)} - g^{(j)}\|_{\infty}),$$

then d is a complete metric on $C_{\mathbb{R}}^{\infty}([0,1])$.

(ii) Suppose q is a point of $[0,1]$. Show that

$$E_m(q) = \{f \in C_{\mathbb{R}}^{\infty}([0,1]) : |f^{(j)}(q)| \leq m \times j! \times j^j \text{ for all } j \geq 0\}$$

is closed in the d metric and its complement is dense.

(iii) Deduce that we can find an $F \notin \bigcup_{m=1}^{\infty} E_m(q)$. Show that

$$\limsup_{j \to \infty} \frac{|F^{(j)}(q)||y|^j}{j!} = \infty$$

for all $y \neq 0$. Deduce that the Taylor expansion of F around q diverges except at q (ie has zero radius of convergence).

(iv) If F is as in (iii), explain why F cannot have a power series expansion valid in any open interval containing q.

(v) Up to now we have kept q fixed. Extend our argument to show the existence of an infinitely differentiable function $G : [0, 1] \to \mathbb{R}$ which does have a power series expansion valid in any open interval.

Exercise 19.8. *The object of this exercise is to prove the following form of the Tietze extension theorem.*

If (X, τ) is a topological space and Y is a compact subset of X such that, given any $x, y \in Y$ with $x \neq y$, we can find a $g \in C_{\mathbb{R}}(X)$ with $g(x) \neq g(y)$ then, given any $f \in C_{\mathbb{R}}(Y)$, we can find an $F \in C_{\mathbb{R}}(X)$ such that $F(y) = f(y)$ for all $y \in Y$.

(i) Prove Tietze's extension theorem in the form given in Theorem 7.11 using Urysohn's lemma and the result just stated.

(ii) From now on, we assume that (X, τ) and Y satisfy the hypotheses of our theorem. Let us write A for the collection of $f \in C_{\mathbb{R}}(Y)$ such that we can find an $F \in C_{\mathbb{R}}(X)$ with $F(y) = f(y)$ for all $y \in Y$. Use the Stone–Weierstrass theorem to show that A is uniformly dense in $C_{\mathbb{R}}(Y)$.

(iii) Show that, if $f \in A$, we can find $\tilde{f} \in C_{\mathbb{R}}(X)$ with $\tilde{f}(y) = f(y)$ for all $y \in Y$ and $\|\tilde{f}\|_\infty = \|f\|_\infty$. Use this result to show that A is uniformly closed in $C_{\mathbb{R}}(Y)$. Deduce that $A = C_{\mathbb{R}}(Y)$ and our theorem holds.

Exercise 19.9. *We work in $\mathbb{C}$ and write*

$$\partial D = \{z \in \mathbb{C} : |z| = 1\}.$$

Suppose K is a compact subset of $\mathbb{C}$ and $\phi : K \to \partial D$ is continuous. By considering functions of the form $\Phi(z)/|\Phi(z)|$, or otherwise, show that there is an open set $\Omega \supseteq K$ and a continuous function $\tilde{\phi} : \Omega \to \partial D$.

Explain, giving an explicit K and ϕ and using ideas from complex analysis or topology (you are not asked for proofs), why we cannot always take $\Omega = \mathbb{C}$.

Exercise 19.10. *We work with continuous functions on $[0, 1]$.*

(i) Show that, if $f_n \to f$, uniformly then the set $\mathcal{F} = \{f_n : n \geq 1\}$ is equicontinuous.

(ii) Show that, if $\mathcal{F} = \{f_n : n \geq 1\}$ is equicontinuous and $f_n(q) \to f(q)$ as $n \to \infty$ at each rational point $q \in [0, 1]$, then $f_n \to f$ uniformly.

Exercise 19.11. *Recall the definition of* total boundedness *from the metric and topological spaces course. Let (X, τ) be a compact Hausdorff space. Suppose that $\mathcal{F}$ is a subset of $C(X)$.*

(i) Explain why $\mathcal{F}$ is complete under the natural metric d (that is the restriction of the metric induced by $\|.\|_\infty$) if and only if $\mathcal{F}$ is closed in $C(X)$. Explain why this means that $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is closed and totally bounded.

(ii) Show that $\mathcal{F}$ is totally bounded if and only if it is uniformly bounded and equicontinuous.

(iii) Deduce the Ascoli–Arzelà theorem (Theorem 9.3).

Exercise 19.12. *Find all continuously differentiable functions $x : \mathbb{R} \to \mathbb{R}$ with*

$$x'(t) = 3x(t)^{2/3} \text{ for all } t$$

subject to the condition $x(0) = 0$ and show (formally or informally according to taste) that you have found them all.
[Note that $x \mapsto 2x^{2/3}$ is differentiable (and so satisfies a Lipschitz condition) for $x \neq 0$.]

20 Third example sheet

Students who are unsure of their ground should check that they can do the exercises in the main text. Strong students should at least glance at the supplementary example sheet. The order of the exercises roughly follows the order of the lectures.

Exercise 20.1. *We work with the set $A(D)$ of functions analytic on*

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

(i) Let $1 > R > r$. By considering Cauchy's formula with the contour $z = Re^{i\theta}$, or otherwise, show that

$$\mathcal{F}_R(M) = \{f \in A(D) : |f(z)| \leq M \text{ for all } |z| \leq R\}$$

is equicontinuous at every point z with $|z| \leq r$.

(ii) If $f_n \in \mathcal{F}$ and $|f_n(z)| \leq M$ for all $|z| \leq R$, show that we can find a continuous function

$$F : \{z \in \mathbb{C} : |z| \leq r\} \to \mathbb{C}$$

and $n(j) \to \infty$ such that $f_{n(j)}(z) \to F(z)$ uniformly for $|z| \leq r$. Explain why F is analytic on $D_r = \{z : |z| < r\}$.

(iii) Show that the formula

$$d(f, g) = \sum_{j=1}^{\infty} 2^{-j} \min(1, \sup_{|z| \leq 1 - 2^{-j}} |f(z) - g(z)|)$$

gives a complete metric on $A(D)$. Show that the set

$$\mathcal{F}(M) = \{f \in A(D) : |f(z)| \leq M \text{ for all } |z| < 1\}$$

is compact in this metric.
[Simple developments of these ideas produce the powerful method of normal families *in complex variable theory.]*

Exercise 20.2. *Consider the space of continuous functions $C([0,1])$ with the uniform norm. Suppose that E is a closed subspace consisting of continuously differentiable functions.*

(i) By applying the closed graph theorem, or otherwise, show that the differentiation map $D : E \to C([0,1])$ (given by $Df = f'$) is continuous.

(ii) Deduce, using the Ascoli–Arzelà theorem, or otherwise that the unit ball in E is compact.

(iii) Deduce, quoting the appropriate theorem, that E is finite dimensional.

Exercise 20.3. *Suppose K is a compact Hausdorff space and $C(K)$ is the countable union of equicontinuous sets. Show that $C(K)$ is the countable union of bounded and closed (in the uniform norm) equicontinuous sets. Deduce, using three of the 'big theorems' of the course, or otherwise, that $C(K)$ is finite dimensional and so K is finite.*

Exercise 20.4. *(i) Let U be a complete inner product space. If F is closed convex subset of U and $a \in U$, show that we can find a unique $f_0 \in F$ such that*

$$\|a - f_0\|_2 \leq \|a - f\|_2$$

for all $f \in F$.
[A set F is convex if, whenever $x, y \in F$ and $1 \geq \lambda \geq 0$, it follows that $\lambda x + (1 - \lambda)y \in F$.]

(ii) Give an example of a closed convex subset F of l^∞ and a point $\mathbf{a} \in l^\infty$ with the property that the equation

$$\|\mathbf{a} - \mathbf{g}\|_\infty = \inf_{\mathbf{f} \in F} \|\mathbf{a} - \mathbf{f}\|_\infty$$

has more than one solution with $\mathbf{g} \in F$.

Exercise 20.5. *We work in l^2. Let F be the collection of $\mathbf{x} \in l^2$ such that all but finitely many x_j are zero and let*

$$\mathbf{a} = (1, 1/2, 1/3, \dots).$$

Show that F is a subspace of l^2, but, given $\mathbf{f} \in F$, we can always find a $\mathbf{g} \in F$ such that

$$\|\mathbf{a}-\mathbf{f}\|_2 > \|\mathbf{a}-\mathbf{g}\|_2.$$

Why does this not contradict Theorem 12.1?

Exercise 20.6. *(i) Use Theorem 12.1 and Lemma 12.2 to show that, if U is complete separable inner product space and F is closed subspace of U, then F, considered as a space in its own right (inheriting the inner product from U), is itself a complete separable inner product space.*

(ii) Suppose that U is an inner product space and F is closed subspace of U. Suppose further that F considered as a space in its own right has an orthonormal basis e_1, e_2, ... Use the arguments which we used to establish Bessel's inequality to show that, if $a \in U$ and $\sum_{j=1}^{\infty} |x_j|^2 < \infty$, then

$$\left\| \sum_{j=1}^{\infty} x_j e_j - a \right\|_2 \geq \left\| \sum_{j=1}^{\infty} \langle a, e_j \rangle e_j - a \right\|_2$$

with equality only if $x_j = \langle a, e_j \rangle$. Deduce the conclusions of Theorem 12.1 and Lemma 12.2 in this case.

Exercise 20.7. *(i) Consider the inner product space c_{00} of sequences with only finitely many non-zero terms and inner product*

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^{\infty} a_j b_j^*.$$

Show that the formula

$$T\mathbf{u} = \sum_{j=1}^{\infty} j^{-1} u_j$$

defines a continuous linear map $T : c_{00} \to \mathbb{C}$ but there does not exist a $\mathbf{w} \in c_{00}$ such that

$$T\mathbf{u} = \langle \mathbf{u}, \mathbf{w} \rangle.$$

Why does this not contradict the Riesz representation theorem (Theorem 12.3)?

(ii) Find a closed subspace F of c_{00} and an $\mathbf{a} \in c_{00}$ such that, given $\mathbf{f} \in F$, we can always find a $\mathbf{g} \in F$ such that

$$\|\mathbf{a}-\mathbf{f}\|_2 > \|\mathbf{a}-\mathbf{g}\|_2.$$

Why does this not contradict Theorem 12.1?

Exercise 20.8. *Suppose that U is an inner product space and e_1, e_2, ... are orthonormal. Show that*

$$\langle f, e_n \rangle \to 0$$

as $n \to \infty$.

Use this result to give another proof of the Riemann–Lebesgue lemma (see Exercise 19.2).

Exercise 20.9. *Suppose that e_1, e_2, ... is an orthonormal basis for an inner product space E. Show reasonably carefully that*

$$\langle x, y \rangle = \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle y, e_j \rangle^*.$$

Exercise 20.10. *Consider a Hilbert space H with a closed subspace E and $T : E \to \mathbb{C}$ a continuous linear functional. Show that there exists a continuous linear functional $\tilde{T} : H \to \mathbb{C}$ such that $\tilde{T}|_E = T$ and $\|\tilde{T}\| = \|T\|$.*
[You can prove this using bases but well brought up people do not use bases unless they have to. (However, if they have to, they instantly forget their upbringing.)]

Exercise 20.11. *Let E be an inner product space. We say that a non-zero continuous linear map $P : E \to E$ is a* projection *if $P^2 = P$. We say that P is an* orthogonal projection *if $P(U) \perp P^{-1}(0)$.*

(i) Give a example of an orthogonal projection $Q : l^2 \to l^2$ such that both $Q(U)$ and $Q^{-1}(0)$ are infinite dimensional.

(ii) Give a example of a projection $R : \mathbb{C}^2 \to \mathbb{C}^2$ which is not an orthogonal projection.

(iii) If E is an inner product space and $P : E \to E$ is a projection, show that P is an orthogonal projection if and only if $\|P\| = 1$.

Exercise 20.12. *Suppose that H is a Hilbert space. If C is a subset (NB no algebraic structure is implied) of H we write*

$$C^\perp = \{g \in H \,:\, \langle g, f \rangle = 0 \text{ for all } f \in C\}.$$

(i) Show that $C^\perp$ is a closed subspace of H.
(ii) Show that $C^{\perp\perp} = C$ if and only if C is a closed subspace of H.
(iii) Show that $C^{\perp\perp} = \operatorname{Cl}\operatorname{span} C$.

21 Fourth example sheet

Students who are unsure of their ground should check that they can do the exercises in the main text. Strong students should at least glance at the supplementary example sheet. The order of the exercises roughly follows the order of the lectures.

Exercise 21.1. *Let e_1, e_2, ... be an orthonormal basis for the Hilbert space H. Write*

$$\begin{aligned} u_n &= e_{2n-1} \\ v_n &= e_{2n-1} + n^{-1}e_{2n} \end{aligned}$$

and let U be the closed subspace generated by the u_n (ie the smallest closed subspace containing the u_n) and V be the closed subspace generated by the v_n.

(i) Show that $U \cap V = \{0\}$.

(ii) Show that every $x \in U + V$ has a unique expression as $x = Px + Qx$ with $Px \in U$ and $Qx = V$. Show that $P : U + V \to U$ is linear and satisfies $P^2 = P$ (so P is a projection) but that P is not continuous.

(iii) Show that $U + V$ is dense in H but $U + V \neq H$.

Exercise 21.2. *Let H be Hilbert space. Suppose that $T : H \to H$ is linear and self-adjoint (that is to say $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all x, $y \in H$). Use the the principle of uniform boundedness to show that T is continuous.*

Exercise 21.3. *Let U be a normed space and $T : U \to U$ a continuous linear map. Explain why*

$$\rho(T) = \liminf_{n\to\infty} \|T^n\|^{1/n}$$

is well defined.

By making the observation that

$$\|T^{Nk+r}\| \leq \|T^N\|^k \|T\|^r,$$

or otherwise, show that, in fact

$$\|T^n\|^{1/n} \to \rho(T)$$

as $n \to \infty$. (We call $\rho(T)$ the spectral radius of T.)

Now suppose that U is a Banach space. Show that $\lambda I - T$ is invertible (that is to say, has continuous inverse) for all $|\lambda| > \rho(T)$.

By considering maps of the form

$$(x_1, x_2, x_3, \dots) \mapsto (0, c_1x_1, c_2x_2, \dots)$$

with appropriate c_j, show that we can find $T : l^2 \to l^2$ such that T^n is injective (so, in particular, $T^n \neq 0$) for all $n \geq 1$ but $\rho(T) = 0$.

Exercise 21.4. *Let U and V be Banach spaces. Suppose $T : U \to V$ is a continuous injective linear map such that $T(U)$ is dense in V. Show that the following three statements are equivalent.*

(i) There exists a $c > 0$ with the property that

$$\|Tu\|_V \geq c\|u\|_U$$

for all $u \in U$.

(ii) T is surjective.

(iii) T is invertible (that is to say, has continuous inverse).

Exercise 21.5. *(i) Consider the the shift map $S : l^2 \to l^2$ given by*

$$S(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

Show that S is a well defined continuous linear map and find its adjoint.

(ii) Let U be a finite dimensional space and $T : U \to U$ a linear map. Explain why $\lambda I - T$ fails to be invertible if and only if there exists a $u \neq 0$ such that $Tu = \lambda u$.

(iii) Give an example of a continuous linear map $R_1 : l^2 \to l^2$ which is surjective but not bijective. Give an example of a continuous linear map $R_2 : l^2 \to l^2$ which is injective but not bijective.

(iv) By using Exercise 21.4, or otherwise, show that, if U is a Banach space and $T : U \to U$ is a continuous linear map, then $\lambda \in \sigma(T)$ if and only if at least one of the following conditions hold.

(A) λ is an eigenvalue of T, that is to say, there exists a $u \neq 0$ such that $Tu = \lambda u$.

(B) λ is not an eigenvalue of T but λ is an approximate eigenvalue *of T, that is to say, there exist $u_j \in U$ with $\|u_j\| = 1$ such that $\|Tu_j - \lambda u_j\| \to 0$.*

(C) $(\lambda I - T)(U)$ is not dense in V.

(v) Give an example of a continuous linear map $T : l^2 \to l^2$ such that $\sigma(T) = \{0\}$ but 0 is not an eigenvalue of T.

[The moral of this question is that the spectrum is much more a complicated than at first appears. We find an interesting spectrum in Exercise 22.16.]

Exercise 21.6. *(i) Let U be a Banach space and $T : U \to U$ a continuous linear map. Suppose that λ is in the* frontier *of $\sigma(T)$ (that is to say λ lies in $\sigma(T)$ and in the closure of the complement of $\sigma(T)$). By examining possibility (C) in Exercise 21.5, show that λ is an approximate eigenvalue (this includes* *the possibility that λ is an eigenvalue). In other words, the frontier of the spectrum is composed of approximate eigenvalues.*

(ii) Now suppose $T : H \to H$ is a continuous Hermitian (that is to say, self-adjoint) linear map. Show that the approximate eigenvalues of T are real. Deduce from (i) that the spectrum of T consists of real numbers.

Exercise 21.7. *(This quite long, though instructive. Those who want a shorter, thought still non-trivial, question should ignore parts (i) and (iii), and do that part of (ii) which says that the limit of finite rank operators is compact.)*

Consider Hilbert space H. Let us write $B(H)$ for the space of continuous linear maps $T : H \to H$.

(i) Show that the following statements about a bounded sequence x_n are equivalent.

(A) Every subsequence of the sequence has a convergent subsequence.

(B) Given any $\epsilon > 0$ we can find a finite dimensional subspace E such that, if P is the orthogonal projection onto E, we have $\|(I - P)x_n\|_2 < \epsilon$.

(ii) We say that an $S \in B(H)$ is of finite rank if the image space $S(H)$ is finite dimensional. Show that $T \in B(H)$ is compact if and only if we can find finite rank S_n such that $\|S_n - T\| \to 0$ as $n \to \infty$.
[It may be helpful to recall that H is separable]

(iii) Show that the collection of compact operators is a closed nowhere dense (ie having dense complement) subset of $B(H)$.

Exercise 21.8. *(i) We work on the Hilbert space l^2. Show that*

$$S(a_1, a_2, a_3, \dots) = (a_2, a_3, a_4 \dots)$$

is a continuous linear map from l^2 to l^2 and find its adjoint map S^*. Find $\|S\|$ and $\|S^*\|$.

(ii) We use the notation of Exercise 21.7. Show that the collection of self-adjoint operators is a closed nowhere dense subset of $B(H)$.

Exercise 21.9. *Check that the following statement is equivalent to the spectral theorem for compact self-adjoint operators (Theorem 14.5). Let H be a Hilbert space. If $T : H \to H$ is a continuous linear compact self-adjoint map we can find a finite or infinite sequence $P_1, P_2, \dots$ of orthogonal projections (that is to say continuous linear maps $P_j : H \to H$ with $P_j^{-1}(\{0\}) \perp P_j(H)$) which are mutually orthogonal (that is to say $P_j(H) \perp P_k(H)$ for $j \neq k$) and have finite dimensional non-trivial images (that is to say $1 \leq \dim P_j(H) < \infty$) together with distinct non-zero real λ_j, with $\lambda_n \to 0$ as $n \to \infty$ if the sequence is infinite, such that*

$$T = \sum_j \lambda_j P_j.$$

[Although this is just a simple rewrite of Theorem 14.5 it provides a better jumping off point for generalisation.]

Exercise 21.10. *Suppose $T : H \to H$ is a compact self-adjoint operator on a Hilbert space H. Let λ be a non-zero real number and let $\mathbf{y} \in H$. Consider the two equations*

$$T\mathbf{x} = \lambda\mathbf{x}, \tag{1}$$

$$T\mathbf{x} = \lambda\mathbf{x} + \mathbf{y}. \tag{2}$$

Prove the Fredholm alternative *which states that* either *(1) has no non-zero solutions and (2) has a unique solution* or *the space of solutions E_λ of (1) contains non-zero vectors and (2) has solutions if and only if $\mathbf{y} \perp E_\lambda$. In the second case, if $\mathbf{z}$ is a solution of (2), the set of solutions is*

$$\{\mathbf{z} + \mathbf{w} \ : \ \mathbf{w} \in E_\lambda\}.$$

[Hint: If this was a 1B mathematical methods question you would consider it rather easy.]

Exercise 21.11. *Work through Section 15.*

Exercise 21.12. *In this exercise we work in a Hilbert space H.*

(i) Suppose that T and S are commuting compact self-adjoint continuous maps from H to H. If we write

$$E_\lambda = \{x \in H \ : Tx = \lambda x\}$$

show that $S(E_\lambda) \subseteq E_\lambda$. By repeated use of Theorem 14.5, or otherwise, show that we can find an orthonormal basis f_1, f_2, f_3, ...such that each f_j is an eigenvector of both T and S.

(ii) A continuous linear map $R : H \to H$ is called normal *if $RR^* = R^*R$. By considering $T = (R + R^*)/2$ and $S = (R - R^*)/(2i)$, or otherwise show that, if R a compact normal continuous linear map from H to H, we can find an orthonormal basis f_j of H and $\lambda_j \in \mathbb{C}$ with $\lambda_j \to 0$ such that*

$$Ru = \sum_{j=1}^{\infty} \lambda_j \langle u, f_j \rangle f_j$$

for all $u \in U$.

(iii) Show conversely that if f_j is any orthonormal basis of of H and λ_j any sequence in $\mathbb{C}$ with $\lambda_j \to 0$ then the formula

$$Ru = \sum_{j=1}^{\infty} \lambda_j \langle u, f_j \rangle f_j$$

for all $u \in U$ defines a continuous linear map $R : H \to H$ which is compact and normal.

22 Supplementary exercises

These exercises are not intended to be harder but are less closely linked to the immediate needs of the course. They do, however, provide background and anyone intending to do more analysis should at least glance at them.

Exercise 22.1. *(Revision of 1A) We say that a function $f : [a, b] \to \mathbb{R}$ is strictly concave if, whenever $a \leq t < s \leq b$ and $0 < \lambda < 1$, it follows that*

$$\lambda f(t) + (1 - \lambda)f(s) < f\big(\lambda t + (1 - \lambda)s\big).$$

(i) Show that, if $g : [a, b] \to \mathbb{R}$ is twice differentiable with $g''(t) < 0$ for all $t \in (a, b)$, then g is strictly concave.

(ii) Give an example of a strictly concave function $g : [-1, 1] \to \mathbb{R}$ which is not differentiable at 0.

(iii) Suppose that $f : [a, b] \to \mathbb{R}$ is strictly concave. By using induction, or otherwise, prove Jensen's inequality which states that if $x_1, x_2, \dots, x_n$ are distinct points of $[a, b]$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ are strictly positive real numbers with $\sum_{j=1}^n \lambda_j = 1$, then

$$f\left(\sum_{j=1}^n \lambda_j x_j\right) > \sum_{j=1}^n \lambda_j f(x_j).$$

Deduce that if $x_1, x_2, \dots, x_n$ are points of $[a, b]$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ are strictly positive real numbers with $\sum_{j=1}^n \lambda_j = 1$, then

$$f\left(\sum_{j=1}^n \lambda_j x_j\right) = \sum_{j=1}^n \lambda_j f(x_j)$$

if and only if $x_1 = x_2 = \cdots = x_n$.

(iv) Use Jensen's inequality to show that, if $a_j > 0$, then

$$(a_1 a_2 \dots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

(This is Cauchy's arithmetic-geometric inequality.) What are the conditions for equality?

(v) Suppose that $p > 1$ and let $g(x) = (1 + x^{1/p})^p$. Show that g is a concave function.

Suppose that $a_1, a_2, \dots, a_n > 0$, $\sum_{j=1}^n a_j^p = 1$ and $b_1, b_2, \dots, b_n > 0$. By applying Jensen's inequality with $x_k = b_k^p / a_k^p$ and λ_k chosen appropriately, prove Minkowski's inequality.

$$\left(\sum_{j=1}^n (a_j + b_j)^p\right)^{1/p} \leq \left(\sum_{j=1}^n a_j^p\right)^{1/p} + \left(\sum_{j=1}^n b_j^p\right)^{1/p}$$

and obtain the conditions for equality. Why does the result flow for general values of $\sum_{j=1}^n a_j^p$?

Exercise 22.2. *Obtain Minkowski's inequality by applying Hölder's inequality to the observation*

$$\sum_{j=1}^n |x_j + y_j|^p \leq \sum_{j=1}^n |x_j||x_j + y_j|^{p-1} + \sum_{j=1}^n |y_j||x_j + y_j|^{p-1}.$$

Is this really a different proof to the one given in the lectures using the reverse Hölder inequality?

Exercise 22.3. *The results of Exercise 2.2 depend on clever inequalities*[13] *but there are other ways of arriving at the results.*

Let $\infty \geq s > r \geq 1$. Investigate maxima and minima of $\sum_{j=1}^n x_j^s$ subject to $x_j \geq 0$, $\sum_{j=1}^n x_j^r = 1$ using the calculus of variations. (Unless we take care, which you are not asked to do, the results will not be rigorous but, once we know what is happening, it is much easier to prove that it happens by some other technique.)

Exercise 22.4. *If V is a vector space over $\mathbb{F}$, we say that E is an* algebraic basis *(that is to say a basis in the sense of 1B algebra) if every $v \in V$ can be written uniquely as a finite sum*

$$v = \sum_{j=1}^n \lambda_j e_j$$

with $e_1, e_2, \ldots, e_n$ distinct elements of E and $\lambda_j \in \mathbb{F}$. The collection V^* of linear maps $\alpha : V \to \mathbb{F}$ is called the algebraic dual *(that is to say, the dual space in the sense of 1B algebra). The proofs of 1B algebra show that U^* can be given the structure of a vector space.*

Let c_{00} be the vector space of complex sequences with only a finite number of non-zero terms. Explain why c_{00} has a countable basis. Identify c_{00}^* in a natural manner with the space $\mathbb{C}^{\mathbb{N}}$ of all complex sequences. Show that c_{00}^* does not have a countable basis. (The argument is not difficult but you should not sleep walk through it.)

Although this question deals only with one space the reader should require little convincing that, if we only deal with algebraic duals, the algebraic dual of an infinite dimensional space will be very much bigger than the the space (and the dual of the dual will be even bigger).

[13] To the writer all inequalities seem clever.

Exercise 22.5. *(i) Prove the parallelogram law*

$$\|\mathbf{a}+\mathbf{b}\|_2^2 + \|\mathbf{a}-\mathbf{b}\|_2^2 = 2(\|\mathbf{a}\|_2 + \mathbf{b}\|_2^2)$$

for all $\mathbf{a}$, $\mathbf{b} \in l^2$.

(ii) Use induction to show that, for each n, *we can find* $\zeta_{jk}(n) = \pm 1$ *such that*

$$\sum_{j=1}^{2^n} \left\| \sum_{k=1}^{2^n} \zeta_{jk}(n)\mathbf{a}(k) \right\|_2^2 = 2^n \sum_{k=1}^{2^n} \|\mathbf{a}(k)\|_2^2$$

for all $\mathbf{a}(k) \in l^2$.

(iii) If $(U, \|.\|)$ *is isomorphic to* $(l^2, \|.\|_2)$ *explain why there is a constant* K *independent of* n *such that*

$$K \sum_{j=1}^{2^n} \left\| \sum_{k=1}^{2^n} \zeta_{jk}(n)\mathbf{u}(k) \right\|_U^2 \geq 2^n \sum_{j=1}^{2^n} \|\mathbf{u}(k)\|_U^2 \geq K^{-1} \sum_{j=1}^{2^n} \left\| \sum_{k=1}^{2^n} \zeta_{jk}(n)\mathbf{u}(k) \right\|_U^2$$

for all $\mathbf{u}(k) \in U$.

(iv) Show that l^2 *is not isomorphic to* l^p *when* $p \neq 2$.

Exercise 22.6. *Consider the space of* $n \times n$ *complex matrices with the operator norm. Prove the Cayley–Hamilton theorem by using the fact proved in Exercise 6.2 that the set of of matrices with* n *distinct eigenvalues is dense.*

Exercise 22.7. *Suppose that* $g_n : \mathbb{T} \to \mathbb{R}$ *is continuous and satisfies the conditions set out in Exercise 6.11 as follows.*

(i) There exists a constant K *such that*

$$\frac{1}{2\pi} \int_{\mathbb{T}} |g_n(t)| \, dt \leq K$$

for all $n \geq 1$.

(ii) If $\delta > 0$ *and* f *is a continuous function with* $f(t) = 0$ *for* $|t| < \delta$, *then*

$$\frac{1}{2\pi} \int_{\mathbb{T}} f(t) g_n(t) \, dt \to 0$$

as $n \to \infty$.

(iii) We have

$$\frac{1}{2\pi} \int_{\mathbb{T}} g_n(t) \, dt \to 1$$

as $n \to \infty$.

If $f : \mathbb{T} \to \mathbb{R}$ is continuous and $\delta > 0$ observe that

$$\frac{1}{2\pi}\int_{\mathbb{T}} g_n(t)f(t)\,dt - f(0) = \frac{1}{2\pi}\int_{|t|<\delta} g_n(t)(f(t)-f(0))\,dt$$
$$+\frac{1}{2\pi}\int_{|t|\geq\delta} g_n(t)(f(t)-f(0))\,dt + \left(\frac{1}{2\pi}\int_{\mathbb{T}} g_n(t)\,dt - 1\right)\times f(0),$$

and, by estimating the three terms separately, show that

$$\frac{1}{2\pi}\int_{\mathbb{T}} g_n(t)f(t)\,dt \to f(0)$$

as $n \to \infty$.

Show that condition (ii) is implied by

(ii)′ If $\delta > 0$ then $g_n(t) \to 0$ uniformly for $t \notin (-\delta, \delta)$.

Show also, by considering the Riemann–Lebesgue lemma or otherwise, that condition (ii) does not imply (ii)′.

Exercise 22.8. *Let $f : \mathbb{T} \to \mathbb{C}$ be continuous. We write*

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\exp(-int)\,dt.$$

(i) Show that $\sum_{n=-N}^{N} r^{|n|}\exp(int)$ converges uniformly, to $P_r(t)$ say, for $t \in \mathbb{T}$ as $N \to \infty$ for each fixed r with $0 < r < 1$. Deduce that

$$\sum_{n=-\infty}^{\infty} r^{|n|}\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)P_r(-t)\,dt = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)P_r(t)\,dt.$$

Show that

$$P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}.$$

(ii) Show that

(A) $P_r(t) \geq 0$ for all t.

(B) $\displaystyle\frac{1}{2\pi}\int_{\mathbb{T}} P_r(t)\,dt = 1$.

(C) $P_r(t) \to 0$ uniformly as $r \to 1-$ for $t \notin (-\delta, \delta)$.

(iii) Deduce, by the arguments of Exercise 22.7, or otherwise, that

$$\sum_{n=-\infty}^{\infty} r^n\hat{f}(n) \to f(0)$$

as $r \to 1-$. *By considering the Fourier coefficients of* f_a *given by* $f_a(t) = f(t-a)$, *or otherwise, show that*

$$\sum_{n=-\infty}^{\infty} r^n \hat{f}(n) \exp(int) \to f(t)$$

as $r \to 1-$.

Exercise 22.9. *Let* $f : \mathbb{T} \to \mathbb{C}$ *be continuous. We write*

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \exp(-int)\, dt.$$

(i) We write $D_n(t) = \sum_{j=-n}^{n} \exp(ijt)$. *Show that, if* $t \neq 0$,

$$D_n(t) = \frac{\sin\big((n+\frac{1}{2})t\big)}{\sin\frac{t}{2}}.$$

What is the value of $D_n(0)$*?*

(ii) Show that

$$\sum_{j=-n}^{n} \hat{f}(j) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) D_n(-t)\, dt = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) D_n(t)\, dt.$$

(iii) Show that

$$|D_n(t)| \geq \frac{A(n+\frac{1}{2})}{r+1} |\sin\big((n+\tfrac{1}{2})t\big)|$$

for all $r\pi/(n+\frac{1}{2}) \leq |t| \leq (r+1)\pi/(n+\frac{1}{2})$ *and* $1 \leq r \leq n-1$ *for some* $A > 0$ *(to be chosen to fit your convenience) independent of* n.

(iv) Deduce that

$$\frac{1}{2\pi} \int_{\mathbb{T}} |D_n(t)|\, dt \geq B \log n$$

for all $n \geq N$ *and some* $B > 0$ *(where both* N *and* B *are chosen to fit your convenience).*

(v) Use Exercise 6.11 to show that there exists a continuous function f *whose Fourier sum does not converge to* f *at* 0.

Exercise 22.10. *Let* (X, τ) *be a topological space. Write* $x \sim y$ *if* $f(x) = f(y)$ *whenever* $f \in C(X)$ *the space of real valued continuous functions on* X.

(i) Show that $\sim$ *is an equivalence relation.*

(ii) Show that, if we give $X/\sim$ the quotient topology $\tau/\sim$, then the function $J : C(X) \to C(X/\sim)$ given by $J(f)[x] = f(x)$ (where $[x]$ denotes the equivalence class of x) is a well defined linear isometry.

(iii) Show that $(X/\sim, \tau/\sim)$ is Hausdorff.

(iv) If (X, τ) is compact, show that $(X/\sim, \tau/\sim)$ is also compact.

Exercise 22.11. *In this exercise we discuss various* separation axioms *for topological spaces (X, τ).*

A topological space (X, τ) is said to be T_0 if, given $a, b \in X$ with $a \neq b$, at least one of these statements is true. (α) There exists a $U \in \tau$ such that $a \in U$ and $b \notin U$. (β) There exists a $V \in \tau$ such that $a \notin V$ and $b \in V$. A topological space (X, τ) is called T_1 if given $a, b \in X$ with $a \neq b$, at both of these statement are true.

(i) Show that (X, τ) is T_1 if and only if every singleton set $\{x\}$ is closed.

(ii) Consider $\mathbb{N}$. If the elements of σ_0 are $\varnothing$ and sets of the form $\{m \in \mathbb{N} : m \geq n\}$ show that σ_1 is a topology and $(\mathbb{N}, \sigma_1)$ is T_0 but not T_1.

(iii) Consider $\mathbb{N}$. If the elements of $U \in \sigma_1$ if and only if $A = \varnothing$ or $\mathbb{N} \setminus A$ is finite show that σ_1 is a topology and $(\mathbb{N}, \sigma_1)$ is T_1 but not Hausdorff. (In this classification, Hausdorff spaces are called T_2.)

(iv) Show that every T_1 normal space is Hausdorff but give an example of a normal space which is not Hausdorff. (Recall that a topological space is said to be normal if, given A and B non-empty disjoint closed sets, we can find disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.)
[There are very small and simple examples.]

(v) Carry out us much of the proof of Urysohn's lemma and the Tietze extension theorem as you need[14] *to show that Urysohn's lemma holds with the hypothesis 'compact and Hausdorff' replaced by 'normal' and the Tietze extension theorem holds in the following form:-*

If Y is closed subset of a normal topological space (X, τ), then, given any bounded continuous real valued function f on Y (where Y has the subspace topology), we can find a bounded continuous real valued function F on X such that $F(y) = f(y)$ for all $y \in Y$.

Exercise 22.12. *We work with a real vector space to simplify the algebra but similar results can be obtained in the complex case.*

(i) Consider a real inner product space. Obtain the formulae

$$4\langle u, v\rangle = \|u + v\|_2^2 - \|u - v\|_2^2$$

and

$$\|u + v\|_2^2 + \|u - v\|_2^2 = 2\big(\|u\|_2^2 + \|v\|_2^2\big).$$

[14] Either do the whole thing as a revision exercise or just cast your eyes over your notes.

(ii) Suppose that V is a real normed vector space such that

$$\|u+v\|^2 + \|u-v\|^2 = 2\big(\|u\|^2 + \|v\|^2\big).$$

Show that if we set

$$p(u,v) = (\|u+v\|^2 - \|u-v\|^2)/4$$

then

$$p(u+w,v) = p(u,v) + p(w,v).$$

[It may or may not be helpful to consider the parallelepiped associated with u, v and w.]

(iii) Continuing with the hypotheses and notation of (ii), show that

$$p(nu,v) = np(u,v)$$

for n a strictly positive integer. Deduce that $p(\lambda u,v) = \lambda p(u,v)$ when λ is rational. Show that p is an inner product and $\|.\|$ is the norm associated with p.

Exercise 22.13. *In some of the exercises we used metrics rather than norms. This exercise shows why the need may arise. We work in $C^\infty([0,1])$.*

(i) Show that $d(f,g) = \sum_{j=0}^\infty 2^{-j}\min(1, \|f^{(j)} - g^{(j)}\|_\infty)$ is a metric. Show that $f_n^{(j)} \to f^{(j)}$ uniformly on $[0,1]$ for each j if and only if $d(f_n,f) \to 0$. Show that d is a complete metric.

(ii) The object of the rest of this question is to show that there is no norm with this property. To this end, suppose that $\|.\|$ is a norm on $C^\infty([0,1])$. such that $\|f_n - f\| \to 0$ implies $f_n^{(j)} \to f^{(j)}$ uniformly on $[0,1]$ for each j. Show, by reductio ad absurdum, or otherwise, that there must exist $\epsilon_j > 0$ with the property that $\|f^{(j)}\|_\infty \geq 1$ implies $\|f\| \geq \epsilon_j$.

(iii) Show that we can find a sequence of $f_n \in C^\infty([0,1])$ such that

$$\|f_n^{(j)}\|_\infty \leq 2^{-n} \text{ for } 0 \leq j \leq n-1$$
$$\|f_n^{(n)}\|_\infty \geq 2^n \epsilon_n^{-1}.$$

Show that $f_n^{(j)} \to f^{(j)}$ uniformly on $[0,1]$ for each j but $\|f_n\| \to \infty$.

Exercise 22.14. *Suppose that we have a sequence of metrics d_j on a space X.*

(i) Show that

$$d(x,y) = \sum_{j=1}^\infty 2^{-j}\min\big(1, d_j(x,y)\big)$$

is a metric on X such that the identity map $\iota : (X, d) \to (X, d_j)$ is continuous for each j.

(ii) Show that if ρ is a metric on X such that the identity map $\iota : (X, \rho) \to (X, d_j)$ is continuous for each j then $\iota : (X, \rho) \to (X, d)$ is continuous.

Exercise 22.15. *(This easy exercise discusses anti-isomorphism[15].) If $(U, +, ., \mathbb{C})$ is a vector space show that, if we write*

$$\lambda \bullet u = \lambda^* u,$$

then $(U, +, \bullet, \mathbb{C})$ is a vector space. Show that $(U, +, ., \mathbb{C})$ and $(U, +, \bullet, \mathbb{C})$ are algebraically anti-isomorphic.

Exercise 22.16. *(i) (This is about difference equations in the sense of the 1A differential equations course.) If $\lambda \in \mathbb{C}$, $\lambda \neq 0$, find the general solution of the difference equation*

$$\lambda u_n - u_{n+1} = 0 \ [n \in \mathbb{Z}].$$

(ii) Find the general solution of the the difference equation.

$$\lambda u_n - u_{n+1} = 0 \ [n \in \mathbb{Z},\ n \neq 0].$$

(Note that there are two arbitrary constants.)

(iii) Find the general solution of the the difference equation.

$$\lambda u_n - u_{n+1} = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0. \end{cases}$$

(iv) From now on we work in $l^2(\mathbb{Z})$ the vector space of two sided *square summable sequences of complex numbers*

$$\mathbf{a} = (\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots)$$

with norm

$$\|\mathbf{a}\|_2 = \left(\sum_{j=-\infty}^{\infty} |a_j|^2 \right)^{1/2}.$$

Show that, if $0 < |\lambda| < 1$, there is exactly one $\mathbf{u} \in l^2(\mathbb{Z})$ such that

$$\lambda u_n - u_{n+1} = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0, \end{cases}$$

[15] And shows that there is not much to discuss

and that the same is true if $|\lambda| > 1$.

(iv) Show that the shift operator S defined by $S\mathbf{a} = \mathbf{b}$ with $b_j = a_{j+1}$ is a well defined continuous linear map from $l^2(\mathbb{Z})$ to itself. Show that, if $|\lambda| \neq 1$, then $\lambda \notin \sigma(S)$.

(v) Show that 1 is not an eigenvalue of S. By considering $\mathbf{v}$ with

$$v_n = \begin{cases} N^{-1/2} & \text{if } 1 \leq n \leq N \\ 0 & \text{otherwise,} \end{cases}$$

show that 1 is an approximate eigenvalue of S.

(vi) Show that

$$\sigma(S) = \{\lambda \in \mathbb{C} \,:\, |\lambda| = 1\},$$

that all the points of $\sigma(S)$ are approximate eigenvalues and that none of them are eigenvalues.

Exercise 22.17. *The object of this exercise is to exhibit a complete inner product space which is not separable.*

(i) Let X be a non-empty set and $a : X \to \mathbb{R}$ a function such that $a(x) \geq 0$ for all $x \in X$ and such that there exists a K with

$$\sum_{x \in F} a(x) \leq K$$

whenever F is a finite subset of X. Show that

$$Y = \{x \in X \,:\, a(x) \neq 0\}$$

is countable.

Using the fact that the sum of a sequence of positive numbers is unaffected by the order of summation, this means that we can define

$$\sum_{x \in X} a(x) = \sum_{x \in Y} a(x).$$

Suppose $b : X \to \mathbb{R}$ a function such that $b(x) \geq 0$ for all $x \in X$. In what follows we say that $\sum_{x \in X} b(x)$ converges if $\sum_{x \in F} b(x) \leq K$ whenever F is a finite subset of X and we write

$$\sum_{x \in X} b(x) = \sum_{b(x) \neq 0} b(x).$$

(ii) If X is a non-empty set and $a : X \to \mathbb{C}$, we say that $a \in l^2(X)$ if $\sum_{x \in X} |a(x)|^2$ converges. Show that $l^2(X)$ is a vector space. Show further that

$$\langle a, b \rangle = \sum_{x \in X} a(x)b(x)^*$$

gives a well defined inner product on $l^2(X)$.

(iii) Show that $l^2(X)$ *is separable if and only if* X *is countable.*

(iv) Show that $l^2(X)$ *is complete.*