Sketch Solutions for Some Exercises in Calculus for the Ambitious

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Introduction

When I was young, I used to be surprised when the answer in the back of the book was wrong. I could not believe that the wise and gifted people who wrote textbooks could possibly make mistakes. I am no longer surprised.

Here are what I believe to be sketch solutions to the bulk of the exercises. By the nature of things, they cannot be guaranteed free of error. I would appreciate the opportunity to remedy problems. Please tell me of any errors, unbridgeable gaps, misnumberings etc. I welcome suggestions for additions.

ALL COMMENTS GRATEFULLY RECEIVED.

If you can, please use LaTeX 2_ε or its relatives for mathematics. If not, please use plain text. My e-mail is **twk@dpmms.cam.ac.uk**. You may safely assume that I am both lazy and stupid, so that a message saying 'Presumably you have already realised the mistake in Exercise Z' is less useful than one which says 'I think you have made a mistake in Exercise Z because you have have assumed that the sum is necessarily larger than the integral. One way round this problem is to assume that f is decreasing.'

It may be easiest to navigate this document by using the table of contents which follow on the next few pages. To avoid disappointment, observe that those exercises marked ★ have no solution given.

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Exercise 1.2.1

We have

$$\begin{aligned}20\,019 \times 299\,987 &= (20\,000 + 19) \times (300\,000 - 13)\\ &= 20\,000 \times 300\,000 + 19 \times 300\,000 - 13 \times 20\,000 - 19 \times 13\end{aligned}$$

so, to zeroth order,

$$20\,019 \times 299\,987 \approx 20\,000 \times 300\,000 = 6\,000\,000\,000$$

and, to first order,

$$\begin{aligned}20\,019 \times 299\,987 &= (20\,000 + 19)\\ &\approx 20\,000 \times 300\,000 + 19 \times 300\,000 - 13 \times 20,000\\ &= 6\,000\,000\,000 + 5700\,000 - 260\,000 = 6\,005\,440\,000.\end{aligned}$$

The exact answer is

$$20\,019 \times 299\,987 = 6\,005\,439\,753.$$

Exercise 1.2.2

By the binomial theorem (see Exercise 5.3.3) or direct calculation,

$$(x + \delta x)^3 = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 \approx x^3 + 3x^2\delta x.$$

Alternatively, apply the product rule,

$$(a + \delta a) \times (b + \delta b) \approx a \times b + a \times \delta b + b \times \delta a$$

to first order, twice.

By the binomial theorem, induction or repeated use of the product rule,

$$(x + \delta x)^n \approx x^n + nx^{n-1}\delta x$$

to the first order.

Exercise 1.2.3

$$(a + \delta a) - (b + \delta b) = a - b + (\delta a - \delta b)$$

exactly and thus certainly to first order.

Exercise 1.2.4

With the notation given,

$$a = 1\,000\,000,\ v = 1\,000,\ \delta a = 3$$

so

$$\delta v \approx \frac{1}{2\sqrt{a}} \times \delta a = \frac{3}{2\,000}.$$

Thus

$$\sqrt{1\,000\,003} = v + \delta v \approx 1000.0015.$$

in agreement with my calculator to these decimal places (and actually rather more).

Exercise 1.2.5

Let us write

$$v = a^{1/3} \text{ and } v + \delta v = (a + \delta a)^{1/3}.$$

If δa is small in magnitude compared to a, then $a + \delta a$ will be close to a and so $v + \delta v$ will be close to v. In other words, δv will be small in magnitude compared to v. Thus, working to first order,

$$a + \delta a = (v + \delta v)^3 \approx v^3 + 3v^2 \times \delta v = a + 3(a^{1/3})^2 \times \delta v.$$

Subtracting a from both sides and rearranging, we get

$$\delta a \approx 3a^{2/3} \times \delta v,$$

(where $a^{2/3} = (a^{1/3})^2$) so

$$\delta v \approx \frac{1}{3a^{2/3}} \times \delta a$$

to the first order.

Take

$$a = 1\,000\,000, \;\; \delta a = 3.$$

To first order

$$(1\,000\,003)^{1/3} = v + \delta v \approx 100 + \frac{\delta a}{3a^{2/3}} = 100 + \frac{1}{10\,000} = 100.0001$$

in agreement with my calculator.

(Working to more figures it gets 100.00009999900000166666333341 . . ..)

Exercise 1.2.7

Since $u = s - a$, we have

$$u = (b + c - a)/2$$

and, by the addition rule,

$$\delta u = (\delta b + \delta c - \delta a)/2.$$

Similar formulae hold for v and w, whilst

$$\delta s = (\delta a + \delta b + \delta c)/2.$$

Thus

$$\delta A \approx \frac{\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}}{8} \times \left(\frac{\delta a + \delta b + \delta c}{a+b+c} + \frac{-\delta a + \delta b + \delta c}{-a+b+c} + \frac{\delta a - \delta b + \delta c}{a-b+c} + \frac{\delta a + \delta b - \delta c}{a+b-c}\right)$$

to first order.

If we observe that

$$u + v + w = 3s - a - b - c = s,$$

we obtain

$$\delta s = \delta u + \delta v + \delta w$$

and

$$\delta A \approx \sqrt{(u+v+w)uvw} \times \left(\frac{\delta u + \delta v + \delta w}{u+v+w} + \frac{\delta u}{u} + \frac{\delta v}{v} + \frac{\delta w}{w}\right)$$

to first order.

Exercise 1.2.8

If $S = b^2 - 4c$ and $S + \delta S = (b + \delta b)^2 - 4(c + \delta c)$, then, to first order,

$$\delta S = 2b\delta b - 4\delta c$$

so, if

$$\begin{aligned} u &= \frac{-b + \sqrt{b^2 - 4c}}{2}, \\ u + \delta u &= \frac{-(b + \delta b) + \sqrt{(b + \delta b)^2 - 4(c + \delta c)}}{2}, \end{aligned}$$

we have

$$\begin{aligned} \delta u &= \frac{1}{2}\left(-\delta b + \frac{\delta S}{2\sqrt{S}}\right) \\ &= \frac{1}{2}\left(-\delta b + \frac{b\delta b - 2\delta c}{\sqrt{b^2 - 4c}}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} &\frac{-(b + \delta b) - \sqrt{(b + \delta b)^2 - 4(c + \delta c)}}{2} \\ &\qquad = \frac{-b - \sqrt{b^2 - 4c}}{2} - \frac{1}{2}\left(\delta b + \frac{b\delta b - 2\delta c}{\sqrt{b^2 - 4c}}\right) \end{aligned}$$

to first order.

Exercise 1.2.9

Let R be the radius of the moon in meters. The intended length is $2\pi(R+1)$, so the actual length is $10+2\pi(R+1)$ and the radius of the ring will be

$$\frac{10+2\pi(R+1)}{2\pi} = \frac{10}{2\pi} + 1 + R.$$

The extra height is $10/2\pi \approx 1.6$ meters.

Exercise 1.2.10

Using the formula $(x+y)(x-y) = x^2 - y^2$, with $x = 1$ and $100y$ the percentage rise, we see that, after one rise and fall, Pumpkin shares have fallen by .01%. Since .01 is very small compared with 100, we expect an overall fall of roughly $40 \times .01 = .4$ percent (binomial theorem or Exercise 1.2.2). The value is essentially unchanged.

Melon shares will be worth $1^2 - .4^2 = .84$ times their original value, that is to say 84% of their original value, a substantial change.

Exercise 1.3.1★

Exercise 1.3.2

(i) Let $x \geq y \geq 0$. Then

$$\begin{aligned}|x| + |y| &= x + y = |x + y|,\\ |x| + |-y| &= x + y \geq x - y = |x + (-y)|,\\ |-x| + |y| &= x + y \geq x - y = |(-x) + y|,\\ |-x| + |-y| &= x + y = |-(x + y)| = |(-x) + (-y)|.\end{aligned}$$

(ii) Let x, $y \geq 0$. Then

$$\begin{aligned}|x||y| &= xy = |x||y|,\\ |x||-y| &= xy = |x(-y)|,\\ |-x||y| &= xy = |(-x)y|,\\ |-x||-y| &= xy = |(-x)(-y)|.\end{aligned}$$

(iii) By (ii),

$$|a - b| + |b - c| \geq |(a - b) + (b - c)| = |a - c|.$$

(iv) We have

$$|3 - 2| + |2 - 1| = 2 = |3 - 1|,$$

but

$$|2 - 3| + |2 - 1| = 2 > 1 = |2 - 1|.$$

Exercise 1.3.4★

Exercise 1.4.1★

Exercise 1.4.2

We have
$$150 = f(32) - f(31) = A,$$
so
$$f(31 + \tfrac{2}{3}) = f(31) + 2A/3 = f(31) + 100$$
as required.

Exercise 1.4.4★

Exercise 1.4.5

(i) $g(x+h) = g(x) = g(x) + 0h + o(h)$, so $g'(x) = 0$.

(ii) If $g(t) = a$, then

$$u'(t) = g'(t)f(t) + g(t)f'(t) = 0 \times f(t) + a \times f'(t) = af'(t).$$

(iii) $g_1(x+h) = x + h = g(x) + 1 \times h + o(h)$, so $g_1'(x) = 1$.

(iv) $g_2'(t) = g_1(t)g_1'(t) + g_1'(t)g_1(t) = t \times 1 + 1 \times t = 2t$.

(v) $g_3'(t) = g_1(t)g_2'(t) + g_1'(t)g_2(t) = 2t^2 + t^2 = 3t^2$.

(vi) If $g_{n-1}'(t) = (n-1)g_{n-2}(t)$, then, since $g_n(t) = g_1(t)g_{n-1}(t)$,

$$g_n'(t) = g_1(t)g_{n-1}'(t) + g_1'(t)g_{n-1}(t) = (n-1)g_{n-1}(t) + 1 \times g_{n-1}(t) = ng_{n-1}(t),$$

so the result follows by induction.

(vii) Write $u(t) = 1$. We have $u(t) = g_n(t)g_{-n}(t)$ and

$$\begin{aligned} 0 = u'(t) &= g_n'(t)g_{-n}(t) + g_n(t)g_{-n}'(t) \\ &= ng_{n-1}(t)g_{-n}(t) + g_n(t)g_{-n}'(t) \\ &= ng_{-1}(t) + g_n(t)g_{-n}'(t) \end{aligned}$$

so

$$g_n(t)g_{-n}'(t) = -ng_{-1}(t)$$

and

$$g_{-n}'(t) = -\frac{n}{t^{n+1}}.$$

Combining our results for positive and negative integers, we get

$$g_n'(t) = ng_{n-1}(t)$$

for all integers $n \neq 0$.

By (i), $g_0'(t) = 0$.

(viii) Using the rule $(f+g)' = f' + g'$,

$$P'(t) = na_n t^{n-1} + (n-1)a_{n-1}t^{n-2} + \ldots + a_1.$$

(ix) $h(t) = g_{-1} \circ f(t)$ so

$$h'(t) = f'(t)g_{-1}' \circ f(t) = -\frac{f'(t)}{f(t)^2}.$$

(x) $0 = f'(t)h(t) + f(t)h'(t)$, so

$$h'(t) = -\frac{f'(t)h(t)}{f(t)} = -\frac{f'(t)}{f(t)^2}.$$

(x) By the product rule and the quotient rule,

$$h'(t) = \frac{f'(t)}{g(t)} - \frac{f(t)g'(t)}{g(t)^2}.$$

Exercise 1.4.7

(i) If $h(t) = t^p$, then $g_{1/p} = h^{-1}(t)$, so

$$g'_{1/p}(t) = (h^{-1})'(t) = \frac{1}{h'(h^{-1}(t))} = \frac{1}{p(h^{-1}(t))^{p-1}} = \frac{1}{pt^{(p-1)/p}} = \frac{1}{pt}g_{1/p}(t).$$

(ii) By the product rule (or applying the function of a function rule to $u \circ g_{1/p}$ with $u(t) = t^q$), we have

$$g'_{q/p}(t) = q(g_{1/p}(t))^{q-1}g'_{1/p}(t) = \frac{q}{p}g_{(q/p)-1}(t).$$

Exercise 1.4.8

(i) We know that

$$a'(x) = 0,\ S'(x) = \frac{1}{2\sqrt{x}}$$

and, by the addition rule,

$$u'(x) = 2x.$$

We have $c(x) = S(u(x))$, so, by the function of a function rule,

$$c'(x) = u'(x)S'(u(x)) = \frac{x}{\sqrt{1+x^2}}.$$

By the addition rule,

$$h'(x) = a'(x) + c'(x) = \frac{x}{\sqrt{1+x^2}}.$$

We also know that

$$b'(x) = \frac{1}{3x^{2/3}}$$

so, by the addition rule,

$$g'(x) = a'(x) + b'(x) = \frac{1}{3x^{2/3}}.$$

The quotient rule now gives

$$\begin{aligned} f'(x) &= \frac{g'(x)}{h(x)} - \frac{g(x)h'(x)}{h(x)^2} \\ &= \frac{1}{3x^{2/3}(1+\sqrt{1+x^2})} - \frac{x(1+x^{1/3})}{(1+\sqrt{1+x^2})^2\sqrt{1+x^2}}. \end{aligned}$$

Exercise 1.4.9

If $h > 0$,

$$m(h) - m(0) = h = h + o(h),$$

so $a = 1$.

If $h < 0$,

$$m(h) - m(0) = -h = -h + o(h),$$

so $b = -1$.

Exercise 1.5.1

There are many ways of explaining this. One way (and there is no claim that this is better than any other) is to observe that

$$\angle ABX = 2r - \angle CBX$$

(where r is a right angle), so

$$\begin{aligned}\tan(2r-\theta) &= \frac{\sin(2r-\theta)}{\cos(2r-\theta)} = \frac{\cos(r-\theta)}{-\sin(r-\theta)}\\ &= \frac{-\sin(-\theta)}{-\cos(-\theta)} = \frac{\sin(-\theta)}{\cos(-\theta)}\\ &= -\frac{\sin\theta}{\cos\theta} = -\tan\theta.\end{aligned}$$

Exercise 1.5.2★

Exercise 1.5.3

(i) $\sin(x+4r) = -\cos(x+3r) = -\sin(x+2r) = -\cos(x+r) = \sin x$ and

$$\cos(x+4r) = \sin(x+3r) = \sin(x-r) = \cos x.$$

Since $\sin 0 = -\sin(-0) = -\sin 0$, we have $\sin 0 = 0$.

Finally,

$$\cos(-x) = \sin(-x+r) = -\sin(x-r) = \cos x.$$

(ii) We have

$$\begin{aligned}\cos(u+v) &= \sin(u+v+r) = \sin(u+r)\cos v + \cos(u+r)\sin v \\ &= \cos u \cos v - \sin u \sin v.\end{aligned}$$

(iii) We have

$$\begin{aligned}1 &= \sin r = \cos 0 = \cos(u+(-u)) = \cos u \cos(-u) - \sin u \sin(-u) \\ &= (\cos u)^2 + (\sin u)^2.\end{aligned}$$

(iv) We have

$$\sin 2x = \sin(x+x) = \sin x \cos x + \sin x \cos x = 2 \sin x \cos x,$$

and

$$\begin{aligned}\cos 2x &= \cos(x+x) = \cos x \cos x - \sin x \sin x \\ &= (\cos x)^2 - (\sin x)^2 = (1-(\sin x)^2) - (\sin x)^2 \\ &= 1 - 2(\sin x)^2 = 1 - 2(1-(\cos x)^2) = 2(\cos x)^2 - 1.\end{aligned}$$

(v) We have

$$\begin{aligned}\tan(u+v) &= \frac{\sin(u+v)}{\cos(u+v)} \\ &= \frac{\sin u \cos v + \cos u \sin v}{\cos u \cos v - \sin u \sin v} \\ &= \frac{\tan u + \tan v}{1 - \tan u \tan v}.\end{aligned}$$

Setting $u = v = x$, we obtain

$$\tan 2x = \frac{2 \tan x}{1 - (\tan x)^2}.$$

Exercise 2.2.1

$$
\begin{aligned}
N(f) &= N_1(f) + N_2(f) + \ldots + N_n(f),\\
N(g) &= N_1(g) + N_2(g) + \ldots + N_n(g),\\
N(f+g) &= N_1(f+g) + N_2(f+g) + \ldots + N_n(f+g).
\end{aligned}
$$

Now observe that

$$f(t) \geq N_r(f)s \text{ and } g(t) \geq N_r(g)s$$

so

$$f(t) + g(t) \geq \big(N_r(f) + N_r(g)\big)s$$

whenever $a + (r-1)s \leq t \leq a + rs$, so

$$N_r(f+g) \geq N_r(f) + N_r(g).$$

Exercise 2.2.2★

Exercise 2.2.3

(i) We have

$$g(t) = \begin{cases} f(t) \geq 0 & \text{if } f(t) \geq 0, \\ 0 \geq 0 & \text{otherwise} \end{cases}$$

and

$$h(t) = \begin{cases} 0 \geq 0 & \text{if } f(t) \geq 0, \\ -f(t) \geq 0 & \text{otherwise.} \end{cases}$$

whilst

$$\begin{aligned} g(t) - h(t) &= \begin{cases} f(t) + 0 & \text{if } f(t) \geq 0, \\ 0 - (-f(t)) & \text{otherwise} \end{cases} \\ &= f(t) \end{aligned}$$

as required.

(ii) $g(t)-h(t) = |f(t)|-(|f(t)|-f(t)) = f(t)$ and, since $|f(t)| \geq f(t)$, we have $g(t)$, $h(t) \geq 0$.

(iii) We have $g(t) - h(t) = (M + f(t)) - M = f(t)$,

$$g(t) = M + f(t) \geq M - M \geq 0$$

and $g(t) = M \geq 0$.

Exercise 2.2.4

We have $G(t)$, $H(t) \geq 0$ and $G(t) - H(t) = F(t)$, so

$$\begin{aligned} I_a^b F(t)\,dt &= \int_a^b G(t)\,dt - \int_a^b H(t)\,dt \\ &= \int_a^b F(t)\,dt - 0 = \int_a^b F(t)\,dt. \end{aligned}$$

Exercise 2.2.5

Choosing g and h so that $g(t)$, $h(t) \geq 0$ and $f(t) = g(t) - h(t)$, we have $-f(t) = h(t) - g(t)$. Thus

$$\begin{aligned}\int_a^b \big(-f(t)\big)\,dt &= \int_a^b h(t)\,dt - \int_a^b g(t)\,dt \\ &= -\left(\int_a^b g(t)\,dt - \int_a^b h(t)\,dt\right) \\ &= -\int_a^b f(t)\,dt.\end{aligned}$$

Exercise 2.2.6

Let $f(t) = f_1(t) - f_2(t)$, $g(t) = g_1(t) - g_2(t)$ with $f_j(t), g_j(t) \geq 0$ $[j = 1, 2]$. We have

$$f(t) + g(t) = (f_1(t) + g_1(t)) - (f_2(t) + g_2(t))$$

and

$$(f_1(t) + g_1(t)),\ (f_2(t) + g_2(t)) \geq 0,$$

so

$$\begin{aligned}
\int_a^b f(t) + g(t)\,dt &= \int_a^b (f_1(t) + g_1(t))\,dt - \int_a^b (f_2(t) + g_2(t))\,dt \\
&= \left(\int_a^b f_1(t)\,dt + \int_a^b g_1(t)\,dt\right) - \left(\int_a^b f_2(t)\,dt + \int_a^b g_2(t)\,dt\right) \\
&= \left(\int_a^b f_1(t)\,dt - \int_a^b f_2(t)\,dt\right) + \left(\int_a^b g_1(t)\,dt - \int_a^b g_2(t)\,dt\right) \\
&= \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.
\end{aligned}$$

Using the first part of the question and the result of Exercise 2.2.5, we have

$$\begin{aligned}
\int_a^b f(t) - g(t)\,dt &= \int_a^b f(t) + (-g(t))\,dt \\
&= \int_a^b f(t)\,dt + \int_a^b (-g(t))\,dt \\
&= \int_a^b f(t)\,dt - \int_a^b g(t)\,dt.
\end{aligned}$$

Exercise 2.2.7

(i) We have

$$\begin{aligned}\int_a^b pf(t)\,dt &= \int_a^b \underbrace{f(t)+f(t)+\ldots+f(t)}_{p}\,dt \\ &= \underbrace{\int_a^b f(t)\,dt + \int_a^b f(t)\,dt + \ldots + \int_a^b f(t)\,dt}_{p} \\ &= p\int_a^b f(t)\,dt.\end{aligned}$$

If $p=0$, we have

$$\int_a^b 0\times f(t)\,dt = \int_a^b 0\,dt = 0 = 0\times\int_a^b f(t)\,dt.$$

(ii) By (i), we have

$$\begin{aligned}q\int_a^b \frac{p}{q}f(t)\,dt &= \int_a^b q\frac{p}{q}f(t)\,dt \\ &= \int_a^b pf(t)\,dt \\ &= p\int_a^b f(t)\,dt,\end{aligned}$$

so

$$\int_a^b \frac{p}{q}f(t)\,dt = \frac{p}{q}\int_a^b f(t)\,dt.$$

(iii) Since

$$\frac{p}{q}f(t) \leq uf(t) \leq \frac{p+1}{q}f(t),$$

the inequality rule for integrals yields

$$\frac{p}{q}\int_a^b f(t)\,dt \leq \int_a^b uf(t)\,dt \leq \frac{p+1}{q}\int_a^b f(t)\,dt.$$

Thus

$$\left(u-\frac{1}{q}\right)\int_a^b f(t)\,dt \leq \int_a^b uf(t)\,dt \leq \left(u+\frac{1}{q}\right)\int_a^b f(t)\,dt.$$

Since we can make q as large as we please,

$$\int_a^b uf(t)\,dt = u\int_a^b f(t)\,dt.$$

(iv) We can write $g(t) = g_1(t) - g_2(t)$ with $g_1(t)$, $g_2(t) \geq 0$. Thus, if $u \geq 0$, part (iii) yields

$$\begin{aligned}\int_a^b ug(t)\,dt &= \int_a^b ug_1(t) - ug_2(t)\,dt \\ &= \int_a^b ug_1(t)\,dt - \int_a^b ug_2(t)\,dt \\ &= u\int_a^b g_1(t)\,dt - u\int_a^b g_2(t)\,dt \\ &= u\left(\int_a^b g_1(t)\,dt - \int_a^b g_2(t)\,dt\right) \\ &= u\int_a^b g(t)\,dt.\end{aligned}$$

Using Exercise 2.2.5, we deduce that

$$\begin{aligned}\int_a^b (-u)g(t)\,dt &= \int_a^b -(ug(t))\,dt = -\int_a^b (ug(t))\,dt \\ &= -\left(u\int_a^b g(t)\,dt\right) = (-u)\int_a^b g(t)\,dt.\end{aligned}$$

The required result now follows.

Exercise 2.2.8

(i) We know the result for $a \leq b$. If $b \leq a$, then

$$\begin{aligned}
\int_a^b (f(t) + g(t))\,dt &= -\int_b^a (f(t) + g(t))\,dt \\
&= -\left(\int_b^a f(t)\,dt + \int_b^a g(t)\,dt\right) \\
&= -\int_b^a f(t)\,dt - \int_b^a g(t)\,dt \\
&= \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.
\end{aligned}$$

(ii) We know the result for $a \leq b$. If $b \leq a$, then

$$\int_a^b vf(t)\,dt = -\int_b^a vf(t)\,dt = -v\int_b^a f(t)\,dt = v\int_a^b f(t)\,dt.$$

(iii) If $A \geq B$, then $-B \geq -A$.

Exercise 2.2.9

Write $f(t) = f_1(t) - f_2(t)$ with f_1 and f_2 positive functions. We have

$$\begin{aligned}
\int_a^b f(t)\,dt + \int_b^c f(t)\,dt &= \left(\int_a^b f_1(t)\,dt - \int_a^b f_2(t)\,dt\right) + \left(\int_b^c f_1(t)\,dt - \int_b^c f_2(t)\,dt\right) \\
&= \left(\int_a^b f_1(t)\,dt + \int_b^c f_1(t)\,dt\right) - \left(\int_a^b f_2(t)\,dt + \int_b^c f_2(t)\,dt\right) \\
&= \int_a^c f_1(t)\,dt - \int_a^c f_2(t)\,dt \\
&= \int_a^c f(t)\,dt.
\end{aligned}$$

Exercise 2.2.10

(i) We have

$$\int_a^c f(t)\,dt + \int_b^c f(t)\,dt = -\int_c^a f(t)\,dt - \int_b^c f(t)\,dt$$
$$= -\left(\int_b^c f(t)\,dt + \int_c^a f(t)\,dt\right) = -\int_b^a f(t)\,dt = \int_a^b f(t)\,dt.$$

(ii) We have

$$\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^b f(t)\,dt - \int_c^b f(t)\,dt$$
$$= \int_a^b f(t)\,dt - \int_c^a f(t)\,dt - \int_a^b f(t)\,dt = -\int_c^a f(t)\,dt = \int_a^c f(t)\,dt.$$

(iii) If $a \leq c \leq b$,

$$\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt.$$

We thus have the result for $c \leq b \leq a$, $c \leq a \leq b$, $a \leq c \leq b$ and $a \leq b \leq c$. The remaining cases when $b \leq a$ follow as in (i).

(iv) We know the result for $a \leq b$ and $k \geq 0$ (see note below), so, if $k < 0$,

$$\int_a^b k\,dt = -\int_a^b (-k), dt = -(-k)(b-a) = k(b-a)$$

If $b \leq a$, then

$$\int_a^b k\,dt = -\int_b^a k\,dt = -k(b-a) = k(a-b).$$

(Note: So far as this book is concerned, the result for $b \geq a$ and $k \geq 0$ is obvious. We could go one step further back and remark that if $b > a$ and $k > 0$, our argument about splitting into squares shows that

$$\int_a^b (b-a)\,dt = (b-a)^2$$

and so

$$\int_a^b k\,dt = \int_a^b \frac{k}{b-a}(b-a)\,dt = \frac{k}{b-a}\int_a^b (b-a)\,dt = k(b-a).$$

However, this is all a bit hair splitting.)

Exercise 2.2.11

(i) We have

$$(r+1)r - r(r-1) = r\big((r+1)-(r-1)\big) = 2r.$$

Thus

$$\begin{aligned}1+2+\ldots+n &= \frac{1}{2}\big((2\times 1 - 1\times 0)+\ldots\\ &\quad + ((r+1)r - r(r-1))\\ &\quad + ((r+2)(r+1)-(r+1)r)+\ldots\\ &\quad + ((n+1)n - n(n-1))\big)\\ &= \frac{1}{2}((n+1)n - 1\times 0) = \frac{1}{2}n(n+1).\end{aligned}$$

(ii) We have

$$(r+1)r(r-1) - r(r-1)(r-2) = r(r-1)\big((r+1)-(r-2)\big) = 3r(r-1)$$

Thus

$$\begin{aligned}&1\times 0 + 2\times 1 + \ldots + r(r-1) + \ldots + n(n-1)\\ &\quad = \frac{1}{3}\big((2\times 1\times 0 - 2\times 0\times(-1))+\ldots\\ &\qquad + ((r+1)r(r-1) - r(r-1)(r-2))\\ &\qquad + ((r+2)(r+1)r - (r+1)r(r-1))+\ldots\\ &\qquad + ((n+1)n(n-1) - n(n-1)(n-2))\big)\\ &\quad = \frac{1}{3}\big((n+1)n(n-1) - 1\times 0\times(-1)\big) = \frac{1}{3}(n+1)n(n-1).\end{aligned}$$

(iii) We have

$$\begin{aligned}&1^2+2^2+\ldots+r^2+\ldots+n^2\\ &\qquad = (1+1\times 0)+\ldots+(r(r-1)+r)+\ldots+(n(n-1)+n)\\ &\qquad = (1\times 0 + 2\times 1+\ldots+r(r-1)+\ldots+n(n-1))\\ &\qquad\quad + (1+2+\ldots+n)\\ &\qquad = \frac{1}{3}\Big((n+1)n(n-1)+\frac{1}{2}n(n+1)\Big) = \frac{1}{6}n(n+1)(2n+1).\end{aligned}$$

(iv) Observe that

$$\frac{r^2}{n^2} \le t^2 \le \frac{(r+1)^2}{n^2}$$

for $r/n \le t \le (r+1)/n$ and so

$$\frac{r^2}{n^3} = \int_{r/n}^{(r+1)/n} \frac{r^2}{n^2}\,dt \le \int_{r/n}^{(r+1)/n} t^2\,dt \le \int_{r/n}^{(r+1)/n} \frac{(r+1)^2}{n^2}\,dt = \frac{(r+1)^2}{n^3}.$$

(v) Thus, adding,

$$\frac{0^2}{n^3} + \ldots + \frac{r^2}{n^3} + \ldots + \frac{(n-1)^2}{n^3}$$
$$\leq \int_0^{1/n} t^2\,dt + \ldots + \int_{r/n}^{(r+1)/n} t^2\,dt + \ldots + \int_{(n-1)/n}^{1} t^2\,dt$$
$$\leq \frac{1^2}{n^3} + \ldots + \frac{(r+1)^2}{n^3} + \ldots + \frac{n^2}{n^3},$$

so

$$\frac{0^2 + 1^2 + \ldots + (n-1)^2}{n^3} \leq \int_0^1 t^2\,dt \leq \frac{1^2 + 2^2 + \ldots + n^2}{n^3}$$

and, by (iii),

$$\frac{(1-n^{-1})(1-\frac{1}{2}n^{-1})}{3} \leq \int_0^1 t^2\,dt \leq \frac{(1+n^{-1})(1+\frac{1}{2}n^{-1})}{3}$$

Since we can make n as large as we please,

$$\int_0^1 t^2\,dt = \frac{1}{3}.$$

(iv) Observe that

$$\frac{r^2a^2}{n^2} \leq t^2 \leq \frac{(r+1)^2a^2}{n^2}$$

for $ra/n \leq t \leq (r+1)a/n$ and so

$$\frac{r^2a^3}{n^3} = \int_{ra/n}^{(r+1)a/n} \frac{r^2a^2}{n^2}\,dt \leq \int_{r/n}^{(r+1)a/n} t^2\,dt \leq \int_{ra/n}^{(r+1)a/n} \frac{(r+1)^2a^2}{n^2}\,dt = \frac{(r+1)^2a^3}{n^3}.$$

Thus, adding,

$$\frac{(0^2 + 1^2 + \ldots + (n-1)^2)a^3}{n^3} \leq \int_0^a t^2\,dt \leq \frac{(1^2 + 2^2 + \ldots + n^2)a^3}{n^3}$$

and, by (iii),

$$a^3\frac{(1-n^{-1})(1-\frac{1}{2}n^{-1})}{3} \leq \int_0^a t^2\,dt \leq a^3\frac{(1+n^{-1})(1+\frac{1}{2}n^{-1})}{3}.$$

Since we can make n as large as we please,

$$\int_0^a t^2\,dt = a^3\frac{1}{3}.$$

We observe (using symmetry) that, if $a > 0$

$$\int_0^{-a} t^2\,dt = -\int_{-a}^{0} t^2\,dt = -\int_0^a t^2\,dt = -\frac{a^3}{3} = \frac{(-a)^3}{3}.$$

Thus (since the result for $a = 0$ is trivial)

$$\int_0^a t^2\,dt = \frac{a^3}{3}$$

for all values of a whether positive, negative or zero.

Thus

$$\int_a^b t^2\,dt = \int_0^b t^2\,dt - \int_0^a t^2\,dt = \frac{b^3 - a^3}{3}.$$

Exercise 2.2.12

(i) Observe that

$$\begin{aligned}(r+1)r(r-1)(r-2)\dots(r-k+1) - r(r-1)(r-2)\dots(r-k)\\ &= r(r-1)(r-2)\dots(r-k+1)\times\bigl(r+1-(r-k)\bigr)\\ &= (k+1)u_r,\end{aligned}$$

so, adding, we have

$$(k+1)(u_1+u_2+\dots+u_n) = (n+1)n(n-1)(n-2)\dots(n-k+1).$$

(ii) The result is true when $k=0$, since

$$1^0+2^0+\dots+n^0 = n = \frac{n^{0+1}}{0+1}.$$

Suppose it is true for all $k \leq K-1$. Then, writing

$$u_r = (r+1)r(r-1)(r-2)\dots(r-K+1)$$

and

$$s_p = 1^p+2^p+\dots+n^p,$$

we have

$$r^K = u_r + a_0 1^0 + a_1 r + a_2 r^2 + \dots + a_{K-1}r^{K-1}$$

and so

$$\begin{aligned}1^K+2^K+\dots+n^K &= (u_1+u_2+\dots+u_n) + (a_0s_0+a_1s_1+a_2s_2+\dots+a_{K-1}s_{k-1}\\ &= (n+1)n(n-1)(n-2)\dots(n-k+1) + a_0\frac{n^1}{1} + a_1\frac{n^2}{2} + a_{K-1}\frac{n^K}{K}\\ &\qquad + a_0P_0(n) + a_1P_1(n) + a_{K-1}P_{K-1}(n)\\ &= \frac{n^{K+1}}{K+1} + P_K(n)\end{aligned}$$

where P_K is some polynomial of degree at most K. The result now follows by induction.

(iii) Observe that, if $a>0$,

$$\frac{a^kr^k}{n^k} \leq t^k \leq \frac{a^k(r+1)^k}{n^k}$$

for $ar/n \leq t \leq a(r+1)/n$ and so

$$\frac{a^{k+1}r^k}{n^{k+1}} = \int_{ar/n}^{a(r+1)/n}\frac{a^kr^k}{n^k}\,dt \leq \int_{ar/n}^{(ar+1)/n} t^k\,dt \leq \int_{ar/n}^{a(r+1)/n}\frac{a^k(r+1)^k}{n^k}\,dt = \frac{a^{k+1}(r+1)^k}{n^{k+1}}.$$

Thus, adding,

$$a^{k+1}\left(\frac{0^k}{n^{k+1}}+\ \ldots+\frac{r^k}{n^{k+1}}+\ldots+\frac{(n-1)^k}{n^{k+1}}\right)$$
$$\leq\int_0^{a/n}t^k\,dt+\ldots+\int_{ar/n}^{a(r+1)/n}t^k\,dt+\ldots+\int_{a(n-1)/n}^{a}t^k\,dt$$
$$\leq a^{k+1}\left(\frac{1^k}{n^{k+1}}+\ldots+\frac{(r+1)^k}{n^{k+1}}+\ldots+\frac{n^k}{n^{k+1}}\right),$$

so

$$a^{k+1}\frac{0^k+1^k+\ldots+(n-1)^k}{n^{k+1}}\leq\int_0^a t^k\,dt\leq a^{k+1}\frac{1^k+k^k+\ldots+n^k}{n^{k+1}}$$

Thus

$$a^{k+1}\frac{(n-1)^{k+1}/(k+1)+P_k(n-1)}{n^{k+1}}\leq\int_0^a t^k\,dt\leq a^{k+1}\frac{n^{k+1}/(k+1)+P_k(n)}{n^{k+1}},$$

i.e.

$$a^{k+1}\frac{1}{k+1}+Q_1(1/n)\leq\int_0^1 t^k\,dt\leq a^{k+1}\frac{1}{k+1}+Q_2(1/n)$$

where Q_1 and Q_2 are polynomials with zero constant term.

Since we can make n as large as we please,

$$\int_0^a t^k\,dt=\frac{a^{k+1}}{k+1}.$$

Using symmetry or antisymmetry

$$\int_0^{-a}t^k\,dt=-\int_{-a}^0 t^k\,dt=(-1)^{k+1}\int_0^a t^k\,dt=\frac{(-a)^{k+1}}{k+1}.$$

Thus

$$\int_0^a t^k\,dt=\frac{a^{k+1}}{k+1}$$

regardless of the sign of a and so

$$\int_a^b t^k\,dt=\int_0^b t^k\,dt-\int_0^a t^k\,dt=\frac{b^k-a^k}{3}$$

for all a and b.

Exercise 2.3.1

(Only part (ii).)

If $\delta t > 0$ and $f(s)$ is negative for $t \le s \le t + \delta t$, then

$$\begin{aligned}\int_t^{t+\delta t} f(x)\,dx &= -\int_t^{t+\delta t} (-f(x))\,dx \\ &= -(-f(t)\delta t + o(\delta t)) \\ &= f(t)\delta t + o(\delta t).\end{aligned}$$

Exercise 2.3.2

We have

$$f(t) - u \leq f(x) \leq f(t) + u$$

whenever $|x - t| \leq v$. Thus, if $0 \leq \delta t \leq v$, we have

$$\int_{t-\delta t}^{t} (f(t) - u)\, dx \leq \int_{t-\delta t}^{t} f(x)\, dx \leq \int_{t-\delta t}^{t} (f(t) + u)\, dx$$

so that

$$(f(t) - u)\delta t \leq \int_{t-\delta t}^{t} f(x)\, dx \leq (f(t) + u)\delta t$$

and, multiplying through by -1,

$$(f(t) - u)(-\delta t) \geq \int_{t}^{t-\delta t} f(x)\, dx \geq (f(t) + u)(-\delta t).$$

Exercise 2.3.3

If
$$g(x) = \big(F(x) - G(x)\big) - \big(F(a) - G(a)\big),$$
then
$$g'(x) = F'(x) - G'(x) - 0 = f(x) - f(x) = 0,$$
so
$$g(x) = c$$
a constant, whence $G(x) = F(x) + c$ for all x.

Conversely, if $F'(x) = f(x)$ and $G(x) = F(x) + c$, then
$$G'(x) = F'(x) + 0 = f(x).$$

Exercise 2.3.4

(i) If

$$F(t) = a_0 t + a_1 \frac{t^2}{2} + a_2 \frac{t^3}{3} + \ldots + a_n \frac{t^{n+1}}{n+1},$$

then

$$F'(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n.$$

Thus

$$\int_0^x (a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n)\, dt = [F(t)]_0^x$$
$$= a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \ldots + a_n \frac{x^{n+1}}{n+1}.$$

(ii) Write $k = p/q$. If

$$F(t) = \frac{t^{k+1}}{k+1},$$

then

$$F'(t) = t^k.$$

Thus

$$\int_a^b t^k\, dt = [F(t)]_a^b = \frac{1}{k+1}(b^{k+1} - a^{k+1}).$$

Exercise 2.3.5

(i) Observe that
$$\frac{r^a}{n^a} \leq t^a \leq \frac{(r+1)^a}{n^a}$$
for $r/n \leq t \leq (r+1)/n$ and so
$$\frac{r^a}{n^{a+1}} = \int_{r/n}^{(r+1)/n} \frac{r^a}{n^a}\,dt \leq \int_{r/n}^{(r+1)/n} t^a\,dt = \int_{r/n}^{(r+1)/n} \frac{(r+1)^a}{n^a}\,dt \leq \frac{(r+1)^a}{n^{a+1}}.$$

Thus, adding,
$$\begin{aligned}\frac{0^a}{n^{a+1}} + \ldots + &\frac{r^a}{n^{a+1}} + \ldots + \frac{(n-1)^a}{n^{a+1}}\\ &\leq \int_0^{1/n} t^a\,dt + \ldots + \int_{r/n}^{(r+1)/n} t^a\,dt + \ldots + \int_{(n-1)/n}^{1} t^a\,dt\\ &\leq \frac{1^a}{n^{a+1}} + \ldots + \frac{(r+1)^a}{n^{a+1}} + \ldots + \frac{n^a}{n^{a+1}},\end{aligned}$$
so
$$\frac{0^a + 1^a + \ldots + (n-1)^a}{n^{a+1}} \leq \int_0^1 t^a\,dt \leq \frac{1^a + 2^a + \ldots + n^a}{n^{a+1}}.$$

(ii) Since
$$\int_0^1 t^a\,dt = \left[\frac{t^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1},$$
this gives us
$$0^a + 1^a + \ldots + (n-1)^a \leq \frac{n^{a+1}}{a+1} \leq 1^a + 2^a + \ldots + n^a.$$

(iii) Thus
$$\frac{n^{a+1}}{a+1} \leq 1^a + 2^a + \ldots + n^a \leq \frac{n^{a+1}}{a+1} + n^\alpha$$
and
$$0 \leq \frac{1^a + 2^a + \ldots + n^a}{n^{a+1}} - \frac{1}{a+1} \leq \frac{1}{n}.$$

Exercise 2.4.1

Observe that

$$b^3 - a^3 = (b-a)(a^2+ab+b^2) = (b-a)\big((a+b/2)^2 + 3b^2/4\big) > 0$$

if $b > a$.

However, $f'(t) = 3t^2$, so $f'(0) = 0$.

Exercise 2.4.2

If $g(t) = f(t) - Mt$, then $g'(t) = f'(t) - M \leq 0$ so g is decreasing and $g(b) - g(a) \leq 0$. In other words,

$$(f(b) - Mb) - (f(a) - Ma) \leq 0,$$

so

$$f(b) - f(a) \leq M(b - a).$$

If $h(t) = Mt + f(t)$, then $h'(t) = M + f'(t) \geq 0$, so h is increasing and $h(b) - h(a) \geq 0$. In other words,

$$(Mb + f(b)) - (Ma + f(a)) \geq 0$$

so

$$f(b) - f(a) \geq -M(b - a).$$

Exercise 2.4.3

By the definition of differentiation, given $v > 0$, we can find a $u > 0$ with $u \leq (t-s)/2$ such that

$$|f(y+h) - f(y) - f'(y)h| \leq v|h|$$

whenever $|h| \leq u$. Since $f'(y) > 0$, we can take $v = f'(y)/2$.

Now

$$\begin{aligned} f(s) &\geq f(y+u) = (f(y) + f'(y)u) + (f(y+u) - f(y) - f'(y)u) \\ &\geq (f(y) + f'(y)u) - |f(y+u) - f(y) - f'(y)u| \\ &\geq (f(y) + f'(y)u) - \frac{f'(y)}{2}u = f(y) + \frac{f'(y)u}{2} \\ &> f(y) \geq f(t) \end{aligned}$$

as stated.

Exercise 2.5.1

(i) If $s < 0$, then Supergirl will run a longer distance and swim a longer distance than if she had taken $s = 0$.

(ii) We have

$$\begin{aligned} f(s) &= \text{time running} + \text{time swimming} \\ &= \frac{\text{distance run}}{u} + \frac{\text{distance swum}}{v} \\ &= \frac{s}{u} + \frac{\sqrt{(a-s)^2 + b^2}}{v}. \end{aligned}$$

Exercise 2.5.2

(i) $f(s) = h(s) + g(s)/v$ with $h(s) = s/u$, $g(s) = G(H(s))$ where $G(s) = \sqrt{s}$ and $H(s) = (a-s)^2 + b^2$.

Now $H'(s) = -2(s-a)$,

$$g'(s) = H'(s)G'(H(s)) = -2(s-a) \times \frac{1}{2\sqrt{H(s)}} = -\frac{s-a}{\sqrt{(a-s)^2+b^2}}$$

and

$$f'(s) = \frac{g'(s)}{v} + h'(s) = \frac{1}{u} - \frac{1}{v}\frac{a-s}{((a-s)^2+b^2)^{1/2}}.$$

(ii) We have

$$f(s) = \frac{1}{u} - \frac{F(s)}{vg(s)},$$

with g as in (i) and $F(s) = a - s$, so

$$\begin{aligned} f'(s) &= -\frac{1}{v}\left(\frac{F'(s)}{g(s)} - \frac{F(s)g'(s)}{g(s)^2}\right) \\ &= \frac{1}{v}\left(\frac{1}{\big((a-s)^2+b^2\big)^{1/2}} - \frac{(a-s)^2}{((a-s)^2+b^2)^{3/2}}\right) \\ &= \frac{1}{v}\frac{b^2}{\big((a-s)^2+b^2\big)^{3/2}} > 0. \end{aligned}$$

Exercise 2.5.3

(i) Since

$$a \leq (a^2 + b^2)^{1/2},$$

we have

$$\frac{1}{u} - \frac{1}{v}\frac{a}{(a^2 + b^2)^{1/2}} \geq 0$$

whenever $v \geq u$.

Since Supergirl can swim faster than she runs and a straight line is the path of shortest length, it is clear that she should dive in at once.

(ii) This is just the observation that

$$\frac{v}{u} = \frac{a - s_0}{((a - s_0)^2 + b^2)^{1/2}}$$

and so the line joining (a, b) and $(a - s_0, 0)$ is at angle θ_0 with

$$\sin\theta_0 = \frac{a - s_0}{((a - s_0)^2 + b^2)^{1/2}} = \frac{v}{u}.$$

(iii) As s decreases, running along the bank for a fixed time becomes less effective in closing the distance to Superman.

Exercise 2.5.4★

Exercise 2.5.5

a is local maximum if and only if there is a $u > 0$ with $b - a \geq u$ such that $f(a) \geq f(s)$ for $a \leq s \leq a + u$ or, equivalently, for all s with $a \leq s \leq b$, $|s - a| \leq u$.

b is local maximum if and only if there is a $u > 0$ with $b - a \geq u$ such that $f(b) \geq f(s)$ for $b - u \leq s \leq b$ or, equivalently, for all s with $a \leq s \leq b$, $|s - b| \leq u$.

If $a < t < b$, then t is a local maximum if and only if we can find a $u > 0$ with $u \leq t - a, b - t$ such that $f(s) \leq f(t)$ whenever $t - u \leq s \leq t + u$ or, equivalently, for all s with $a \leq s \leq b$, $|s - t| \leq u$.

Exercise 2.5.6

(Parts (ii) and (iii) only.)

(ii) If $f(s) \geq f(t)$ for all $t \in [a, b]$, then, automatically, $f(s) \geq f(t)$ for all $t \in [a, b]$ and $|t - s| \leq 1$.

(iii) We say that f has a *local minimum* at s with $a \leq s \leq b$ if we can find a $u > 0$ such that, whenever $a \leq t \leq b$ and $|t - s| \leq u$, we have $f(s) \leq f(t)$.

Exercise 2.5.7

We give two arguments.

First argument

$$f(c + \delta t) = f(c) + f'(c)\delta t + o(\delta t).$$

If $f'(c) < 0$, then $f(c + \delta t) < f(c)$ when δt is strictly positive and sufficiently small, so f does not have a minimum at c. If $f'(c) > 0$, then $f(c+\delta t) < f(c)$ when δt is strictly negative and sufficiently small (in absolute value), so f does not have a minimum at c. Thus, if f attains a local minimum at c, we must have $f'(c) = 0$.

Second argument If f attains a local minimum at c, then $-f$ attains a local maximum, so $-f'(c) = (-f)'(c) = 0$ and $f'(c) = 0$.

Exercise 2.5.8

We have
$$f(a+\delta t) = f(a) + f'(a)\delta t + o(\delta t),$$
so, if $f'(a) > 0$, then $f(a+\delta t) > f(a)$ when δt is strictly positive and sufficiently small, and so f does not have a maximum at a. Thus, if f has a local maximum at a, we have $f'(a) \leq 0$.

We have
$$f(b+\delta t) = f(b) + f'(b)\delta t + o(\delta t)$$
so, if $f'(b) < 0$, then $f(b+\delta t) > f(b)$ when δt is strictly negative and sufficiently small, so f does not have a maximum at b. Thus, if f has a local maximum at b, we have $f'(b) \geq 0$.

Exercise 2.5.9

This is just a repeat of Exercise 2.4.1. Observe that

$$b^3 - a^3 = (b-a)(a^2 + ab + b^2) = (b-a)\big((a+b/2)^2 + 3b^2/4\big) > 0$$

if $b > a$, so f is everywhere increasing and can have neither a local maximum nor a local minimum.

However, $f'(t) = 3t^2$, so $f'(0) = 0$.

Exercise 2.5.10

If c is an interior point with $f'(c) = 0$ and we can find a $u > 0$ such that $f'(t) \leq 0$ for $c - u \leq t \leq c$ and $f'(t) \geq 0$ for $c \leq t \leq c + u$, then we know that $f(t)$ decreases as t runs from $c - u$ to c and increases as t runs from c to $c + u$. Thus f attains a local minimum at c.

(We could obtain the same result by observing that f has a local minimum at c if and only if $-f$ has a local maximum at c.)

Exercise 2.5.11

If we can find a $u > 0$ such that $f'(t) \leq 0$ for $a \leq t \leq a + u$, then f is decreasing between a and $a+u$ so $f(a) \geq f(t)$ for $a \leq t \leq a+u$ and a is a local maximum.

If we can find a $u > 0$ such that $f'(t) \geq 0$ for $b - u \leq t \leq b$, then f is increasing between $b - u$ and b so $f(b) \geq f(t)$ for $b - u \leq t \leq b$ and b is a local maximum.

Exercise 2.5.12★

Exercise 2.5.13

If $j + k$ is odd and $1 \leq j \leq n$, then $f'(t) > 0$ for $x_{j-1} < t < x_j$, so $f(t)$ is strictly increasing as t runs from x_{j-1} to x_j and $f'(t) < 0$ for $x_j < t < x_{j+1}$, so $f(t)$ is strictly decreasing as t runs from x_j to x_{j+1}. Thus f has a local maximum at x_j and no local maxima or minima at s with $x_{j-1} < s < x_j$. or with $x_j < s < x_{j+1}$.

If $k = 1$, then $f'(t) < 0$ for $x_0 < t < x_1$ so $f(t)$ is strictly decreasing as t runs from x_0 to x_1. Thus f has a local maximum at x_0 and no local maxima or minima at s with $x_0 < s < x_1$.

If $k = 0$, then $f'(t) > 0$ for $x_0 < t < x_1$, so $f(t)$ is strictly increasing as t runs from x_0 to x_1. Thus f has a local minimum at x_0 and no local maxima or minima at s with $x_0 < s < x_1$.

The remaining cases are dealt with similarly.

Exercise 2.5.14

Observe that, by the statement proved in Exercise 2.5.13,

$$e_j = (-1)^{j+k+1},$$

so

$$\begin{aligned}
&\tfrac{1}{2}e_0 + e_1 + e_2 + \ldots + e_{n-1} + e_n + \tfrac{1}{2}e_{n+1} \\
&\qquad = (-1)^{k+1}\left(\tfrac{1}{2}(-1)^0 + (-1)^1 + (-1)^2 + \ldots + (-1)^n + \tfrac{1}{2}(-1)^{n+1}\right) \\
&\qquad = \begin{cases} (-1)^{k+1}(\frac{1}{2} + 0 - \frac{1}{2}) = 0 & \text{if } n \text{ is even,} \\ (-1)^{k+1}(\frac{1}{2} - 1 + \frac{1}{2}) = 0 & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Exercise 2.5.15

(i) f does not satisfy the conditions since f' has infinitely many zeros.

(ii) f does not satisfy the conditions since f' does not change sign as it passes through its zero.

Exercise 2.6.1

We have

$$\begin{aligned} f(s) &= \text{time running} + \text{time swimming} \\ &= \frac{\text{distance run}}{u} + \frac{\text{distance swum}}{v} \\ &= \frac{1}{u}\sqrt{(a-s)^2+b^2} + \frac{1}{v}\sqrt{(c-s)^2+d^2}. \end{aligned}$$

(ii) When $|s| \geq 2|a|, 2|c|$, then $|a-s|, |c-s| \geq |s|/2$ and so

$$\sqrt{(a-s)^2+b^2},\ \sqrt{(c-s)^2+d^2} \geq |s|/2.$$

Exercise 2.6.2

If $A(s) = a - s$, then $A'(s) = -1$.

If $B(s) = (a-s)^2 = A(s)^2$, then the function of a function rule gives $B'(s) = 2A'(s)A(s) = -2(a-s)$.

If $C(s) = (a-s)^2 + b^2 = B(s) + b^2$, then the addition rule gives $C'(s) = B'(s) + 0 = -2(a-s)$.

If $D(s) = \sqrt{(a-s)^2 + b^2} = \sqrt{C(s)}$, then the function of a function rule gives

$$D'(s) = \frac{C'(s)}{2\sqrt{C(s)}} = -\frac{a-s}{\sqrt{(a-s)^2 + b^2}}.$$

The result now follows.

Exercise 2.6.3

The time taken is

$$f(x) = \frac{1}{u}\big(\sqrt{(a-x)^2+b^2} + \sqrt{(p-x)^2+q^2}\big).$$

We observe that $f(x)$ is large when $|x|$ is large.

Now

$$f'(x) = \frac{-1}{u}\left(\frac{a-x}{\sqrt{(a-x)^2+b^2}} + \frac{p-x}{\sqrt{(p-x)^2+q^2}}\right).$$

We observe that $f'(x) < 0$ when x is large and negative and $f'(x) > 0$ when x is large and positive.

If

$$g(x) = \frac{a-x}{\sqrt{(a-x)^2+b^2}},$$

then, by the product rule,

$$g'(x) = \frac{-1}{\sqrt{(a-x)^2+b^2}} + \frac{(a-x)^2}{((a-x)^2+b^2)^{3/2}} \quad = \frac{-b^2}{((a-x)^2+b^2)^{3/2}},$$

so

$$f''(x) = \frac{1}{u}\left(\frac{b^2}{((a-x)^2+b^2)^{3/2}} + \frac{q^2}{((p-x)^2+q^2)^{3/2}}\right) > 0.$$

Thus f' is strictly increasing.

It follows that f' has a unique zero at z say and this z gives a minimum. Let $X = (z, 0)$.

Since

$$\frac{z-a}{\sqrt{(z-x)^2+b^2}} = \frac{p-z}{\sqrt{(p-z)^2+q^2}},$$

elementary trigonometry shows that that AX and XC make equal angles with the brook.

Exercise 2.6.4

If one side has length x, the adjacent sides have length $2a-x$. Thus

$$f(x) = x(2a-x) = 2ax - x^2.$$

Since $f'(x) = 2(a-x)$ we have $f'(a) = 0$, $f'(x) > 0$ for $0 \leq x < a$ and $f'(x) < 0$ for $a < x \leq 2a$. Thus $f(x)$ has a global maximum at $x = a$ and nowhere else.

In other words, among all rectangles of given perimeter, the squares have greatest area.

Remark Completing the square gives a non-calculus proof. Observe that

$$f(x) = x(2a-x) = -(a-x)^2 + a^2 \leq a^2$$

with equality when $x = a$.

Exercise 2.6.5

In the notation of Example 1.2.6,

$$b = 2s - a - c,$$

so the area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-c)(s-a)(a+c-s)}.$$

Thus the area is maximised when $f(s) = (s-a)(a+c-s)$ is maximised. Since

$$f(a) = -a^2 + (2s-c)a - s^2 + sc,$$

we have $f'(a) = -2a + (2s-c)$, $f'(s - \frac{1}{2}c) = 0$, $f'(a) > 0$ for $0 \leq a < s - \frac{1}{2}c$ and $f'(a) < 0$ for $s - \frac{1}{2}c < a \leq 2s - c$. Thus $f(a)$ has a global maximum at $a = s - \frac{1}{2}c$ (i.e. when $a = b$) and nowhere else. Thus, among the triangles with fixed perimeter $2s$ and one side of fixed length c, the triangles with the other two sides of equal length have greatest area.

Now consider such triangles and allow c to vary. The area of such triangles is

$$\sqrt{s(s-c)(\tfrac{1}{2}c)^2} = \tfrac{1}{2}\sqrt{s(s-c)c^2}.$$

Thus the area is maximised when $g(c) = (s-c)c^2$ is maximised.

Using the product rule

$$g'(c) = -c^2 + 2(s-c)c = (2s-3c)c.$$

Thus $g'(\frac{2}{3}s) = 0$, $g'(c) > 0$ for $0 \leq c < \frac{2}{3}s$ and $g'(c) < 0$ for $\frac{2}{3}s < c \leq s$. Thus $g(c)$ has a global minimum at $c = \frac{2}{3}s$ (i.e. when $a = b = c$) and nowhere else.

Thus, among the triangles with fixed perimeter, the equilateral triangles have greatest area.

[Here again we could have used completing the square rather than differentiation.]

Exercise 2.6.6

(i) We have

$$f(x) = \frac{1}{(n-1)^{n-1}} x(na - x)^{n-1}$$

and

$$\begin{aligned} f'(x) &= \frac{1}{(n-1)^{n-1}} \left((na - x)^{n-1} - (n-1)x(na - x)^{n-2} \right) \\ &= \frac{(na - x)^{n-2}}{(n-1)^{n-1}} (na - x - (n-1)x) \\ &= n\frac{(na - x)^{n-2}}{(n-1)^{n-1}} (a - x). \end{aligned}$$

Thus $f'(a) = 0$, $f'(x) > 0$ for $0 \leq x < a$ and $f'(x) < 0$ for $a < x \leq na$. It follows that $f(x)$ has a global maximum at $x = a$ and nowhere else.

(ii) If $n = 2$, then $f(x)$ is the area of a rectangle sides x, $(2a - x)$ (i.e. perimeter $4a$).

(iii) By (i) with $n = 2$, if z is fixed and $x + y + z = 3a$, $x, y, z \geq 0$, then

$$xy \leq \left(\frac{3a - z}{2} \right)^2$$

with equality only if $x = y = (3a - z)/2$.

Thus if $x + y + z = 3a$, $x, y, z \geq 0$, (i) with $n = 3$, yields

$$xyz \leq z \left(\frac{3a - z}{2} \right)^2 \leq a^3$$

with equality only if $x = y = z = a$.

(iv) Let $P(n)$ be the proposition that, if the $x_j \geq 0$ and

$$x_1 + x_2 + \ldots + x_n = na,$$

then

$$x_1 x_2 \ldots x_n \leq a^n$$

with equality only if $x_1 = x_2 = \ldots = x_n$. We know that $P(2)$ is true.

If $P(n-1)$ is true, then if $x_j \geq 0$ and

$$x_1 + x_2 + \ldots + x_n = na,$$

the inductive hypothesis tells us that

$$x_1 x_2 \ldots x_{n-1} x_n \leq x_n \left(\frac{na - x_n}{n - 1} \right)^{n-1}$$

with equality only if $x_1 = x_2 = \ldots = x_{n-1}$ and part (i) tells us that

$$\left(\frac{na - x_n}{n-1}\right)^{n-1} \leq a^n$$

with equality only if $x_n = a$ so

$$x_1 x_2 \ldots x_{n-1} x_n \leq a^n$$

with equality only if $x_1 = x_2 = \ldots = x_{n-1} = x_n = a$

Thus $P(n)$ is true. The full result follows by induction.

(v) Setting $x_1 + x_2 + \ldots + x_n = na$, we obtain

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \leq (x_1 x_2 \ldots x_n)^{1/n}$$

with equality only if

$$x_1 = x_2 = \ldots = x_n.$$

Exercise 3.1.1

If $g(t) = 1 + t^2$, then $g'(t) = 2t$.

If $G(t) = 1/g(t)$ the quotient rule gives

$$G'(t) = -\frac{g'(t)}{g(t)^2} = -\frac{2t}{(1 + t^2)^2}.$$

Since $F(t) = (-1/2)G(t)$, we have

$$F'(t) = \frac{t}{(1 + t^2)^2}.$$

Exercise 3.1.2★

Exercise 3.1.3

If $a \geq 1$, $\log a$ is the area under $y = f(x)$ between $x = 1$ and $x = a$.

If $0 < a \leq 1$, $\log a$ is minus the area under $y = f(x)$ between $x = a$ and $x = 1$.

Exercise 3.1.4

(i) $\log 1 + \log 1 = \log 1$, so $\log 1 = 0$.

$\log a + \log a^{-1} = \log aa^{-1} = \log 1 = 0$, so $\log a = -\log(1/a)$.

(ii) We have

$$\log 1 = \int_1^1 \frac{1}{t}\, dt = 0$$

since $\int_a^a f(t)\, dt = 0$ automatically.

Let $f(x) = G(x) = 1/x$ in the 'change of variable formula' on page 19. Then

$$\begin{aligned}\log(1/a) &= \int_1^{1/a} \frac{1}{t}\, dt = \int_{G(1)}^{G(a)} f(y)\, dy \\ &= \int_1^a f\big(G(x)\big)G'(x)\, dx = \int_1^a x\frac{-1}{x^2}\, dx \\ &= -\int_1^a \frac{1}{x}\, dx = -\log a.\end{aligned}$$

Exercise 3.1.5

Note that $\log' x = 1/x > 0$, so log is strictly increasing.

(i) We have

$$\log x = \int_1^x \frac{1}{t}\,dt \geq \int_1^x \frac{1}{2}\,dt = \frac{x-1}{2}.$$

(ii) We have

$$2^{-1} \leq t^{-1}$$

for $1 \leq t \leq 2$, so

$$\frac{1}{2} = \int_1^2 \frac{1}{2}\,dt \leq \int_1^2 \frac{1}{t}\,dt = \log 2$$

(iii) $\log 2^n = \log 2 + \log 2^{n-1} = \ldots = n \log 2 \geq \frac{n}{2}$, so $\log 2^n > K$ whenever $n \geq 2K$.

(iv) $\log 2^{-n} = -\log 2^n \leq -K$ whenever $n \geq 2K$.

Exercise 3.1.6

By previous results, we know that, given K, we can find an Q such that

$$\log x \geq K$$

when $x \geq Q$.

Now we can find a L such that

$$\log x \geq Q$$

for $x \geq L$.

Thus $\log \log x \geq \log Q \geq M$ when $x \geq L$,

Experiment shows that $K = 10^{24}$ will do but $K = 10^{23}$ will not.

(In the next section we see that $\log \log \exp \exp 5 = 5$.)

Exercise 3.1.7

Observe that

$$f(x) = \frac{1}{2a}(\log(a+x) - \log(a-x))$$

so, using the function of a function rule,

$$\begin{aligned} f'(x) &= \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right) \\ &= \frac{1}{2a} \times \frac{(a-x)+(a+x)}{a^2-x^2} \\ &= \frac{1}{a^2-x^2}. \end{aligned}$$

EXERCISE 3.2.1

(i) If a, $b > 0$, then

$$E(\log a + \log b) = E(\log ab) = ab = E(\log a)E(\log b),$$

so, setting $a = E(x)$, $b = E(y)$, we have

$$E(x + y) = E(x)E(y).$$

(ii) $E(0) = E(\log 1) = 1$, so $E(x)E(-x) = E(x-x) = E(0) = 1$ and $1/E(x) = E(-x)$.

(iii) There are lots of ways. Probably best to look at definition, but we could also note that $E(x) = E(x/2)^2 \geq 0$ and (by (ii)) $E(x) \neq 0$.

(iv) $E'(x) = E(x) > 0$, so E is strictly increasing.

(v) If $x > \log a$, then $E(x) > E(\log a) = a$. If $\log b > x > 0$, then $b = E(\log b) > \log x$.

Exercise 3.2.2

If $f(x) = x^{-n} \exp x$, then, using the product rule,

$$f'(x) = -nx^{-n-1} \exp x + x^{-n} \exp x = x^{-n-1}(x - n) \exp x > 0$$

when $x > n$, so $f(x) = x^{-n} \exp x$ is an increasing function of x for $x \geq n$.

Exercise 3.2.3

Let

$$f(y) = ay^2 \log \frac{1}{y} = -ay^2 \log y.$$

Then, using the product rule,

$$f'(y) = -a\left(2y\log y + \frac{y^2}{y}\right) = -ay(2\log y + 1)$$

so $f'(\exp(-1/2)) = 0$, $f'(y) > 0$ for $0 < y < \exp(-1/2)$ and $f'(y) < 0$ for $\exp(-1/2) < y$. Thus $f(y)$ has a unique maximum at $y = \exp(-1/2)$.

Exercise 3.2.4

(i) Observe that

$$\begin{aligned} n\log x &= \underbrace{\log x + \log x + \ldots + \log x}_{n} \\ &= \log(\underbrace{x \times x \times \ldots \times x}_{n}) \\ &= \log x^n \end{aligned}$$

for $n \geq 1$.

We have also

$$0\log x = 0 = \log 1 = \log x^0.$$

(ii) If $n \geq 0$ then

$$(-n)\log x = -(n\log x) = -\log x^n = \log(1/x^n) = \log x^{-n}.$$

Thus $n\log x = \log x^n$ for all integers n.

(iii) Using part (ii),

$$q\log x^{p/q} = \log(x^{p/q})^q = \log x^p = p\log x$$

and so

$$\frac{p}{q}\log x = \log x^{p/q}.$$

Thus

$$\exp\left(\frac{p}{q}\log x\right) = \exp(\log x^{p/q}) = x^{p/q}.$$

Exercise 3.2.5

(i) Observe that

$$\begin{aligned} x^{a+b} &= \exp\big((a+b)\log x\big) = \exp(a\log x + b\log x)\\ &= \exp(a\log x)\exp(b\log x) = x^a x^b. \end{aligned}$$

(ii) Observe that

$$\begin{aligned} (xy)^a &= \exp\big(a\log(xy)\big) = \exp(a\log x + a\log y)\\ &= \exp(a\log x)\exp(a\log y) = x^a y^a. \end{aligned}$$

(iii) Observe that

$$\log(x^a)^b = b\log x^a = ba\log x = ab\log x = \log x^{ab},$$

so that

$$(x^a)^b = \exp\log(x^a)^b = \exp\log x^{ab} = x^{ab}.$$

(iv) $e^a = \exp(a\log e) = \exp a.$

(v) If $f(t) = t^a$, then

$$f(t) = \exp(a\log t)$$

and, using the function of a function rule,

$$f'(t) = \frac{a}{t}\exp(a\log t) = at^{-1}t^a = at^{a-1}.$$

Exercise 3.2.6

Observe that, if $x \geq \exp(\exp(\exp 4))$ then (since log is an increasing function)

$$\begin{aligned}\log(\log(\log x)) &\geq \log(\log(\log(\exp(\exp(\exp 4)))))\\ &= \log(\log(\exp(\exp 4))) = \log(\exp 4) = 4\end{aligned}$$

Similarly if $x < \exp(\exp(\exp 4))$, then $\log\log\log x < 4$ (or, if x is small, $\log\log\log x$ is not even defined).

My calculator gives

$$\exp\exp 4 > 5 \times 10^{23}$$

so

$$\begin{aligned}\exp\exp\exp 4 &> \exp(5 \times 10^{23}) = e^{5\times 10^{23}}\\ &= (e^5)^{10^{23}} \geq 100^{10^{23}} = 10^{2\times 10^{23}}.\end{aligned}$$

Thus M has at least 2×10^{23} figures.

70 years contain roughly 2.2×10^9 seconds. Thus M is far too large to be written down in a lifetime.

Exercise 3.2.7

We have $g(t) = f(h(t))$ with $f(t) = \exp t$, $h(t) = (\log a)t$, so

$$g'(t) = h'(t)f'(h(t)) = (\log a)f(h(t)) = (\log a)a^t.$$

Exercise 3.2.8

If $g(t) = t^{1/t}$, then $g(t) = f(h(t))$ with $f(t) = \exp t$, $h(t) = t^{-1}(\log t)$, so

$$g'(t) = h'(t)f'(h(t)) = (t^{-2} - t^{-2}\log t)f(h(t)) = t^{-2}(1 - \log t)t^{1/t}.$$

Thus $g'(e) = 0$, $g'(y) > 0$ for $0 < y < e$ and $g'(y) < 0$ for $y > e$. Thus $g(t)$ is strictly increasing as t increases up to a maximum at $t = e$ and thereafter is strictly decreasing.

Thus, if $g(a) = g(b)$ with $a < b$, we have $a < e < b$. Since $2 < e < 3$, we must have $a \leq 2$, $b \geq 3$.

If $n^m = m^n$ with $n > m$, then $m \log n = n \log m$ and $g(n) = g(m)$. Thus m is a positive integer with $m \leq 2$. By inspection $m = 1$ is not a solution, but $m = 2$ is a solution ($2^4 = 4^2$). Thus there is exactly one solution.

Exercise 3.2.9★

Exercise 3.2.10

Let $f(t) = \log t$ and $g(t) = t$.

We have

$$\begin{aligned}
\int_1^n \log t \, dt &= \int_1^n f(t)g'(t) \, dt \\
&= [f(t)g(t)]_1^n - \int_1^n f'(t)g(t) \, dt \\
&= [t \log t]_1^n - \int_1^n \frac{1}{t} \times t \, dt \\
&= n \log n - \int_1^n 1 \, dt \\
&= n \log n - (n-1).
\end{aligned}$$

Exercise 3.2.11

(i) Observe that

$$\begin{aligned}\int_1^{n+1} \log t\,dt &= \int_1^n \log t\,dt + \int_n^{n+1} \log t\,dt\\ &\leq \int_1^n \log t\,dt + \int_n^{n+1} \log(n+1)\,dt = \int_1^n \log t\,dt + \log(n+1).\end{aligned}$$

(ii) Recall that

$$\log n! \leq \int_1^{n+1} \log t\,dt \leq \log(n+1)!$$

so

$$\int_1^n \log t\,dt \leq \log n!$$

and, using (i),

$$\int_1^n \log t\,dt \leq \log n! \leq \int_1^{n+1} \log t\,dt \leq \int_1^n \log t\,dt + \log(n+1).$$

Exercise 3.2.10 now gives

$$(n\log n) - (n-1) \leq \log n! \leq \big(n\log n) - (n-1) + \log(n+1).$$

(iii) I get

$$6908.76 \leq \log 1000! \leq 6913.37.$$

(iv) Observe that, between 1 and 1000 inclusive, there are

200 integers divisible by 5
40 integers divisible by 5^2
8 integers divisible by 5^3
1 integer divisible by 5^4
0 integers divisible by 5^5.

Thus 1000! is divisible by 5^N with

$$N = 200 + 40 + 8 + 1 = 249,$$

but not by 5^{N+1}.

Between 1 and 1000 inclusive, there are 500 integers divisible by 2, so 1000! is divisible by 2^{500}. Thus 1000! is divisible by 10^{249}, but not by 10^{250}.

Thus 1000! ends in exactly 249 zeros.

(v) We have

$$\frac{\log 1000!}{\log 10} \approx 2957$$

so 1000! has about 2957 digits.

(vi) Applying exp to the inequality in (ii) yields

$$n^n e^{1-n} \leq n! \leq (n+1) n^n e^{1-n}.$$

Exercise 3.2.12

(i) It is plausible that using the value of f at a mid-point rather than one of the end points will give a better idea of the behaviour of f on a given interval.

(ii) If
$$f(t) = 2\log r - \big(\log(r-t) + \log(r+t)\big),$$
then
$$f'(t) = 0 - \left(-\frac{1}{r-t} + \frac{1}{r+t}\right) = \frac{1}{r-t} - \frac{1}{r+t}$$
and
$$f''(t) = \frac{1}{(r-t)^2} + \frac{1}{(r+t)^2} \geq 0$$
for $0 \leq t \leq 1/2$.

Thus $f'(t)$ is increasing as t increases from 0 to 1/2 so, since $f'(0) = 0$, $f'(t) \geq 0$ for $0 \leq t \leq 1/2$. Thus
$$0 \leq f'(t) \leq f'(1/2)$$
and, since $f(0) = 0$, the mean value inequality gives
$$0 \leq f(t) \leq \frac{1}{2}\left(\frac{1}{r-\frac{1}{2}} - \frac{1}{r+\frac{1}{2}}\right)$$
for $0 \leq t \leq 1/2$.

(iii) Now
$$\begin{aligned}\int_0^{1/2} f(t)\,dt &= \int_0^{1/2} 2\log r - \big(\log(r-t) + \log(r+t)\big)\,dt\\ &= \log r - \int_0^{1/2} \log(r-t)\,dt - \int_0^{1/2} \log(r+t)\,dt\\ &= \log r - \int_{r-1/2}^{r} \log t\,dt - \int_r^{r+1/2} \log t\,dt\\ &= \log r - \int_{r-1/2}^{r+1/2} \log t\,dt\end{aligned}$$
and, by (ii),
$$0 \leq \int_0^{1/2} f(t)\,dt \leq \int_0^{1/2} \frac{1}{2}\left(\frac{1}{r-\frac{1}{2}} - \frac{1}{r+\frac{1}{2}}\right)\,dt = \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right).$$
Thus
$$0 \leq \log r - \int_{r-1/2}^{r+1/2} \log t\,dt \leq \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

(iv) Adding these inequalities yields

$$0 \leq \log n! - \int_{1/2}^{n+1/2} \log t\, dt \leq \frac{1}{2}.$$

Now the argument of Exercise 3.2.10 shows that

$$\int_{1/2}^{n+1/2} \log t\, dt = (n + \tfrac{1}{2}) \log(n + \tfrac{1}{2}) - \tfrac{1}{2} \log \tfrac{1}{2} - n,$$

so

$$(n+\tfrac{1}{2}) \log(n+\tfrac{1}{2}) - \tfrac{1}{2} \log \tfrac{1}{2} - n \leq \log n! \leq (n+\tfrac{1}{2}) \log(n+\tfrac{1}{2}) - \tfrac{1}{2} \log \tfrac{1}{2} - n + \tfrac{1}{2}$$

and

$$2^{-1/2} e^{-n} (n + \tfrac{1}{2})^{(n+\frac{1}{2})} \leq n! \leq e^{1/2} 2^{-1/2} e^{-n} (n + \tfrac{1}{2})^{(n+\frac{1}{2})}.$$

(v) The same arguments give

$$0 \leq \log n! - \log m! - \int_{m-1/2}^{n+1/2} \log t\, dt \leq \frac{1}{(m - 1/2)(m + 1/2)}$$

so

$$\begin{aligned} &(n + \tfrac{1}{2}) \log(n + \tfrac{1}{2}) - (m - \tfrac{1}{2}) \log(m - \tfrac{1}{2}) + m - n \\ &\qquad \leq \log n! - \log m! \\ &\qquad \leq (n + \tfrac{1}{2}) \log(n + \tfrac{1}{2}) - (m - \tfrac{1}{2}) \log(m - \tfrac{1}{2}) + m - n + \frac{1}{m - \frac{1}{2}} \end{aligned}$$

and

$$m! e^{m-n} (n+\tfrac{1}{2})^{(n+\frac{1}{2})} (m-\tfrac{1}{2})^{-(m-\frac{1}{2})} \leq n! \leq e^{1/(m-\frac{1}{2})} m! e^{m-n} (n+\tfrac{1}{2})^{(n+\frac{1}{2})} (m-\tfrac{1}{2})^{-(m-\frac{1}{2})}.$$

Exercise 3.2.13

(i) If $0 \geq a > -1$, then
$$(r-1)^a \geq t^a \geq r^a$$
for $r-1 \leq t \leq r$, so
$$\int_{r-1}^{r} (r-1)^a \, dt \geq \int_{r-1}^{r} t^a \, dt \geq \int_{r-1}^{r} r^a \, dt$$
and so
$$(r-1)^a \geq \int_{r-1}^{r} t^a \, dt \geq r^a.$$
Summing these inequalities, we obtain
$$1^a + 2^a + \ldots + (n-1)^a \geq \int_{1}^{n} t^a \, dt \geq 2^a + \ldots + n^a$$
so
$$1^a + 2^a + \ldots + (n-1)^a \geq \frac{n^{a+1}-1}{a+1} \geq 2^a + \ldots + n^a$$
and
$$\left| (1^a + 2^a + \ldots + n^a) - \frac{n^{a+1}}{a+1} \right| \leq \frac{1}{a+1}.$$

(ii) Similarly,
$$(r-1)^{-1} \geq \int_{r-1}^{r} t^{-1} \, dt \geq r^{-1}.$$
so
$$1^{-1} + 2^{-1} + \ldots + (n-1)^{-1} \geq \int_{1}^{n} t^{-1} \, dt \geq 2^{-1} + \ldots + n^{-1}$$
and
$$1^{-1} + 2^{-1} + \ldots + (n-1)^{-1} \geq \log n \geq 2^{-1} + \ldots + n^{-1}$$
whence
$$\left| (1^{-1} + 2^{-1} + \ldots + n^{-1}) - \log n \right| \leq 1.$$

(iii) If $a \geq 0$,
$$(r-1)^a \leq t^a \leq r^a$$
for $r-1 \leq t \leq r$ so
$$\int_{r-1}^{r} (r-1)^a \, dt \leq \int_{r-1}^{r} t^a \, dt \leq \int_{r-1}^{r} r^a \, dt$$
and so
$$(r-1)^a \leq \int_{r-1}^{r} t^a \, dt \leq r^a.$$
Summing these inequalities, we obtain
$$1^a + 2^a + \ldots + (n-1)^a \leq \int_{1}^{n} t^a \, dt \leq 2^a + \ldots + n^a$$

so

$$1^a + 2^a + \ldots + (n-1)^a \leq \frac{n^{a+1} - 1}{a+1} \leq 2^a + \ldots + n^a$$

and

$$\left| (1^a + 2^a + \ldots + n^a) - \frac{n^{a+1}}{a+1} \right| \leq n^a.$$

(iv) If $-1 > a$, the result of (i) remains true, but the inequality does not improve as we increase n.

Exercise 3.3.1

If an angle is θ in the A system, then it is $k_B k_A^{-1}\theta$ in the B system. Thus, taking $\theta = k_A t$, we have

$$\sin_A k_A t = \sin_B k_B t.$$

Exercise 3.3.2

(i) We have an isosceles triangle with base angle T_N and side 1, so altitude $\cos_R T_N/2$ and base $2\sin_R T_N/2$, so area

$$\frac{1}{2} \times 2\sin_R(T_N/2)\cos_R(T_N/2) = \frac{1}{2} \times \sin_R T_N.$$

(ii) We have N equal angles adding up to 4 right angles so $T_N = 4k_R/N$ and the total area

$$P_N = \frac{N}{2} \times \sin_R(4k_R/N).$$

(iii) We have

$$\sin_R \delta t = \sin_R'(0)\delta t + o(\delta t) = \delta t + o(\delta t).$$

so

$$P_N = \frac{N}{2} \times \left(\frac{4k_R}{N} + o\left(\frac{4k_R}{N}\right)\right) = 2k_R + o(1)$$

(where $o(1)$ represents a quantity which becomes arbitrarily small provided N is large enough).

Thus P_N gets arbitrarily close to $2k_R$ as N increases.

(iv) The area of our polygon becomes arbitrarily close to that of our circle i.e. arbitrarily close to π. Thus $2k_R = \pi$.

Exercise 3.3.3

(i) We have already seen that X_0X_1 has length

$$2\sin(T_N/2) = 2\sin(2k_R/N),$$

so the perimeter of the polygon is

$$2N\sin(2k_R/N).$$

For large N, the perimeter is close to the length of the circle, so

$$2\pi \approx 2N\sin(2k_R/N) \approx 2N \times \frac{2k_R}{N} \approx 4k_R,$$

with the approximation improving as N gets larger. Thus

$$2\pi = 4k_R$$

and $k_R = \pi/2$.

(ii) There are several ways of seeing this. One is to place points X_0, X_1, $\ldots X_N$ at equal distances along the arc. Then

$$\begin{aligned}\text{length arc} &\approx N \times \text{length line segment } X_0X_1\\ &= N(2a\sin\big(\theta/(2N)\big) \approx N(2a\big(\theta/(2N)\big) = a\theta\end{aligned}$$

The approximation improves as N increases, so the length of the arc of the circle is $a\theta$.

Exercise 3.3.4

We will need the following remark. If
$$f(x) = \sqrt{1-x},$$
then $f(0)=1$ and
$$f'(x) = -\frac{1}{2} \times \frac{1}{\sqrt{1-x}}$$
so $f'(0) = -1/2$ and
$$\sqrt{1-\delta x} = f(\delta x) = f(0) + f'(0)\delta x = 1 - \tfrac{1}{2}\delta x$$
to first order.

Let O be the centre of the earth, A the top of the lighthouse, B the crow's nest and X the point where the straight line AB touches the surface of the earth. Writing $|AX|$ for the length of AX, we have (since $\angle OXA$ is a right angle)
$$|OA|^2 = |OX|^2 + |XA|^2$$
and (taking R to be radius of the earth)
$$\begin{aligned}|AX| &= \sqrt{(R+h)^2 - R^2} = \sqrt{2Rh - h^2}\\ &= 2^{1/2}R^{1/2}h^{1/2}\sqrt{1 - h/(2R)} = 2^{1/2}R^{1/2}h^{1/2}(1 - h/(4R) + o(h/R)).\end{aligned}$$
Similarly
$$|BX| = 2^{1/2}R^{1/2}H^{1/2}(1 - h/(4r) + o(H/R))$$
so
$$|AB| = |AX| + |BX| = 2^{1/2}R^{1/2}(h^{1/2} + H^{1/2})$$
to the zeroth order and so (since h/R and H/R very small indeed) the distance required is $2^{1/2}R^{1/2}(h^{1/2} + H^{1/2})$.

(ii) Observe that quadrupling the height of the towers only doubles the viewing distance (and so only halves the number of towers required) so it will be cheaper to build fairly short towers rather than very tall ones.

Exercise 3.3.5

(i) Let $g(t) = \sin t$, $h(t) = t + \pi/2$ so that $\cos t = g(h(t))$. By the function of a function rule,

$$\cos' t = h'(t)g'(h(t)) = \cos(t + \pi/2) = -\sin t.$$

(ii) We have $\tan t = \sin t/\cos t$, so, by the product and quotient rules,

$$\begin{aligned}\tan' t &= \frac{\sin' t}{\cos t} - \frac{\sin t \cos' t}{(\cos t)^2} = \frac{\cos t}{\cos t} - \frac{-(\sin t)^2}{(\cos t)^2} \\ &= 1 + \frac{(\sin t)^2}{(\cos t)^2} = \frac{(\cos t)^2 + (\sin t)^2}{(\cos t)^2} = \frac{1}{(\cos t)^2} \\ &= (\sec t)^2.\end{aligned}$$

Since $\cot t = -\tan(t + \pi/2)$, we have

$$\cot' t = -\big(\sec(t + \pi/2)\big)^2 = -(\operatorname{cosec} t)^2.$$

Since $\operatorname{cosec} t = 1/\sin t$, the quotient rule gives

$$\operatorname{cosec}'(t) = -\frac{\sin' t}{(\sin t)^2} = -\frac{\cos t}{(\sin t)^2} = -\cot t \operatorname{cosec} t.$$

Since $\sec t = \operatorname{cosec}(t + \pi/2)$, we have

$$\sec' t = -\cot(t + \pi/2)\operatorname{cosec}(t + \pi/2) = \tan t \sec t.$$

Exercise 3.3.6

The time taken is

$$\begin{aligned} f(\theta) &= \text{time running} + \text{time swimming} \\ &= \frac{\text{distance run}}{u} + \frac{\text{distance swum}}{v} \\ &= \frac{\text{distance run}}{u} + \frac{2a\sin\big((\pi-\theta)/2\big)}{v} \\ &= \frac{a\theta}{u} + \frac{2a\cos(\theta/2)}{v}. \end{aligned}$$

Now

$$f'(\theta) = \frac{a}{u} - \frac{a}{v}\sin\theta/2$$

so $f'(\theta)$ decreases from $\frac{a}{u}$ to $\frac{a}{u} - \frac{a}{v}$ as θ goes from 0 to π.

If $u \leq v$, then $f'(\theta) \geq 0$ for all $0 \leq \theta \leq \pi$ and f increases as θ increases, so she should dive in at once.

If $u > v$, then there is a θ_0 such that $f'(\theta) > 0$ for $0 \leq \theta < \theta_0$ and $f'(\theta) < 0$ for $\theta_0 < \theta \leq \pi$. Thus f attains its global maximum at an end point. If $u > \pi v$, she should run all the way, if $u < \pi v$, she should swim all the way. If $u = \pi v$, she should do one or the other.

Exercise 3.3.7

(i) $\sin' x = \cos x > 0$ for $-\pi/2 < x < \pi/2$, so $\sin x$ is strictly increasing as x runs from $-\pi/2$ to $\pi/2$.

Since $-\pi/2 \leq \sin^{-1} y \leq \pi/2$,

$$\cos(\sin^{-1} y) \geq 0$$

so, since

$$\big(\cos(\sin^{-1} y)\big)^2 = 1 - \big(\sin(\sin^{-1} y)\big)^2 = 1 - y^2,$$

we have

$$\cos(\sin^{-1} y) = \sqrt{1-y^2}.$$

By the inverse function rule,

$$(\sin^{-1})'(y) = \frac{1}{\sin'(\sin^{-1} y)} = \frac{1}{\cos(\sin^{-1} y)} = \frac{1}{\sqrt{1-y^2}}.$$

(ii) $\cos' x = -\sin x < 0$ for $0 < x < \pi$, so $\cos x$ is a strictly decreasing as x runs from 0 to π.

Since $0 \leq \cos^{-1} y \leq \pi/2$, it follows that

$$\sin(\cos^{-1} y) \geq 0$$

so, since

$$\big(\sin(\cos^{-1} y)\big)^2 = 1 - \big(\cos(\cos^{-1} y)\big)^2 = 1 - y^2,$$

we have

$$\sin(\cos^{-1} y) = \sqrt{1-y^2}.$$

By the inverse function rule,

$$(\cos^{-1})'(y) = \frac{1}{\cos'(\cos^{-1} y)} = \frac{-1}{\sin(\cos^{-1} y)} = \frac{-1}{\sqrt{1-y^2}}.$$

(iii) $\tan'(x) = \operatorname{cosec}^2 x$, so $\tan x$ is a strictly increasing as x runs from $-\pi/2$ to $\pi/2$.

We observe that

$$\frac{1}{1+(\tan x)^2} = \frac{(\cos x)^2}{(\cos x)^2 + (\sin x)^2} = (\cos x)^2$$

so that

$$\frac{1}{1+y^2} = \frac{1}{1+\big(\tan(\tan^{-1} y)\big)^2} = \big(\cos(\tan^{-1} y)\big)^2.$$

By the inverse function rule,

$$(\tan^{-1})'(y) = \frac{1}{\tan'(\tan^{-1} y)} = \big(\cos(\tan^{-1} y)\big)^2 = \frac{1}{1+y^2}.$$

Exercise 3.3.8

(Parts (i) to (iii) only.)

(i) We have

$$\cosh(-x) = \frac{\exp(-x) + \exp(x)}{2} = \cosh x$$

and

$$\sinh(-x) = \frac{\exp(-x) - \exp(x)}{2} = -\sinh x.$$

(ii) We have

$$\cosh'(x) = \frac{\exp(x) - \exp(-x)}{2} = \sinh x$$

and

$$\sinh'(x) = \frac{\exp(x) + \exp(-x)}{2} = \cosh x.$$

(iii) $\sinh' x = \cosh x \geq 1/2 > 0$ for all x, so $\sinh x$ is strictly increasing as x increases.

$\cosh' x = \sinh x > 0$ for all $x > 0$, so $\cosh x$ is strictly increasing as x increases from 0.

Exercise 3.3.9

(i) If $f(t) = m^{-1}\sin mt$, then $f'(t) = \cos mt$. Thus

$$\int_a^b \cos mt\,dt = \int_a^b f'(t)\,dt = [f(t)]_a^b = \frac{\sin mb - \sin ma}{m}.$$

(ii) We have

$$\begin{aligned}\cos(u+v) + \cos(u-v) &= \cos u\cos v - \sin u\sin v + \cos u\cos v + \sin u\sin v\\ &= 2\cos u\cos v.\end{aligned}$$

(iii) We have, using (ii),

$$\int_{-\pi}^{\pi} \cos nx\cos mx\,dx = \frac{1}{2}\left(\int_{-\pi}^{\pi} \cos(n+m)x\,dx + \int_{-\pi}^{\pi} \cos(n-m)x\,dx\right)$$

By (i),

$$\int_{-\pi}^{\pi} \cos qt\,dt = \frac{\sin(q\pi) - \sin(-q\pi)}{q} = \frac{0-0}{q} = 0$$

whenever q is a non-zero integer.

Thus if $n \neq m$,

$$\int_{-\pi}^{\pi} \cos nx\cos mx\,dx = 0.$$

If $n = m \neq 0$,

$$\int_{-\pi}^{\pi} \cos nx\cos mx\,dx = \frac{1}{2}\left(\int_{-\pi}^{\pi} \cos(2nx)\,dx + \int_{-\pi}^{\pi} 1\,dx\right) = \frac{1}{2}(0+2\pi) = \pi$$

and, if $n = m = 0$,

$$\int_{-\pi}^{\pi} \cos nx\cos mx\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

Exercise 4.1.1

$$y(t) = -\frac{gt^2}{2},$$

so

$$\begin{aligned}
\frac{y\big((N+1)T\big) - y(NT)}{y\big((M+1)T\big) - y(MT)} &= \frac{\big((N+1)T\big)^2 - (NT)^2}{\big((M+1)T\big)^2 - (MT)^2} \\
&= \frac{(N+1)^2 - N^2}{(M+1)^2 - M^2} \\
&= \frac{2N+1}{2M+1}.
\end{aligned}$$

Exercise 4.1.2

(i) If

$$y''(t) = P(t),$$

then, writing

$$z(t) = y'(t) - \left(a_0 t + \frac{a_1}{2}t^2 + \ldots + \frac{a_n}{n+1}t^{n+1}\right),$$

we have

$$z'(t) = 0,$$

so $z(t) = c_1$ for some constant c_1 and, writing

$$w(t) = y(t) - \left(c_1 t + a_0\frac{t^2}{2\times 1} + a_1\frac{t^3}{3\times 2} + \ldots + a_n\frac{t^{n+2}}{(n+2)\times(n+1)}\right),$$

we have

$$w'(t) = z(t) - c_1 = 0,$$

so $w(t) = c_0$ for some constant c_0 and

$$y(t) = c_0 + c_1 t + a_0\frac{t^2}{2\times 1} + a_1\frac{t^3}{3\times 2} + \ldots + a_n\frac{t^{n+2}}{(n+2)\times(n+1)}.$$

(ii) If

$$y'''(t) = P(t),$$

then, by part (i),

$$y'(t) = c_1 + 2c_2 t + a_0\frac{t^2}{2\times 1} + a_1\frac{t^3}{3\times 2} + \ldots + a_n\frac{t^{n+2}}{(n+2)\times(n+1)},$$

for some constants c_1, c_2.

Thus writing

$$u(t) = y(t) - \left(c_1 t + c_2 t^2 + a_0\frac{t^3}{3\times 2\times 1} + a_1\frac{t^4}{4\times 3\times 2} + \ldots + a_n\frac{t^{n+3}}{(n+3)\times(n+2)\times(n+1)}\right)$$

we have

$$u'(t) = 0,$$

so $u(t) = c_0$ for some constant c_0 and

$$y(t) = c_0 + c_1 t + c_2 t^2 + a_0\frac{t^3}{3\times 2\times 1} + a_1\frac{t^4}{4\times 3\times 2} + \ldots + a_n\frac{t^{n+3}}{(n+3)\times(n+2)\times(n+1)}.$$

Exercise 4.1.3

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{V\sin\theta}{V\cos\theta} = \frac{v}{u}.$$

Exercise 4.1.4

Completion of the square with $a > 0$ gives

$$-as^2 + bs = -(a^{1/2}s - b/(2a^{1/2}))^2 + b^2/(4a),$$

so

$$y = -\left(\frac{g^{1/2}\sec\theta}{2^{1/2}V}x - \frac{V\sin\theta}{2^{1/2}g^{1/2}}\right)^2 + \frac{V^2\sin^2\theta}{2g}.$$

Since squares are positive, the maximum of $-(a^{1/2}s - b/(2a^{1/2}))^2 + b^2/(4a)$ occurs when

$$a^{1/2}s - b/(2a^{1/2}) = 0$$

(i.e. $s = b/(2a)$) and is $b^2/(4a)$.

Thus the highest point on the trajectory is $V^2\sin^2/(2g)$ above its initial height and the projectile travelled a horizontal distance

$$x = \tan\theta \times \frac{V^2}{g\sec^2\theta} = \cos\theta\sin\theta \times \frac{V^2}{g} = \frac{V^2}{g}\sin 2\theta.$$

(This is, as we shall see, half the total range.)

If $V\sin\theta < 0$, we must be firing from the top of a cliff and the particle is at highest point when it is fired.

Exercise 4.1.5

$$y = Y - b = CX^2 - b = C(x + a)^2.$$

If $C \neq 0$, $Y = CX^2$ is a parabola, so $y = C(x + A)^2 + B$ is a parabola with respect to different axes. Thus equation ★★ is the equation of a parabola.

Exercise 4.2.1

Set $f(t) = A \exp t$, $g(t) = -kt$. If $u(t) = f(g(t))$, the function of a function rule yields

$$u'(t) = g'(t)f'(g(t)) = -ku(t).$$

If $w(t) = (\exp kt)u(t)$, the product rule gives

$$w'(t) = k(\exp kt)u(t) + (\exp kt)u'(t) = (\exp kt)(u'(t) + ku(t)) = 0.$$

Exercise 4.2.2

(i) Using the fundamental theorem of the calculus,

$$\begin{aligned} f'(t) &= -a\exp(-at)g(t) + \exp(-at)g'(t) - \exp(-at)h(t) \\ &= \exp(-at)(g'(t) - ag(t) - h(t)) = 0, \end{aligned}$$

so $f(t) = A$ a constant and so

$$A = \exp(-at)g(t) - \int_0^t \exp(-ax)h(x)\,dx,$$

whence

$$g(t) = A\exp at + (\exp at)\int_0^t \exp(-ax)h(x)\,dx.$$

(ii) If $h(t) = C$, we have

$$\begin{aligned} g(t) &= A\exp at + \exp at\int_0^t C\exp(-ax)\,dx \\ &= A\exp at + \left[-\frac{C}{a}\exp(-ax)\right]_0^t \exp at \\ &= A\exp at + \frac{C}{a}\big(1 - \exp(-at)\big)\exp at \\ &= B\exp at - \frac{C}{a} \end{aligned}$$

for some constant B.

If $a = 0$, the fundamental theorem of the calculus tells us that

$$g(t) = Ct + A$$

for some constant A.

If $h(t) = C\exp bt$ for all t and $a \neq b$, then

$$\begin{aligned} g(t) &= A\exp at + (\exp at)\int_0^t C\exp(-ax)\exp(bx)\,dx \\ &= A\exp at + \exp at\int_0^t C\exp\big((b-a)x\big)\,dx \\ &= A\exp at + \left[\frac{C}{b-a}\exp\big((b-a)x\big)\right]_0^t \exp at \\ &= A\exp at + \frac{C}{b-a}\Big(\exp\big((b-a)t\big) - 1\Big)\exp at \\ &= B\exp at + \frac{C}{b-a}\exp bt \end{aligned}$$

for some constant B.

If $h(t) = C\exp at$ for all t,

$$\begin{aligned} g(t) &= A\exp at + (\exp at)\int_0^t C\exp(-ax)\exp(ax)\,dx \\ &= A\exp at + \exp at\int_0^t C\,dx \\ &= A\exp at + Ct\exp at \end{aligned}$$

for some constant A.

(iii) We have

$$u'(t)-bu(t) = (g''(t)-ag'(t))-b(g'(t)-ag(t)) = g''(t)+pg'(t)+qg(t) = 0.$$

Thus

$$u(t) = C\exp bt$$

for some constant C.

It follows that

$$g'(t) - ag(t) = C\exp bt$$

and, by (ii),

$$g(t) = A\exp at + \frac{C}{b-a}\exp bt = A\exp at + B\exp bt$$

for some constants A and B.

(iv) We have

$$u'(t)-au(t) = (g''(t)-ag'(t))-a(g'(t)-ag(t)) = g''(t)-2ag'(t)+a^2g(t) = 0.$$

Thus

$$u(t) = B\exp at$$

for some constant B, so

$$g'(t) - ag(t) = B\exp at$$

and

$$g(t) = (A + Bt)\exp at$$

for some constant A.

(v) We have

$$u'(t)-bu(t) = (g''(t)-ag'(t))-b(g'(t)-ag(t)) = g''(t)+pg'(t)+qg(t) = 1.$$

Thus

$$u(t) = C\exp bt - \frac{1}{b}$$

for some constant C, so

$$g'(t) - ag(t) = C\exp bt - \frac{1}{b}.$$

Thus

$$\begin{aligned} g(t) &= A\exp at + \exp at \int_0^t \exp(-ax)(C\exp bx - b^{-1})\,dx \\ &= A\exp at + B\exp bt + \frac{1}{ab} \\ &= A\exp at + B\exp bt + \frac{1}{q} \end{aligned}$$

for some constants A and B.

(vi) Let $u(t) = g'(t) - ag(t)$ and $v(t) = u'(t) - bu(t)$. Then

$$\begin{aligned} &v'(t) - cv(t) = (u''(t) - bu'(t)) - c(u'(t) - bu(t)) \\ &\quad = (g'''(t) - ag''(t)) - b(g''(t) - ag'(t)) - c(g''(t) - ag'(t)) + cb(g'(t) - ag(t)) \\ &\quad = g'''(t) + pg''(t) + qg'(t) + rg(t) = 0. \end{aligned}$$

Thus $v(t) = A_1\exp(ct)$, so $u(t) = B_1\exp(bt) + B_2\exp(ct)$ and, using the formula of (i) again,

$$g(t) = C_1\exp(at) + C_2\exp(bt) + C_3\exp(ct)$$

where A_1, B_i, C_j are constants.

Exercise 4.3.1

Whatever f we choose,

$$f'(x)^2 \geq 0 > -1 - x^2.$$

Exercise 4.3.2

If $u(t) = 0$, then

$$u'(t) = 0 = 3 \times 0^{2/3} = 3 \times u(t)^{2/3}.$$

If $v(t) = t^3$, then

$$v'(t) = 3t^2 = 3 \times v(t)^{2/3}.$$

Exercise 4.3.3

Observe that, if $v(t) > V$, then $v'(t) < 0$. Thus, if $v(S) \leq V + b$, we have $v(t) \leq V + b$ for all $t \geq S$ and, if $v(T) \geq V + b$, we have $v(t) \geq V + b$ for all $t \leq T$.

Suppose that $v(T) \geq V + b$. Then $v(t) \geq V + b$, $h(v(t)) \geq h(V + b)$ and

$$v'(t) = g - h(v(t)) \leq g - h(V + b) \leq g - (g + a) = -a$$

for $0 \leq t \leq T$. By the mean value inequality, this gives

$$v(0) - v(T) \leq -aT$$

and

$$T \leq (v(0) - V)/a.$$

Thus

$$v(t) \leq V + b$$

for $t \geq (v(0) - V)/a$.

Exercise 4.3.4

(i) Using Exercise 4.2.2, we have

$$\begin{aligned} x(t) &= -A_1 \exp(-kt) + A_2 \\ y(t) &= -B_1 \exp(-kt) + B_2 - gt/k \end{aligned}$$

with A_j, B_j constants. We have

$$\begin{aligned} x'(t) &= kA_1 \exp(-kt) \\ y'(t) &= kB_1 \exp(-kt) - g/k. \end{aligned}$$

Thus, taking $t = 0$,

$$A_1 = u_0/k,\ B_1 = (v_0 + g/k)/k,$$
$$A_2 = -A_1 = -u_0/k,\ B_2 = -B_1 = -(v_0 + g/k)/k.$$

We have

$$\begin{aligned} x(t) &= \frac{u_0}{k}(1 - \exp(-kt)) \\ y(t) &= \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - \exp(-kt)) - \frac{gt}{k}. \end{aligned}$$

(ii) When t is large, $\exp(-kt) \approx 0$ so, since

$$\begin{aligned} x'(t) &= -u_0 \exp(-kt), \\ y'(t) &= -\left(v_0 + \frac{g}{k}\right)\exp(-kt) - \frac{g}{k}, \end{aligned}$$

we have

$$\begin{aligned} x'(t) &\approx 0 \\ y'(t) &\approx -\frac{g}{k} \end{aligned}$$

when t is large.

Observing that $\exp(-kt) > 0$ for all t, we see that $x(t) \approx u_0/k$ for t large, but $x(t) < u_0/k$ whenever $t \geq 0$.

(iii) We have

$$y(t) = \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - \exp(-kt)) - \frac{gt}{k} = \left(v_0 + \frac{g}{k}\right)\frac{x(t)}{u_0} - \frac{gt}{k}.$$

Now

$$\exp(-kt) = \frac{kx(t)}{u_0} - 1,$$

so

$$-kt = \log\frac{kx(t) - u_0}{u_0}$$

and

$$t = \frac{1}{k}\log\frac{u_0}{u_0 - kx(t)}.$$

Thus the equation of the path is

$$y = \left(\frac{c + v_0}{u_0}\right) x - \frac{c}{k} \log\left(\frac{u_0}{u_0 - kx}\right),$$

where $c = g/k$.

Exercise 4.3.5

To first order, the weight of air in the column at height between x and $x + \delta x$ is

$$-W'(x)\delta x = W(x) - W(x + \delta x) = KP(x)\delta x$$

and

$$W'(x) = -KP(x),$$

whence

$$P'(x) = -cP(x)$$

for some constants K and c.

We thus have

$$P(x) = A\exp(-cx)$$

for some constant A and so

$$P(x) = P(0)\exp(-cx).$$

Exercise 5.1.1★

Exercise 5.1.2

$$x \log\left(1 + \frac{a}{x}\right) = x \log\left(\frac{a+x}{x}\right) = x(\log(a+x) - \log x).$$

Exercise 5.1.3

By the product rule,

$$\begin{aligned}
f'(x) &= (\log(a+x) - \log x) + x\left(\frac{1}{a+x} - \frac{1}{x}\right) \\
&= \log(a+x) - \log x + x\left(\frac{x-(a+x)}{x(a+x)}\right) \\
&= \log(a+x) - \log x - \frac{a}{a+x}.
\end{aligned}$$

Thus

$$\begin{aligned}
f''(x) &= \frac{1}{a+x} - \frac{1}{x} + \frac{a}{(a+x)^2} \\
&= \frac{x(a+x) - (a+x)^2 + ax}{x(a+x)^2} \\
&= \frac{-a(a+x) + ax}{x(a+x)^2} \\
&= -\frac{a^2}{x(a+x)^2} < 0.
\end{aligned}$$

Exercise 5.1.4

We shall see that the argument we have given for $a > 0$ will, in fact, work for $0 > a$. (However, we need a condition on x to make sure that $1 + a/x > 0$.)

More specifically, we shall show that

$$g(x) = \left(1 + \frac{a}{x}\right)^x$$

increases as x increases for $x > \max\{0, -a\}$. This gives the result required.

Observe that it is sufficient to show that

$$f(x) = \log g(x) = x \log\left(1 + \frac{a}{x}\right) = x\big(\log(x + a) - \log x\big)$$

increases as x increases for $x > \max\{0, -a\}$.

Now

$$\begin{aligned} f'(x) &= (\log(x + a) - \log x) + x\left(\frac{1}{x + a} - \frac{1}{x}\right) \\ &= (\log(x + a) - \log x) - \frac{a}{x + a} \end{aligned}$$

so

$$\begin{aligned} f''(x) &= \frac{1}{x + a} - \frac{1}{x} + \frac{a}{(a + x)^2} \\ &= -\frac{a}{(a + x)x} + \frac{a}{(a + x)^2} \\ &= -\frac{a^2}{(a + x)^2 x} > 0 \end{aligned}$$

for $x > \max\{0, -a\}$.

Since $f''(x) < 0$ for $x > \max\{0, -a\}$, we know that f' is decreasing. Thus, if $f'(x_0) = -u_0 < 0$ for some $x_0 > \max\{0, -a\}$, we will have

$$\log\left(1 + \frac{a}{x}\right) - \frac{a}{a + x} = f'(x) \leq -u_0$$

whenever $x \geq x_0$ and this is clearly false when x is very large. Thus $f'(x) \geq 0$ whenever $x > \max\{0, -a\}$ and $f(x)$ is increasing for $x > \max\{0, -a\}$, as we hoped.

Exercise 5.1.5

We have

$$f(0) = \log 1 + 0 = 0$$

and

$$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x},$$

so

$$|f'(x)| \leq \frac{|x|}{1/2} = 2|x|$$

for $|x| \leq 1/2$, whence, by the mean value inequality,

$$|f(h)| = |f(h) - f(0)| \leq 2|h||h| = 2h^2$$

for $|h| \leq 1/2$.

EXERCISE 5.1.6

Since

$$0 < \exp t = \exp' t = \exp t \leq \exp(a + 1)$$

for $a - 1 \leq t \leq a + 1$, the mean value inequality gives

$$|\exp(a + k) - \exp a| \leq \exp(a + 1)|k|$$

for $|k| \leq 1$.

Exercise 5.1.7

(i) The advantage of compounding is not large enough for the ordinary saver to press for it. The institution which offers the saving system prefers to keep its costs down and to tie the saver to a longer period.

For high rates of interest, compounding is much more advantageous. Institutions which lend at high interest generally have the whip hand over their borrowers and take every advantage of this.

(ii) We need to invest for a period of N years, where

$$(1 + c/100)^N \approx 2$$

i.e.

$$N \log(1 + c/100) \approx \log 2$$

i.e.

$$N \approx \frac{\log 2}{\log(1 + c/100)}.$$

If c is small, then

$$\log(1 + c/100) \approx \log'(1)c/100 = c/100$$

so

$$N \approx \frac{100 \log 2}{c} = \frac{69.3}{c}.$$

(iii) About 4.7×10^{25} by my calculations. But the question requires the institution to survive 2000 years, to be able to find the appropriate investments for 2000 years, not to spend the money and to keep other institutions' hands off it. (The problem of inflation can be considered under the previous heads.)

Exercise 5.2.1★

Exercise 5.2.2

Set $f(x) = (\exp x - 1)/x$.

We have

$$f'(x) = \frac{\exp x}{x} - \frac{\exp x - 1}{x^2} = \frac{x \exp x - \exp x + 1}{x^2} = \frac{g(x)}{x^2},$$

where

$$g(x) = x \exp x - \exp x + 1.$$

Now

$$g'(x) = x \exp x > 0,$$

so g is strictly increasing for $x > 0$. Since $g(0) = 0$, we have $g(x) > 0$ for $x > 0$, so $f'(x) > 0$ for $x > 0$ and f is strictly increasing.

It follows that

$$\frac{\exp kx - 1}{x} = k^{-1} f(x)$$

is strictly increasing and so f is increasing.

(ii) We have

$$L(b) = \frac{w}{T} f(Tb) + c \exp bT$$

so L is the sum of two strictly increasing functions and so strictly increasing.

If interest rates are higher, projects incur higher borrowing costs and must be more expensive.

Exercise 5.2.3

We have, whilst we are in debt, to first order,

$$h(t+\delta t) = h(t) - u\delta t + bh(t)\delta t$$

so, to first order,

$$h(t+\delta t) = h(t) + (bh(t) - u)\delta t$$

and

$$h'(t) = bh(t) - u.$$

Thus

$$h(t) = A\exp(bt) + \frac{u}{b}$$

for some constant A. Taking $t = 0$, we see that $A = L - u/b$ and

$$h(t) = \left(L - \frac{u}{b}\right)\exp(bt) + \frac{u}{b}$$

whilst we are in debt. We pay of our debt at time x given by $h(x) = 0$, i.e.

$$\left(L - \frac{u}{b}\right)\exp(bx) = -\frac{u}{b}$$

i.e.

$$\exp(bx) = \frac{u}{u - bL},$$

so at time

$$\frac{1}{b}\log\left(\frac{u}{u - bL}\right).$$

Exercise 5.2.4

(i) We have

$$\big(\exp(bT/2)-1\big)^2=\exp(bT)-2\exp(bT/2)+1,$$

so

$$(c+w/b)\big(\exp(bT)-2\exp(bT/2)\big)+\frac{w}{b}=(c+w/b)\big(\exp(bT/2)-1\big)^2-c$$

and our inequation becomes

$$\big(\exp(bT/2)-1\big)^2>\frac{c}{c+w/b}=\frac{cb}{cb+w}.$$

(ii) If $T>0$,

$$\big(\exp(bT/2)-1\big)^2>0,$$

so, if $c=0$, we will always be better off starting from both ends.

The project takes half the time (so reducing interest charges) and the cost, not including interest, is unchanged.

(iii) If $4T>\frac{2}{b}\log 2$,

$$\big(\exp(bT/2)-1\big)^2>1^2=1=\frac{cb}{cb}>\frac{cb}{cb+w},$$

so we should always dig from both ends.

If interest rates are very high it is cheaper to incur the cost and associated interest on two base camps, two tunnelling machines and two of everything else for half the time than on one for the whole time.

Exercise 5.3.1

$$\begin{array}{cccc}
1 & 2 & * & * \\
1 & * & 2 & * \\
1 & * & * & 2 \\
* & 1 & 2 & * \\
* & 1 & * & 2 \\
* & * & 1 & 2 \\
2 & 1 & * & * \\
2 & * & 1 & * \\
2 & * & * & 1 \\
* & 2 & 1 & * \\
* & 2 & * & 1 \\
* & * & 2 & 1
\end{array}$$

and, if we cannot distinguish 1 and 2,

$$\begin{array}{cccc}
B & B & * & * \\
B & * & B & * \\
B & * & * & B \\
* & B & B & * \\
* & B & * & B \\
* & * & B & B
\end{array}$$

Exercise 5.3.2

$$q_0 = (1-p)^n = 1 \times 1 \times (1-p)^n = \binom{n}{0} p^0 (1-p)^{n-0}.$$

Exercise 5.3.3

The number ways in which we can place r indistinguishable balls in n holes is the same as the number of ways in which we can take r indistinguishable things from n places (taking at most one thing from any particular place). Thus, if we multiply out $(x+y)^n$ in full, the term $x^{n-r}y^r$ obtained by choosing y from r of the bracketed terms and x from the remainder occurs $\binom{n}{r}$ times. Thus

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}y^n.$$

Taking $x=1$, we obtain

$$(1+y)^n = \binom{n}{0} + \binom{n}{1}y + \binom{n}{2}y^2 + \ldots + \binom{n}{n}y^n.$$

Taking $x=y=1$, we obtain

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$$

Taking $x=1$, $y=-1$, we obtain

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0.$$

Exercise 5.3.4

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n,$$

so, differentiating,

$$n(1+x)^{n-1} = 1 \times \binom{n}{1} + 2 \times \binom{n}{2}x + \ldots + n \times \binom{n}{n}x^{n-1}.$$

If $p \neq 1$ and we set $x = p/(1-p)$, we get

$$n\frac{1}{(1-p)^{n-1}} = 1 \times \binom{n}{1} + 2 \times \binom{n}{2}\frac{p}{(1-p)} + \ldots + n \times \binom{n}{n}\frac{p^{n-1}}{(1-p)^{n-1}}.$$

Multiplying both sides by $p(1-p)^{n-1}$ yields

$$\begin{aligned} np &= 1 \times \binom{n}{1}p(1-p)^{n-1} \\ &\qquad + 2 \times \binom{n}{2}p^2(1-p)^{n-2} + \ldots + n \times \binom{n}{n}p^n \\ &= \mathcal{E}, \end{aligned}$$

as required.

If $p = 1$ the result is immediate from the definition.

Exercise 6.1.1

We have
$$f(x) = \sin x + 10^{-20} \sin 10^{12}x \approx \sin x$$
and
$$f'(x) = \cos x + 10^{-8} \cos 10^{12}x \approx \cos x,$$
so both f and f' can graphed on standard scales, but
$$f''(x) = -\sin x - 10^{2} \sin 10^{12}x \approx -10^{2} \sin 10^{12}x,$$
so the scale for the y axis must be chosen so as to show y with $|y| \leq 10^2+1$ and the scale for the x axis so that points a distance 10^{-13} apart are readily distinguishable.

Exercise 6.2.1

If $u(t) = -t$, then the function of a function rule gives
$$h'(t) = (g \circ u)'(t) = u'(t)(g' \circ u)(t) = -g'(-t)$$
Repeating the argument (or reusing the result) r times gives $h^{(r)}(t) = (-1)^r g^{(r)}(-t)$.

Thus, if $|g^{(n)}(t)| \leq M$ for $-a \leq t \leq 0$, we have $|h^{(n)}(t)| \leq M$ for $0 \leq t \leq a$ and
$$h(0) = h'(0) = h''(0) = \ldots = h^{(n-1)}(0) = 0,$$
so
$$|h(t)| \leq M \frac{t^n}{n!}$$
whenever $0 \leq t \leq a$. It follows that
$$|g(t)| \leq M \frac{|t|^n}{n!}$$
whenever $-a \leq t \leq 0$.

Exercise 6.2.2

$$g^{(r)}(t) = f^{(r)}(t) - f^{(r)}(0) - \frac{f^{(r+1)}(0)(r+1) \times r \times \ldots \times 2}{r!}t - \ldots$$
$$- \frac{f^{(n-1)}(0)(n-1) \times (n-2) \ldots \times (n-r-1)}{(n-1)!}t^{n-r},$$

for $0 \leq r \leq n-1$ and

$$g^{(n)}(t) = f^{(n)}(t),$$

so

$$g(0) = g'(0) = g''(0) = \ldots = g^{(n-1)}(0) = 0$$

and $|g^{(n)}(t)| \leq M$ whenever $|t| \leq a$.

Exercise 6.2.3

This is just a translation.

Set $f(t) = F(t-y)$, so $f^{(r)}(t) = F^{(r)}(t-y)$. Then $|f^{(n)}(s)| \leq M$ whenever $|s| \leq a$, so

$$\left| f(s) - f(0) - \frac{f'(0)}{1!}\ s - \frac{f''(0)}{2!} s^2 - \ldots - \frac{f^{(n-1)}(0)}{(n-1)!} s^{n-1} \right| \leq M \frac{|s|^n}{n!}.$$

whenever $|s| \leq a$ and

$$\left| F(t) - F(y) - \frac{F'(y)}{1!}\ (t-y) - \frac{F''(y)}{2!}(t-y)^2 - \ldots - \frac{F^{(n-1)}(t)}{(n-1)!}(t-y)^{n-1} \right| \leq M \frac{|t-y|^n}{n!}.$$

whenever $|t-y| \leq a$.

Exercise 6.2.4

If

$$g(h) = a_0 + a_1 h + a_2 h^2 + \ldots + a_{n-1}h^{n-1} + o(h^{n-1})$$

and

$$g(h) = b_0 + b_1 h + b_2 h^2 + \ldots + b_{n-1}h^{n-1} + o(h^{n-1}),$$

then

$$0 = g(h) - g(h) = u_0 + u_1 h + u_2 h^2 + \ldots + u_{n-1}h^{n-1} + o(h^{n-1})$$

where $u_j = a_j - b_j$.

We need to prove that $u_j = 0$ for $0 \leq j \leq n-1$. But, if $u_0 = u_1 = \ldots = u_{r-1} = 0$, we have

$$0 = u_r h^r + u_{r+1}h^{r+1} \ldots + u_{n-1}h^{n-1} + o(h^{n-1}),$$

so

$$0 = u_r + u_{r+1}h \ldots + u_{n-1}h^{n-r} + o(h^{n-r}),$$

whence

$$0 = u_r + o(h)$$

and $u_r = 0$. By repeating this argument $n-2$ times, we get

$$u_0 = u_1 = \ldots = u_{n-2} = 0$$

and

$$0 = u_{n-1}(h^{n-1}) + o(h^{n-1}),$$

so $u_{n-1} = 0$ and we are done.

To obtain the last part set $c_j = a_j$ and $b_j = F^{(j)}(y)/j!$.

Exercise 6.2.5

If
$$f(a+h) = f(a) + Ah^2 + o(h^2)$$
and $A < 0$, then the o notation tells us that we can find a $u > 0$ such that
$$\left|f(a+h) - (f(a) + Ah^2)\right| \leq \frac{-A}{2}h^2$$
whenever $|h| \leq u$. We thus have
$$f(a+h) - (f(a) + Ah^2) \leq -\frac{A}{2}h^2,$$
so
$$f(a+h) \leq f(a) + \frac{A}{2}h^2 = f(a) - \frac{|A|}{2}h^2$$
whenever $|h| \leq u$. Thus f attains a local maximum at a.

Alternatively Look at $-f$.

Exercise 6.2.6

Let us suppose first that $f^{(r)}(a) > 0$. By the local Taylor's theorem, we can find a $u > 0$ such that

$$\left| f(a+h) - \left(f(a) + \frac{f^{(r)}(a)}{r!} h^r \right) \right| \leq \frac{f^{(r)}(a)}{2 \times r!} |h|^r \qquad \bigstar$$

whenever $|h| \leq u$.

If r is even, $\bigstar$ tells us that

$$f(a+h) \geq f(a) + \frac{f^{(r)}(a)}{2 \times r!} |h|^r$$

whenever $|h| \leq u$, so f attains a local minimum at a.

If r is odd, $\bigstar$ tells us that

$$f(a+h) \geq f(a) + \frac{f^{(r)}(a)}{2 \times r!} |h|^r$$

for $0 \leq h \leq u$, but

$$f(a+h) \leq f(a) - \frac{f^{(r)}(a)}{2 \times r!} |h|^r$$

for $-u \leq h \leq 0$, so we have neither a maximum nor a minimum.

Now suppose $f^r(a) < 0$. By looking at $-f$ and applying our previous results we see that:-

If r is even, f attains a local minimum at a. If r is odd, we have neither a maximum nor a minimum.

Exercise 6.3.1

Observe that $E^{(r)}(t) = E(t)$ and so

$$E^{(r)}(0) = E(0) = 1.$$

Now substitute into the formula of Taylor's theorem.

Exercise 6.3.2

Since E is increasing,

$$0 < E^{(n)}(t) = E(t) \leq E(0) = 1$$

for all $t \leq 0$.

Exercise 6.3.3

Our error estimate shows that

$$|\exp(1) - 2.71806| \leq 0.006,$$

so $\exp(1) \leq 2.8$ and the error is less than

$$\frac{\exp(1)}{6!} \leq \frac{2.8}{720} \leq 0.0039.$$

Exercise 6.3.4

(i) We have

$$\begin{aligned} u_0 &= 1\\ u_1 &= 1\\ u_2 &= 0.5\\ u_3 &\approx 0.166666667\\ u_4 &\approx 0.041666667\\ u_5 &\approx 0.008333333\\ u_6 &\approx 0.001388889\\ u_7 &\approx 0.000198413\\ u_8 &\approx 0.000024802\\ u_9 &\approx 0.000002756\\ u_{10} &\approx 0.000000276\\ 2.8\times u_{11} &\approx 0.000000070 \end{aligned}$$

giving

$$\exp(1) \approx u_0 + u_1 + u_2 + \ldots + u_{10} \approx 2.71828180$$

with an error of less than about 0.00000007.

(ii) We take $u_j = (10)^{-j}/j!$ giving

$$\begin{aligned} u_0 &= 1\\ u_1 &= 0.1\\ u_2 &= .05\\ u_3 &\approx 0.001666667\\ u_4 &\approx 0.000041667\\ u_5 &\approx 0.000000833\\ 2.8\times u_6 &\approx 0.000000039 \end{aligned}$$

giving

$$\exp(1/10) \approx u_0 + u_1 + u_2 + \ldots + u_5 \approx 1.10517092$$

with an error of about 0.00000004.

(iii) Note that the error will now be no greater than

$$10^{-6}/6! \approx 0.000000001.$$

We take u_j as in (ii).

$$\exp(-1/10) \approx u_0 - u_1 + u_2 - \ldots - u_5 \approx 0.90483742$$

correct to the number of figures given.

Exercise 6.3.5

By Taylor's theorem,

$$\left|\left(1+\frac{a}{N}+\frac{a^2}{2N^2}\right)-\exp(a/N)\right|\leq\frac{\exp 1}{3!}\left|\frac{a}{N}\right|^3$$

provided only that $N>a$.

Since $\log'(t)=1/t$, the mean value inequality tells us that

$$|\log x-\log y|\leq 2|x-y|$$

whenever $|x-1|$, $|y-1|\leq 1/2$. Thus

$$\left|\log\left(1+\frac{a}{N}+\frac{a^2}{2N^2}\right)-\frac{a}{N}\right|\leq K_1\left|\frac{a}{N}\right|^3$$

(where K_1 is some constant) provided only that N is large enough. Thus

$$\left|\log\left(1+\frac{a}{N}+\frac{a^2}{2N^2}\right)^N-a\right|\leq K_1a\left|\frac{a}{N}\right|^2\leq K_1b\left|\frac{a}{N}\right|^2$$

By the mean value theorem

$$|\exp x-\exp a|\leq\exp(|a|+1)|x-a|$$

for $|x-a|\leq 1$. Thus, taking

$$x=\log\left(1+\frac{a}{N}+\frac{a^2}{2N^2}\right)$$

we see that there exists there exists an L (depending on a) with

$$\left|\left(1+\frac{a}{N}+\frac{a^2}{2N^2}\right)^N-\exp a\right|\leq\frac{L}{N^2},$$

provided that N is large enough.

Exercise 6.3.6★

Exercise 6.3.7★

Exercise 6.3.8

Let $u_j = 10^j/j!$. Working to two decimal places.

$$\begin{aligned}
u_0 &= 1\\
u_1 &= 10\\
u_2 &= 50\\
u_3 &= 166.67\\
u_4 &= 416.67\\
u_5 &= 833.33\\
u_6 &= 1388.89\\
u_7 &= 1984.13\\
u_8 &= 2480.16\\
u_9 &= 2755.73\\
u_{10} &= 2755.73\\
u_{11} &= 2505.21\\
u_{12} &= 2087.68\\
u_{13} &= 1605.90\\
u_{14} &= 1147.07\\
u_{15} &= 764.72\\
u_{16} &= 477.95\\
u_{17} &= 281.15\\
u_{18} &= 156.19\\
u_{19} &= 82.20\\
u_{20} &= 41.10
\end{aligned}$$

Thus

$$\exp 10 \approx u_0 + u_1 = \ldots + u_{20} \approx 21991.48$$

with an error of at most about 41.1 Thus

$$22050 > \exp 10 > 21950.$$

(If the reader feels that we should be more careful about the lower estimate she can observe that $\exp 10 \geq u_0+u_1+\ldots+u_{20}+u_{21} > 22000$.)

Exercise 6.3.9★

Exercise 6.3.10

If we write $u_r = E^{(r)}(0)(-100)^r/r! = (-100)^r/r!$ then, using Stirling's inequality from Exercise 3.2.11, we have

$$100! \leq 101 \times 100^{100}e^{-99},$$

so

$$|u_{100}| \geq \frac{e^{99}}{101} > 10^{40}$$

and working to 32 significant figures will allow errors of size 10^8 and so certainly tell us nothing about $\exp(-100)$.

Exercise 6.3.11

$$\begin{aligned}
\exp 1 &\approx 2.718282\\
\exp 2 &\approx 7.389056\\
\exp 4 &\approx 5.459815 \times 10^{1}\\
\exp 8 &\approx 2.980958 \times 10^{3}\\
\exp 16 &\approx 8.886111 \times 10^{6}\\
\exp 32 &\approx 7.896296 \times 10^{13}\\
\exp 64 &\approx 6.235149 \times 10^{27}
\end{aligned}$$

so

$$\begin{aligned}
\exp 100 &= \exp(64) \times \exp(32) \times \exp(4)\\
&\approx 6.235149 \times 7.896296 \times 5.459815 \times 10^{41}\\
&\approx 2.688117 \times 10^{2} \times 10^{41} = 2.688117 \times 10^{43},
\end{aligned}$$

so

$$\exp(-100) = 1/(\exp 100) \approx (1/2.688117) \times 10^{-43} \approx 3.720076 \times 10^{-44}$$

and $\exp(-100) = 3.270 \times 10^{-44}$ to four significant figures.

Exercise 6.3.12

(i) We have

$$\cos^{(4r)} t = \cos t,\ \cos^{(4r+1)} t = -\sin t,\ \cos^{(4r+2)} t = -\cos t,\ \cos^{(4r+3)} t = \sin t.$$

Thus $|\cos^{(k)}(t)| \leq 1$ for all t and k whilst

$$\cos^{(4r)} 0 = 1,\ \cos^{(4r+1)} 0 = 0,\ \cos^{(4r+2)} 0 = -1,\ \cos^{(4r+3)} t = 0.$$

Thus

$$\left|\cos t - \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots + (-1)^r \frac{t^{2r}}{(2r)!}\right)\right| \leq \min\left\{\frac{|t|^{2r+1}}{(2r+1)!}, \frac{|t|^{2r+2}}{(2r+2)!}\right\}$$

Similarly,

$$\sin^{(4r)} t = \sin t,\ \sin^{(4r+1)} t = \cos t,\ \sin^{(4r+2)} t = -\sin t,\ \sin^{(4r+3)} t = -\cos t$$

and

$$\left|\sin t - \left(t - \frac{t^3}{3!} - \ldots + (-1)^r \frac{t^{2r+1}}{(2r+1)!}\right)\right| \leq \min\left\{\frac{|t|^{2r+2}}{(2r+2)!}, \frac{|t|^{2r+3}}{(2r+3)!}\right\}.$$

(ii) In particular,

$$\left|\sin\frac{1}{10} - \left(10^{-1} - \frac{10^{-3}}{3!}\right)\right| \leq \frac{10^{-5}}{5!} \leq 10^{-7}$$

so, since

$$10^{-1} - \frac{10^{-3}}{3!} \approx .0099833,$$
$$\sin(1/10) = .009983$$

to six decimal places.

(iii) Since

$$\frac{|t|^{2r+4}}{(2r+4)!} \leq \frac{1}{4}\frac{|t|^{2r+2}}{(2r+2)!}$$

for $r \geq 2|t|$ we can make the error term as small as we like. However, if $|t|$ is large, our calculations will give $\sin t$ as the difference of two very large numbers and this is computationally unsound.

(iv) If we know π to high accuracy, we can compute $x/2\pi$ and find an integer n such that $|x - 2\pi n| \leq \pi$. We can now compute $y = x - 2n\pi$ to high accuracy and then $\sin x = \sin y$ using Taylor's theorem to compute $\sin y$.

Exercise 6.3.13

(i) Unless the reader can think of something else, this seems to demand knowing the remainder when 2π is divided into 10^{100}. Presumably we would need to know π to at least 100 places of decimals and to carry out the calculation to at least 100 figure accuracy (in fact, slightly more).

(ii) Calculators do not work to this accuracy.

(iii) Computers can work to this accuracy, but this demands special programming techniques. If we replace $\tan 10^{100}$ by $\tan 10^{10^{10}}$, things become really interesting.

Exercise 6.3.14

We have
$$\sin_D t = \sin_R \frac{2\pi}{360} t$$
so, using the function of a function rule, Taylor's theorem yields
$$\left|\sin t - \left(\frac{2\pi}{360}t - \frac{(2\pi)^3 t^3}{3!(360)^3} - \ldots + (-1)^r \frac{(2\pi)^{2r+1} t^{2r+1}}{(2r+1)!(360)^{2r+1}}\right)\right|$$
$$\leq \min\left\{\frac{(2\pi)^{2r+2}|t|^{2r+2}}{(2r+2)!(360)^{2r+2}}, \frac{(2\pi)^{2r+3}|t|^{2r+3}}{(2r+3)!(360)^{2r+3}}\right\}.$$
(We could also obtain the formula by simple substitution.)

MORAL:- Use radians.

Exercise 6.3.15

We have
$$f^{(r)}(x) = m(m-1)\dots(m-r+1)(x+y)^{m-r}$$
and so
$$f^{(r)}(0) = m(m-1)\dots(m-r+1)y^{m-r}$$
for $0 \leq r \leq m$, whilst
$$f^{(r)}(x) = 0$$
for $r \geq m+1$.

Thus $|f^{(n)}(x)| = 0 \leq 0$ whenever $n \geq m+1$ and our Taylor series with error estimate yields the *equality*
$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(m)}(0)}{m!}x^m$$
when we take sufficiently many terms. Substituting in the values for $f^{(r)}(0)$ already obtained we get
$$(x+y)^m = \binom{0}{0}y^m + \binom{m}{m-1}y^{m-1}x + \binom{m}{2}y^{m-2}x^2 + \dots + \binom{m}{m}x^m.$$
This is the binomial theorem.

Exercise 7.1.1

$|f(x)| \leq u$ if $|x - b| \leq u^{1/3}$. Thus, for example, if we can compute f to six places of decimals but have no other information about b, we can only find b to two places of decimals.

Exercise 7.1.2

We have $a_j < b_j$ and $f(a_j) \le 0 \le f(b_j)$.

Let $c_j = (a_j + b_j)/2$. Either $f(c_j) \le 0$ and we set $a_{j+1} = c_j$, $b_{j+1} = b_j$ or $0 < f(c_j)$ and we set $a_{j+1} = a_j$, $b_{j+1} = c_j$.

Exercise 7.1.3

(i) We have $a_0 = 5/3$, $b_0 = 8/3$, so
$$f(a_0) < 0 < f(b_0).$$
$c_0 = 13/6$, $f(13/6) > 0$, so
$$a_1 = 5/3,\ b_1 = 13/6.$$
$c_1 = 23/12$, $f(c_1) < 0$, so
$$a_2 = 23/12,\ b_2 = 13/6.$$
$c_2 = 49/24$, $f(c_2) > 0$, so
$$a_3 = 23/12,\ b_2 = 49/24.$$
$c_3 = 95/48$, $f(c_3) < 0$, so
$$a_4 = 95/48,\ b_2 = 49/24 = 98/48.$$

(ii) $a_0 = 0$, $b_0 = 2$, so
$$f(a_0) < 0 < f(b_0).$$
$c_0 = 1$, $f(1) < 0$, so
$$a_1 = 1,\ b_1 = 2.$$
$c_1 = 3/2$, $f(c_1) > 0$, so
$$a_2 = 1,\ b_2 = 3/2.$$
$c_2 = 5/4$, $f(c_2) < 0$, so
$$a_3 = 5/4,\ b_2 = 3/2.$$
$c_3 = 11/8$, $f(c_3) < 0$, so
$$a_4 = 11/8,\ b_2 = 3/2 = 12/8.$$

Exercise 7.1.4

Either look at previous argument applied to $-f$ or write out as before:-

The method requires initial values a_0, b_0 such that $a_0 < b_0$ and $f(a_0) \geq 0 \geq f(b_0)$. Notice that this guarantees that there is a root between a_0 and b_0.

Let $c_0 = (a_0 + b_0)/2$. Either $f(c_0) \geq 0$ and we set $a_1 = c_0$, $b_1 = b_0$ or $0 > f(c_0)$ and we set $a_1 = a_0$, $b_1 = c_0$. In either case, $f(a_1) \geq 0 \geq f(b_1)$ and $b_1 - a_1 = (b_0 - a_0)/2$, so we *have halved the length of the interval in which we know there exists a root*. We repeat the process as many times as we want, obtaining intervals with end points a_j, b_j such that $f(a_j) \geq 0 \geq f(b_j)$ and $b_j - a_j = 2^{-j}(b_0 - a_0)$.

Exercise 7.1.5

$$2^{10} = 1024 \approx 10^3$$

(indeed $2^{10} > 10^3$) so doing the iteration 40 times produces an interval of size $2^{-40} \approx 10^{-12}$.

EXERCISE 7.2.1

(i) $f'(x) = 2x$, so

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)} = x_j - \frac{x_j^2 - 4}{2x_j} = \frac{1}{2}(x_j + 4x_j^{-1}).$$

Thus

$$\begin{aligned} x_0 &= 3 \\ x_1 &\approx 2.166666667 \\ x_2 &\approx 2.006410256 \\ x_3 &\approx 2.000010240 \\ x_4 &\approx 2.000000000 \end{aligned}$$

correct to the number of figures shown.

(ii) $f'(x) = 2x$, so

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)} = x_j - \frac{x_j^2 - 2}{2x_j} = \frac{1}{2}(x_j + 2x_j^{-1}).$$

Thus

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 1.5 \\ x_2 &\approx 1.416666667 \\ x_3 &\approx 1.414215686 \\ x_4 &\approx 1.414213562. \end{aligned}$$

My calculator gives $x_4^2 = 1.999999999$.

Exercise 7.2.2

If x_j is accurate to within n places of decimals, then $|x_j - a| \leq 10^{-n}/2$ and $|x_{j+1} - a| \leq M \times 10^{-2n}/4$. If n is large (how large depends on M), we will, more or less, double the number of places of accuracy.

Exercise 7.2.3

If $f(x) = \cos x$, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 + \frac{\cos x_0}{\sin x_0} = x_0 + \cot x_0.$$

Now $\cot x$ decreases from $\cot(1/100) > 10$ to 0 as x runs from $1/100$ to $\pi/2$, so there is a value of x_0 with $1/100 \leq x \leq \pi/2$ such that $\cot x_0 = 2\pi$ and so

$$x_1 = x_0 + 2\pi$$

By periodicity, $x_j = x_0 + 2j\pi$.

Exercise 7.2.4

We have

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)} = x_j - \frac{x_j^3 - 2x_j + 2}{3x_j^2 - 2}.$$

Thus

$$x_0 = 0$$
$$x_1 = 1$$
$$x_2 = 0$$

and so $x_{2j} = 0$, $x_{2j+1} = 1$.

Exercise 7.2.5

Let $f(x) = x^{1/3}$. We have

$$\begin{aligned} x_{j+1} &= x_j - \frac{f(x_j)}{f'(x_j)} \\ &= x_j - \frac{x_j^{1/3}}{3^{-1}x_j^{-2/3}} \\ &= x_j - 3x_j = -2x_j. \end{aligned}$$

Thus $x_j = (-1)^j 2^j x_0$ and the Newton–Raphson method fails.

This does not contradict our argument because f is not differentiable at the origin. (If f were differentiable at 0 with $f'(0) = a$, we would have

$$h^{1/3} = ah + o(h)$$

so

$$1 = ah^{2/3} + o(h^{2/3})$$

which is false.)

(ii) If $|x_0| > u$, then the argument of (i) gives $x_1 = -2x_0$ and $|x_1| > 2u$. Thus $x_j = (-1)^j 2^j x_0$ and the Newton–Raphson method fails.

Exercise 7.2.6

Here is a possible solution but there are many others.

The basic idea is to use the Newton–Raphson method to find x (to the accuracy required) with $f(x) = 0$ where $f(s) = (\exp s) - t$, that is to say, to use the sequence

$$x_{j+1} = x_j - \frac{\exp(x_j) - t}{\exp(x_j) - 1} \qquad \bigstar$$

using Taylor series to obtain $\exp x_j$.

However we will run into problems if $\exp x_j$ is very close to 1 because the formula $\bigstar$ will then involve division by a very small quantity. We will also run into problems if we use the Taylor series to find $\exp x_j$ if $|x_j|$ is large,

We therefore proceed as follows. Let N be the integer such that that $1/10 \leq 10^N t < 1$. Now set $T = 10^N t$ and use the Newton–Raphson method to find X (to the accuracy required) with $F(X) = 0$ where $F(s) = \exp s - T$, that is to say, take

$$x_{j+1} = x_j - \frac{\exp(x_j) - T}{\exp(x_j) - 1}$$

starting from $x_0 + 0$ and using Taylor series to obtain $\exp x_j$. We also compute $\log 1/10$.

We now use the formula

$$\log t = \log T - \log 10^N = X + N \log(1/10).$$

Exercise 7.2.7

(i) We have

$$\begin{aligned}
g'(x) &= 1 - \frac{2f'(x)^2 + 2f(x)f''(x)}{2f'(x)^2 - f(x)f''(x)} \\
&\quad + \frac{2f(x)f'(x)(3f''(x)f'(x) - f(x)f'''(x))}{(2f'(x)^2 - f(x)f''(x))^2} \\
&= -f(x)\frac{3f''(x)}{k(x)} + 2f(x)f'(x)\frac{k'(x)}{k(x)^2}
\end{aligned}$$

where $k(x) = 2f'(x)^2 - f(x)f''(x)$ so, since $f(a) = 0$,

$$g'(a) = 0.$$

Using the fact that $f(a) = 0$ to reduce calculation,

$$\begin{aligned}
g''(a) &= -\frac{3f''(a)f'(a)^2}{k(a)} + 2f'(a)^2\frac{k'(a)}{k(a)^2} \\
&= -\frac{3f'(a)f''(a)}{2f'(a)^2} + \frac{2f'(a)^2(3f''(a)f'(a))}{4f'(a)^4} \\
&= 0.
\end{aligned}$$

(Mark Twain talks of a German sentence as 'Swimming the Atlantic with a verb in your mouth.')

(ii) By the local Taylor's theorem, there exist $u > 0$ and $K > 0$ such that

$$|g(a+h) - g(a) - g'(a)h - g''(a)h^2/2| \leq K|h|^3$$

whenever $|h| \leq u$, that is to say $|g(a+h) - a| \leq K|h|^3$ whenever $|h| \leq u$.

(iii) If $x_j = H(x_{j-1})$ for Newton's method, we can find $v > 0$ and M such that

$$|H(a+h) - a| \leq Mh^2$$

for all h such that $|h| \leq v$. Thus, if $\tilde{u} = M^{-1}v^2$ and $L = M^2$,

$$|G(a+h) - a| = |H(H(a+h)) - a| \leq M|H(a+h) - a| \leq Lh^4$$

whenever $|h| \leq \tilde{u}$.

Exercise 7.3.1

(i) Suppose, if possible, that $\sqrt{2} = u/v$ with u and v strictly positive integers having no common factor. Then

$$2 = (u/v)^2 = u^2/v^2,$$

so

$$2u^2 = v^2$$

and u must be even. Thus $u = 2r$ for some integer r and $2v^2 = 4r^2$, whence

$$v^2 = 2r^2$$

and v must be even. Thus u and v have 2 as a common factor contradicting our original assumption.

Thus our initial assumption is false and $\sqrt{2}$ is irrational.

(ii) Suppose, if possible, that $\sqrt{p} = u/v$ with u and v strictly positive integers having no common factor. Then

$$p = (u/v)^2 = u^2/v^2,$$

so

$$pv^2 = u^2$$

and, since p is a prime, u must be divisible by p. Thus $u = rp$ for some integer r and $pv^2 = p^2r^2$, whence

$$v^2 = pr^2$$

and v must be divisible by p. Thus u and v have p as a common factor contradicting our original assumption.

Thus our initial assumption is false and $\sqrt{p}$ is irrational.

(iii) If n is not a perfect square, there must exist a prime p and an integer $t \geq 0$ such that p^{2t+1} divides n, but p^{2t+2} does not. If $\sqrt{n}$ is rational, then so is $p^{-t}\sqrt{n} = \sqrt{p^{-2t}n}$ so we may assume that p divides n but p^2 does not.

Suppose, if possible, that $\sqrt{n} = u/v$ with u and v strictly positive integers with no common factor. Then

$$n = (u/v)^2 = u^2/v^2,$$

so

$$nv^2 = u^2.$$

Since p divides n but p^2 does not, we can find s such that the left hand side is divisible by of p^{2s+1} but not by p^{2s+2}. Since the right hand side is a square divisible by p^{2s+1} it must be divisible by of p^{2s+2} and we have a contradiction.

Thus our initial assumption is false and $\sqrt{p}$ is irrational.

Exercise 7.3.2

Observe that

$$\begin{aligned}|P'(t)| &= |nb_nt^{n-1} + (n-1)b_{n-1}t^{n-2} + \ldots + b_1|\\ &\leq n|b_n||t|^{n-1} + (n-1)|b_{n-1}||t|^{n-2} + \ldots + |b_1|\\ &\leq n|b_n|R^{n-1} + (n-1)|b_{n-1}|R^{n-2} + \ldots + 2|b_2|R + |b_1|\\ &\leq n|b_n|R^{n-1} + (n-1)|b_{n-1}|R^{n-2} + \ldots + 2|b_2|R + |b_1| + 1\end{aligned}$$

for $|t| \leq R$.

Automatically $K \geq 1$.

Exercise 7.3.3

(i) We chose R so that, if $P(x) = 0$, then $|x| \leq R - 2$, so, certainly, if $|p/q| > R$,

$$|p/q - x| \geq 1 \geq K^{-1}q^{-n}.$$

(ii) Since there are only finitely many roots, we can find a $u > 0$ such that $|x - y| > u$ whenever x and y are distinct roots. If we choose $q_0 \geq (2u^{-1} + 1)$ then, if x is an irrational root and p/q is a root with p, q integers, we have $x \neq p/q$ and

$$|p/q - x| > u > K^{-1}q^{-n}.$$

Exercise 7.3.4

$$x = .110\,001\,000\,000\,000\,000\,000\,001\,000\,000\ldots$$

Exercise 7.3.5

If $r_j = 0$ for all $j \geq N$, then y is not merely algebraic, but rational. If we set $b_1 = 10^{N!}$ and $b_0 = -10^{N!}y$, then b_1 and b_0 are integers, $b_1 \neq 0$ and y is the root of

$$b_1 t + b_0 = 0.$$

If there does not exist an N such that $r_j = 0$ for all $j \geq N$, we can find $j(k)$ such that $j(1) \geq 1$, $j(k+1) \geq j(k) + k + 1$ and $r_{j(k)} \neq 0$ for each $k \geq 1$.

If we now write $q_n = 10^{j(n+2)!}$ and

$$p_n = r_{j(1)}10^{(j(n)+2)!-j(1)!} + r_{j(2)}10^{(j(n)+2)!-j(2)!} + r_{j(3)}10^{(j(n)+2)!-j(3)!} + \dots$$
$$\dots + r_{j(n+1)}10^{(j(n+2)!-j(n+1)!} + r_{j(n+2)},$$

we see, by looking at the decimal expansions of y and p_n/q_n, that

$$\frac{p_n}{q_n} \leq x \leq \frac{p_n}{q_n} + 20 \times 10^{-j(n+3)!}$$

and so

$$\left| x - \frac{p_n}{q_n} \right| \leq 20 \times 10^{-(j(n+3)!} \leq q_n^{-n-1}$$

for every $n \geq 1$. Thus y is transcendental.

[There are many ways we might choose q_n, each producing a minor variation of the proof.]

Exercise 7.3.6

(i) This is just long division.

If the reader wishes to be more formal, observe that
$$a_nt^n+a_{n-1}t^{n-1}+\ldots+a_0 = a_nt^{n-1}(t-u)+(a_{n-1}+ua_n)t^{n-1}+a_{n-2}t^{n-2}+\ldots+a_0,$$
so any polynomial P_n of degree at most n can be written as
$$P_n(t) = a_nt^{n-1}(t-u) + P_{n-1}(t)$$
with P_{n-1} some polynomial of degree at most $n-1$. Repeated use of this observation gives
$$\begin{aligned} P_n(t) &= b_{n-1}t^{n-1}(t-u) + P_{n-1}(t) \\ &= b_{n-1}t^{n-1}(t-u) + b_{n-2}t^{n-2}(t-u) + P_{n-2}(t) \\ &\vdots \\ &= b_{n-1}t^{n-1}(t-u) + b_{n-2}t^{n-2}(t-u) + \ldots + b_1t(t-u) + P_1(t) \\ &= b_{n-1}t^{n-1}(t-u) + b_{n-2}t^{n-2}(t-u) + \ldots + b_1t(t-u) + b_0(t-u) + r \end{aligned}$$
with appropriate polynomials P_j of degree at most j, appropriate b_j and appropriate r.

Taking $P_n = P$ and factorising, we have
$$P(t) = (t-u)Q(t) + r$$
with $Q(t) = b_{n-1}t^{n-1} + b_{n-2}t^{n-2} + \ldots + b_0$.

(ii) Setting $t = u$, we have
$$0 = P(u) = (u-u)Q(u) + r = 0 + r = r$$
so $r = 0$ and
$$P(t) = (t-u)Q(t).$$
If $P(v) = 0$ then $(v-u)Q(v) = 0$, so either $v-u = 0$ or $Q(v) = 0$.

(iii) Thus, if a polynomial P_n of degree n has a root, any further roots must also be roots of some fixed polynomial P_{n-1} of degree $n-1$. If P_{n-1} has a root, any further roots must also be roots of some fixed polynomial P_{n-2} of degree $n-2$. Repeating this argument $n-1$ times, we see that, if P_n has $n-1$ distinct roots, any further roots must also be roots of some fixed polynomial P_1 of degree 1. But P_1 has at most 1 root, so P_n has at most n roots.

[The reader may prefer to set these proofs out as formal inductions.]

(iv) Suppose P has degree exactly r for some r with $n \geq r \geq 0$. If $r \geq 1$ then P vanishes at most r times. Thus $r = 0$, P is a constant and so P is zero.

Exercise 8.1.1

There are many ways of answering the question. It is worth remarking that information costs money and the more information anything contains the more expensive it is likely to be to produce.

Exercise 8.1.3

We have

$$(h^2 + k^2)^{1/2} \geq (h^2)^{1/2} = |h|$$

and, similarly, $(h^2 + k^2)^{1/2} \geq |k|$, so

$$\max\{|h|, |k|\} \leq (h^2 + k^2)^{1/2}.$$

On the other hand,

$$(h^2+k^2)^{1/2} \leq \left((\max\{|h|, |k|\})^2+(\max\{|h|, |k|\})^2\right)^{1/2} = 2^{1/2} \max\{|h|, |k|\}.$$

Exercise 8.1.5

Suppose that we set $h = 0$ and allow k to vary. Then equation ★ becomes

$$g(x, y + k) = g(x, y) + Bk + o(k),$$

so the function h_x of one variable, defined by $h_x(y) = f(x, y)$, is differentiable with $h_x'(y) = B$. We shall write $\partial_2 g(x, y) = h_x'(y)$.

Equation ★ yields

$$\begin{aligned} g(x + h, y + k) &= g(x, y) + Ah + Bk + o\big((h^2 + k^2)^{1/2}\big) \\ &= g(x, y) + \big(\partial_1 g(x, y)h + \partial_2 g(x, y)k\big) + o\big((h^2 + k^2)^{1/2}\big). \end{aligned}$$

Exercise 8.2.1

By translation (that is to say, by writing $f(s,t) = g(x+s, y+t)$ and applying the previous result to f)

$$\begin{aligned} g(x+h, y+k) = g(x,y) &+ \partial_1 g(x,y)h + \partial_2 g(x,y)k \\ &+ \frac{1}{2}(\partial_1\partial_1 g(x,y)h^2 \\ &\quad + (\partial_2\partial_1 g(x,y) + \partial_1\partial_2 g(x,y))hk \\ &\qquad + \partial_2\partial_2 g(x,y)k^2) + o(h^2+k^2). \end{aligned}$$

Exercise 8.2.2

We have

$$\begin{aligned}
\partial_1 G(x,y) &= b_1 + 2c_{11}x + 2c_{12}y \\
\partial_2 G(x,y) &= b_2 + 2c_{12}x + 2c_{22}y \\
\partial_1\partial_1 G(x,y) &= 2c_{11} \\
\partial_1\partial_2 G(x,y) &= 2c_{12} \\
\partial_2\partial_1 G(x,y) &= 2c_{12} \\
\partial_2\partial_2 G(x,y) &= 2c_{22},
\end{aligned}$$

and so

$$\begin{aligned}
g(0,0) &= a_0 \\
\partial_1 G(0,0) &= b_1 \\
\partial_2 G(0,0) &= b_2 \\
\partial_1\partial_1 G(0,0) &= 2c_{11} \\
\partial_1\partial_2 G(0,0) &= 2c_{12} \\
\partial_2\partial_1 G(0,0) &= 2c_{12} \\
\partial_2\partial_2 G(0,0) &= 2c_{22}.
\end{aligned}$$

Thus

$$\begin{aligned}
&G(0,0) + \partial_1 G(0,0)h + \partial_2 G(0,0)k \\
&\qquad + \frac{1}{2}\big(\partial_1\partial_1 G(0,0)h^2 + \partial_2\partial_1 G(0,0)hk + \partial_1\partial_2 G(0,0)kh + \partial_2\partial_2 G(0,0)k^2\big) \\
&\qquad = a_0 + b_1 h + b_2 k + \frac{1}{2}(2c_{11}h^2 + 2c_{12}hk + 2c_{12}kh + 2c_{22}k^2) \\
&\qquad = a_0 + b_1 h + b_2 k + c_{11}h^2 + 2c_{12}hk + c_{22}k^2 = G(h,k).
\end{aligned}$$

Exercise 8.2.3

(Very sketchy solution indeed.)

I shall make things easier for myself by replacing the coordinates (h, k) by (h_1, h_2) and using the summation sign

$$\sum_{i=1}^{2} a_i = a_1 + a_2.$$

Repeating our arguments to obtain $f'''(r)$ from $f''(r)$ and setting $r\cos\theta = h_1$, $r\sin\theta = h_2$, we get

$$\begin{aligned} g(h_1, h_2) = g(0,0) + \sum_{i=1}^{2} \partial_i g(0,0) h_i + \frac{1}{2!} \sum_{i=1}^{2}\sum_{j=1}^{2} n_{ij} \partial_i \partial_j g(0,0) h_j h_i \\ + \frac{1}{3!} \sum_{i=1}^{2}\sum_{j=1}^{2} n_{ijk} \sum_{k=1}^{2} \partial_i \partial_j \partial_k g(0,0) h_k h_j h_i + o\big((h^2+k^2)^{3/2}\big). \end{aligned}$$

with $n_{ij} = 1$ if $i \neq j$, $n_{ii} = 2$, $n_{ijk} = 1$ if i, j and k are all different, $n_{ijk} = 3$ if exactly two of the i,j,k are unequal and $n_{iii} = 6$.

If

$$G(x_1, x_2) = a_0 + \sum_{i=1}^{2} a_i x_i + \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij} x_j x_i + \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} a_{ijk} x_k x_j x_i,$$

then

$$\begin{aligned} G(0,0) &= a_0 \\ \partial_r G(0,0) &= a_r \\ \partial_r \partial_s G(0,0) &= a_{rs} + a_{sr} \\ \partial_r \partial_s \partial_q G(0,0) &= a_{rsq} + a_{rqs} + a_{srq} + a_{sqr} + a_{qrs} + a_{qsr}, \end{aligned}$$

so

$$\begin{aligned} G(h_1, h_2) = g(0,0) + \sum_{i=1}^{2} \partial_i G(0,0) h_i + \frac{1}{2!} \sum_{i=1}^{2}\sum_{j=1}^{2} n_{ij} \partial_i \partial_j G(0,0) h_j h_i \\ + \frac{1}{3!} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} n_{ijk} \partial_i \partial_j \partial_k G(0,0) h_k h_j h_i. \end{aligned}$$

Exercise 8.2.4

Let O be the origin, X the point $(x, 0)$, Y the point $(0, y)$ and Z the point (x, y). We note that $OXZY$ forms a rectangle.

Under the given rotation O is fixed, X goes to X' with coordinates

$$(x \cos\theta, x \sin\theta)),$$

Y goes to Y' with coordinates

$$(-y \sin\theta, y \cos\theta)$$

and Z goes to Z'. Since $OX'Z'Y'$ remains an rectangle, Z' must have coordinates

$$(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$$

Exercise 8.2.5

(i) If $c = a$, take $\theta = \pi/4$. If $c \neq a$ take

$$\tan 2\theta = \frac{2b}{a-c}.$$

(ii) We have

$$\begin{aligned}
G(s,t) &= \frac{1}{2}\big(a(s\cos\theta - t\sin\theta)^2 + 2b(s\cos\theta - t\sin\theta)(s\sin\theta + t\cos\theta)\\
&\qquad + c(s\sin\theta + t\cos\theta)^2\big) + o(s^2+t^2)\\
&= \frac{1}{2}\big(as^2(\cos\theta)^2 - 2ast\cos\theta\sin\theta + t^2(\sin\theta)^2\\
&\quad - 2bt^2\sin\theta\cos\theta + 2bst((\cos\theta)^2 - (\sin\theta)^2) + 2bs^2\sin\theta\cos\theta\\
&\quad + ct^2(\cos\theta)^2 + 2cst\cos\theta\sin\theta + cs^2(\sin\theta)^2\big) + o(s^2+t^2)\\
&= \frac{1}{2}(ut^2 + 2vst + ws^2) + o(s^2+t^2)
\end{aligned}$$

with

$$\begin{aligned}
u &= a(\sin\theta)^2 - 2b\cos\theta\sin\theta + c(\cos\theta)^2,\\
2v &= -2a\cos\theta\sin\theta + 2b((\cos\theta)^2 - (\sin\theta)^2) + 2c\cos\theta\sin\theta,\\
w &= a(\cos\theta)^2 + 2b\cos\theta\sin\theta + c(\sin\theta)^2,
\end{aligned}$$

so

$$2v = -a\sin 2\theta + 2b\cos 2\theta + c\sin 2\theta = (c-a)\sin 2\theta + 2b\cos 2\theta.$$

Exercise 8.2.6★

Exercise 8.2.7

Suppose that $f(x, y)$ is a function of two variables defined for

$$x^2 + y^2 \leq 1.$$

Then $(0, 0)$ is a local maximum, if we can find a u with $0 < u < 1$ such that

$$f(0, 0) \geq f(x, y)$$

for $x^2 + y^2 \leq u^2$.

(Or some variation on this theme.)

Exercise 8.2.8

This is the one-dimensional version of the remarks on marshy hilltops.

Let f be the length of daylight at time t. At a maximum or minimum $f'(t) = 0$, so near a maximum or minimum $f'(t)$ is small.

[The paragraph on units of currency (page 14) is vaguely relevant.]

Exercise 8.3.1

Remove all land masses external to Australia. We have the situation as before with one large lake external to Australia. Thus, for Australia,

$$\text{no. peaks} + \text{no. low points} - \text{no. passes} = 1.$$

Exercise 8.3.2★

Exercise 9.2.1

If $f(u,v) = uv$, then $y'(t) = ty(t)$ can be rewritten as

$$y'(t) = f(y(t), t).$$

If $f(u,v) = (1+v^2)^{-1}$, then $y'(t)\big(1+y(t)^2\big) = 1$ can be rewritten as

$$y'(t) = f(y(t), t).$$

If $f(u,v) = u^3 \sin v$, then $y'(t) = t^3 \sin\big(y(t)\big)$ can be rewritten as

$$y'(t) = f(y(t), t).$$

Exercise 9.2.2

(i) Euler's method gives

$$y_{r+1} - y_r = h \times rh = rh^2,$$

so

$$\begin{aligned} y_n &= (y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \ldots + (y_1 - y_0) + y_0 \\ &= \Big((n-1) + (n-2) + \ldots + 1 + 0\Big)h^2 + 0 = \tfrac{1}{2}n(n-1)h^2. \end{aligned}$$

By the fundamental theorem of the calculus, the exact solution is $y(t) = t^2/2$, so

$$|y_N - y(T)| = \frac{1}{2}|N(N-1)h^2 - (Nh)^2| = \frac{Nh^2}{2} = \frac{Th}{2}.$$

(ii) Euler's method gives

$$y_{r+1} - y_r = bhy_r,$$

so $y_{r+1} = y_r(1 + bh)$ and

$$y_n = (1 + bh)y_{n-1} = (1 + bh)^2 y_{n-2} = \ldots = (1 + bh)^n y_0.$$

If T is fixed, then, when N is large

$$y_N = \left(1 + \frac{bh}{N}\right)^N y_0 \approx y_0 e^{bT}$$

(by the result on page 89).

This reassuring because $y(T) = y_0 e^{bT}$ is the correct answer (see Section 4.2).

Exercise 9.2.3

Euler's method gives

$$y_{r+1} - y_r = -2y_r,$$

so

$$y_{r+1} = -y_r$$

and we get $y_r = (-1)^r$ although the correct solution is $y(t) = e^{-8t}$ so $y(r/4) = e^{-r/2}$.

We have simply taken the step length too large.

Exercise 9.2.4★

Exercise 9.3.1

There are several (more or less equivalent) ways of doing this.

If

$$g(t) = Ay(s+2t) + By(s+t) - ty'(s) - By(s-t) - Ay(s-2t),$$

then

$$\begin{aligned} g'(t) &= 2Ay'(s+2t) + By'(s+t) - y'(s) + By'(s-t) + 2Ay'(s-2t) \\ g''(t) &= 4Ay''(s+2t) + By''(s+t) - By''(s-t) - 4Ay''(s-2t) \\ g'''(t) &= 8Ay'''(s+2t) + By'''(s+t) + By'''(s-t) + 8Ay'''(s-2t) \\ g^{(4)}(t) &= 16Ay^{(4)}(s+2t) + By^{(4)}(s+t) - By^{(4)}(s-t) - 16Ay^{(4)}(s-2t) \\ y^{(5)}(t) &= 32Ay^{(5)}(s+2t) + By^{(5)}(s+t) + By^{(5)}(s-t) + 32Ay^{(5)}(s-2t), \end{aligned}$$

so that

$$\begin{aligned} g(0) &= 0 \\ g'(0) &= (4A+2B-1)y'(s) \\ g''(0) &= 0 \\ g'''(0) &= (16A+2B)y'''(s) \\ g^{(4)}(0) &= 0. \end{aligned}$$

Solving the equations

$$\begin{aligned} 4A+2B &= 1 \\ 16A+2B &= 0, \end{aligned}$$

we see that $g(s) = g'(s) = g''(s) = g'''(s) = g^{(4)}(s) = 0$ for all choices of y if $A = -1/12$, $B = 2/3$ and then Taylor's theorem gives

$$|g(h)| \leq (64|A| + 2|B|)M\frac{h^5}{5!} = \frac{20}{3}M\frac{|h|^5}{5!} = \frac{M}{18}h^5$$

if $|g^{(5)}(x)| \leq M$ over the appropriate range

Translating, we see that, if $A = -1/12$, $B = 2/3$,

$$|Ay(s+2h) + By(s+h) - hy'(s) - By(s-h) - Ay(s-2h)| \leq KMh^5$$

for some appropriate constant K.

Exercise 9.3.2★

Exercise 9.3.3★

Exercise 9.3.4

(i) We observe that, if u_n and u_{n-1} are given,

$$u_{n+1} = -bu_n - cu_{n-1}$$

is determined.

(ii) We have

$$p^2 + bp + c = (p - p)(p - q) = 0$$

and so, multiplying through by p^{n-2},

$$p^n + bp^{n-1} + cp^{n-2} = 0.$$

(iii) Similarly

$$q^n + bq^{n-1} + cq^{n-2} = 0.$$

Thus

$$v_n+bv_{n-1}+cv_{n-2} = A(p^n+bp^{n-1}+cp^{n-2})+B(q^n+bq^{n-1}+cq^{n-2}) = 0+0 = 0.$$

(iv) Subtracting the second equation from p times the first, we get $B(p - q) = pu_0 - u_1$, so that

$$B = \frac{pu_0 - u_1}{p - q}$$

and similarly

$$A = \frac{qu_0 - u_1}{q - p}.$$

By inspection these are solutions.

(v) Choose A and B as in (iv). By (iii), $v_n = Ap^n + Bq^n$ is a solution of our difference equation with $v_0 = u_0$, $v_1 = u_1$. By (i),

$$u_n = v_n = Ap^n + Bq^n.$$

Exercise 9.3.5

The standard formula gives two roots

$$-Kh \pm \sqrt{1 + K^2h^2},$$

so (recalling that $Kh > 0$ and we want $p > 0$) we have

$$p = -Kh + \sqrt{1 + K^2h^2},\ q = -Kh - \sqrt{1 + K^2h^2}$$

and

$$pq = (-Kh)^2 - (1 + K^2h^2) = -1.$$

Exercise 9.3.6

Let $f(t) = (1+t)^{1/2}$. Choose L so that $|f''(t)| \leq L$ for $|t| \leq 1/2$. By Taylor's theorem we can find a $u > 0$ and an $L > 0$ such that

$$|f(t) - f(0) - f'(0)t| \leq Lt^2/2$$

for $|t| \leq 1/2$.

Since $f'(t) = 2^{-1}(1+t)^{1/2}$, this gives

$$|(1+t)^{1/2} - 1 - t/2| \leq Lt^2/2$$

for $|t| \leq 1/2$. If we set $M = LK^4/2$ and consider $t = (Kh)^2$ we obtain

$$|(1+K^2h^2)^{1/2} - 1 - K^2h^2/2| \leq Mh^4$$

for $|h| \leq 2^{-1/2}K^{-1}$.

We thus have

$$|-Kh + (1+K^2h^2)^{1/2} - 1 + Kh - K^2h^2/2| \leq Mh^4$$

The mean value inequality tells us that

$$|x^N - y^N| \leq N(\max\{|x|, |y|\})^N |x-y|$$

so, since $|-Kh + (1+K^2h^2)^{1/2}| \leq 1$ and $|1 - Kh - K^2h^2/2| \leq 1$ for h small,

$$|(-Kh + (1+K^2h^2)^{1/2})^N - (1 + Kh - K^2h^2/2)^N| \leq NMh^4$$

Let $h = T/N$. Provided N is large, Exercise 6.3.5 gives

$$\left|\left(1 - \frac{KT}{N} + \frac{KT^2}{2N^2}\right)^N - e^{-KT}\right| \leq \frac{L}{N^2}$$

for some L ie

$$\left|\left(1 - Kh + \frac{Kh}{2}\right)^N - \exp -KT\right| \leq LT^2h^2.$$

Combining the results of the last two paragraphs

$$|(-Kh+(1+K^2h^2)^{1/2})^N - e^{-KT}| \leq NMh^4 + LT^2h^2 = TMh^3 + LT^2h^2 \leq Ch^2$$

for an appropriate C provided N is large enough.

Exercise 9.3.7

We have

$$q = -Kh - \sqrt{1 + K^2h^2}$$

so

$$p^{-N} = q^N = (-1)^N \big(Kh + \sqrt{1 + K^2h^2}\big)^N \approx (-1)^N (1+Kh)^N = (-1)^N \left(1 + \frac{KT}{N}\right)^N \approx (-1)^N e^{K$$

Exercise 10.1.1

We have

$$\begin{aligned}
F(x) &= (1+x)^{-1} \\
F'(x) &= (-1)(1+x)^{-2} \\
F''(x) &= (-1)\times(-2)\times(1+x)^{-3} \\
&\vdots \\
F^{(n)}(x) &= (-1)\times(-2)\times\ldots(-n)\times(1+x)^{-n-1} = (-1)^n n!(1+x)^{-n-1},
\end{aligned}$$

so

$$F^{(n)}(0) = (-1)^n n!$$

and

$$\frac{1}{1+x} \underset{?}{=} 1 - x + x^2 - x^3 + x^4 + \ldots.$$

Exercise 10.1.2

We know, from Exercise 10.1.1, that
$$F^{(n+1)}(t) = (-1)^n (n+1)! \times (1+t)^{-n-2}.$$
Thus
$$|F^{(n+1)}(0)| = (n+1)!$$
and
$$|F^{(n+1)}(t)| \leq (n+1)!$$
for $0 \leq t \leq x$.

Exercise 10.1.3

Observe that

$$\begin{aligned}(1+x)&\left(1-x+x^2-x^3+x^4-\ldots+(-1)^n x^n+\frac{(-1)^{n+1}x^{n+1}}{1+x}\right)\\&=1-(x-x)+(x^2-x^2)-\ldots\\&\qquad+(-1)^{n-1}(x^n-x^n)+(-1)^n x^{n+1}+(-1)^{n+1}x^{n+1}\\&=1.\end{aligned}$$

Exercise 10.1.4

(i) We know that

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 + \ldots + (-1)^n t^n + \frac{(-1)^{n+1} t^{n+1}}{1+t}$$

for $t \geq 0$, so, setting $t = x^2$, we have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \ldots + (-1)^n x^{2n} + \frac{(-1)^{n+1} x^{2(n+1)}}{1+x^2},$$

which is what we are asked to prove.

(ii) If $|x| < 1$, then x^{2n} and so $R_n(G, x)$ can be made as small as we like by taking n large enough, so we can approximate $G(x)$ arbitrarily well by taking sufficiently many terms in the Taylor expansion. If $|x| \geq 1$, then $R_n(G, x) \geq (1 + x^2)$, so the approximation does not improve.

Exercise 10.1.5

(i) We have

$$G_a(x) = \frac{a^2}{a^2 + x^2} = \frac{1}{1 + (x/a)^2} = G(x/a).$$

(ii) and (iii) The remaining results follow at once from Exercise 10.1.3 on using the formula of part (i).

Exercise 10.1.6

Observe that
$$\frac{t^n}{1+t} \geq \frac{t^n}{2},$$
for $0 \leq t \leq 1$, so, if $0 \leq x \leq 1$,
$$|T_n(x)| = \int_0^x \frac{t^n}{1+t}\,dt \geq \int_0^x \frac{t^n}{2}\,dt = \frac{x^{n+1}}{2(n+1)}.$$
In particular, if $x = 1$,
$$|T_n(1)| \geq \frac{1}{2(n+1)}$$
and we would need at least $10^6/2$ terms to get $\log 2$ correct to 6 decimal places using (D) directly.

If $0 \leq t \leq 1$, then
$$\frac{t^n}{1+t} \leq t^{n+1},$$
so, if $0 \leq x \leq 1$,
$$|T_n(x)| = \int_0^x \frac{t^n}{1+t}\,dt \leq \int_0^x t^n\,dt = \frac{x^n}{n+1}.$$
If we compute $\log(1+x)$ with $0 \leq x \leq 1/2$
$$|T_n(x)| \leq 2^{-n}/(n+1).$$
In particular, if we use the use the first 20 terms of (C) the error will be less than 10^{-7}. The second suggested procedure will thus be much more efficient.

An even better approach would use much smaller values of x and add a correspondingly larger number of evaluations of such $\log(1+x)$.

Exercise 10.1.7

If $0 \leq t \leq 2$, then
$$\frac{t^n}{1+t} \geq \frac{t^n}{3}.$$
Thus, if $x > 1$ and we write $y = \min\{x, 2\}$,
$$\begin{aligned}
|T_n(x)| &= \int_0^x \frac{t^n}{1+t}\,dt \\
&\geq \int_0^y \frac{t^n}{1+t}\,dt \\
&= \int_0^y \frac{t^n}{3}\,dt \\
&= \frac{y^{n+1}}{3(n+1)} = \frac{(\min\{x,2\})^n}{3(n+1)}
\end{aligned}$$
which is large when n is large.

(If $y > 1$, then y^n/n is large when n is large. There are many ways of seeing this. For example, consider $f(t) = y^t/t$ and show that $f'(t) > 2$ for t large.)

Exercise 10.1.8

$$\log 2 \underset{?}{=} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16} \ldots + (-1)^{4n+1}\frac{1}{4n} + \ldots.$$
$$\frac{1}{2}\log 2 \underset{?}{=} \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} \ldots + (-1)^{2n+1}\frac{1}{4n} + \ldots$$

so, summing in the indicated manner, we get

$$\frac{3}{2}\log 2 \underset{?}{=} 1+0+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+0+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+0+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+0+\frac{1}{15}-\frac{1}{8}+\ldots.$$

Before we remove the zeros,
the $4n+1$th term is $1/(4n+1)+0 = 1/(4n+1)$,
the $4n+2$th term is $-1/(4n+2)+1/(2\times(2n+1)) = 0$,
the $4n+3$th term is $1/(4n+3)+0 = 1/(4n+3)$,
the $4n+4$th term is $-1/(4n+4)-1/(2\times(2n+2)) = -1/(2n+2)$.
After we remove the zeros, the $3k+1$th term is $1/(4k+1)$,
the $3k+2$th term is $1/(4k+3)$,
the $3k+3$th term is $-1/(2(k+1))$.
We have shown that the terms of the expression are the numbers $(-1)^{r+1}/r$ with each of these numbers occurring exactly once.

Exercise 10.2.1

We have

$$\begin{aligned} a_{2n-1} &= \frac{-1}{(2n-2)\times(2n-1)}a_{2n-3} \\ &= \frac{1}{(2n-4)\times(2n-3)\times(2n-2)\times(2n-1)}a_{2n-5} \\ &= \ldots = \frac{(-1)^{n-1}}{(2n-1)!}a_1. \end{aligned}$$

Exercise 10.2.2

We have $f'(t) = -A\sin t + B\cos t$ and so

$$f''(t) = -A\cos t - B\sin t = -f(t).$$

Thus

$$f''(t) + f(t) = 0.$$

Exercise 10.2.3

Assuming that we can expand g as

$$g(t) \underset{?}{=} a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots.$$

and that we can differentiate the infinite sum in the same way as we differentiate a polynomial, we get

$$g'(t) \underset{?}{=} 1 \times a_1 + 2 \times a_2 t + 3 \times a_3 t^2 + \ldots$$

and so the equation $g(t) = -2g'(t)$ gives us

$$1 \times a_1 + 2 \times a_2 t + 3 \times a_3 t^2 + \ldots \underset{?}{=} -2 \times a_0 t - 2 \times a_1 t^2 - 2 \times a_2 t^3 - \ldots$$

Assuming that we can equate coefficients in the same way as we do for polynomials we obtain

$$\begin{aligned}
-2a_0 &= 2a_2\\
-2a_1 &= 0\\
-2a_2 &= -4a_4\\
-2a_3 &= 0\\
&\vdots\\
-2a_{2n} &= -(2n+2)a_{2n+2}\\
-2a_{2n+1} &= 0\\
&\vdots
\end{aligned}$$

Thus $a_{2r+1} = 0$ for all r and

$$a_{2r+2} = \frac{-1}{r+1} a_{2r}.$$

The equation just stated gives us

$$a_{2n} = \frac{-1}{n} a_{2(n-1)} = \frac{1}{n(n-1)} a_{2(n-2)} = \ldots = \frac{(-1)^n}{n!} a_0,$$

so, if we write $A = a_0$, we get

$$g(t) \underset{?}{=} A\left(1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} + \ldots\right) \underset{?}{=} A\exp(-t^2).$$

The function of a function rule shows that, if we write $g(t) = A\exp(-t^2)$, then $g'(t) = -2tg(t)$.

[We can do better than this, although the reader was not asked to. Suppose that

$$G'(t) = -2tG(t).$$

Then, writing $F(t) = \exp(t^2)G(t)$, we have, by the product rule, that $F'(t) = 0$ so $F(t) = A$ for some constant A and so $G(t) = A\exp(-t^2)$. Thus we have discovered all the solutions of our differential equation.]

Exercise 10.2.4

We obtain $c_n x^n$ as the sum of terms of the form $a_{n-j}x^{n-j} \times b_j x^j$, so

$$(a_0 + a_1 x + a_2 x^2 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots) \underset{?}{=} (c_0 + c_1 x + c_2 x^2 + \dots)$$

with

$$c_r = a_0 b_r + a_1 b_{r-1} + a_2 a_{r-2} + \dots + a_r b_0.$$

Assuming that

$$f(t) \underset{?}{=} a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots.$$

and that we can differentiate the infinite sum in the same way as we differentiate a polynomial, we get

$$f'(t) \underset{?}{=} 1 \times a_1 + 2 \times a_2 t + 3 \times a_3 t^3 + \dots$$

and so the equation $f'(t) = -2tf(t)^2$ gives us

$$1 \times a_1 + 2 \times a_2 t + 3 \times a_3 t^2 + \dots \underset{?}{=} -2 \times c_0 t - 2 \times c_1 t^2 - 2 \times c_2 t^3 - \dots$$

where

$$c_n = a_0 a_n + a_1 a_{n-1} + \dots + a_n a_0$$

so

$$\begin{aligned} a_1 &= 0 \\ 2a_2 &= -2c_0 \\ 3a_3 &= -2c_1 \\ 4a_4 &= -2c_2 \\ &\vdots \\ na_n &= -2c_{n-1} \\ &\vdots \end{aligned}$$

We have $a_0 = B$ for some B and $a_1 = 0$.

Suppose that $a_r = (-1)^{r/2} B^{(r+2)/2}$ when r is even and $r < n$ and $a_r = 0$ for r odd and $r < n$. If n is odd, $r + s = n$ implies that r is odd or s is odd, so $c_{n-2} = 0$ and the equations above give $a_n = 0$. If n is even, then

$$\begin{aligned} c_{n-2} &= a_0 a_{n-2} + a_2 a_{n-4} + \dots + a_{n-2} a_0 \\ &= (-1)^{(n-2)/2} B^{(n+2)/2} + (-1)^{(n-2)/2} B^{(n+2)/2} + \dots + (-1)^{(n-2)/2} B^{(n+2)/2} \\ &= (-1)^{(n-2)/2} \frac{n}{2} B^{(n+2)/2}, \end{aligned}$$

so

$$na_n = (-1)^{n/2} n B^{(n+2)/2}$$

and $a_n = (-1)^{n/2} B^{(n+2)/2}$.

Thus

$$a_{2n-1} = 0,\ a_{2n} = (-1)^n B^{n+1},$$

where B is a constant, and we have

$$f(x) \underset{?}{=} B - B^2x^2 + B^3x^4 - \ldots \underset{?}{=} \frac{B}{1 + Bx^2}$$

If $B > 0$ and $A = B^{-1}$, we get

$$f(x) = \frac{1}{A + x^2}.$$

If $g(x) = (A + x^2)^{-1}$, then, by the quotient rule,

$$g'(x) = \frac{-2x}{(A + x^2)^2} = -2xg(x)^2.$$

Exercise 11.3.2

(i) f is continuous and $f(0) = -2 < 0 < 2 = f(2)$, so there exists a c with $0 < c < 2$ such that $f(c) = 0$ and so $c^2 = 2$.

(ii) f is continuous and $f(1) = -1 < 0 < f(N)$, so there exists a c with $0 < c < N$ such that $f(c) = 0$ and so $\log c = 1$.

Exercise 11.3.3

Let $f(\theta)$ be the difference between the temperature at longitude θ and the temperature at the point diametrically opposite. Observe that (writing r for a right angle) $f(\theta + 2r) = -f(\theta)$.

In particular, $f(0) = -f(2r)$ and the intermediate value theorem tells us that there must exist a t with $0 \leq t \leq 2r$ and $f(t) = 0$.

Exercise 11.3.4

For example, on page 48 we say (in case (2)) '$f'(0) < 0$ and $f'(s) > 0$ for some large s, so there will be an $s_0 > 0$ with $f'(s_0) = 0$.'

Exercise 11.4.1

Observation 1. If we take $f(x) = x$, then $f(|K| + 1) > K$, so f is unbounded. If $u > v$, then $f(u) > f(v)$, so f has no local maxima.

Observation 2. If we take

$$f(x) = \frac{x^2}{1 + x^2},$$

then $0 \leq f(x) \leq 1$. We note that

$$f(x) = 1 - \frac{1}{1 + x^2}$$

so, if $|u| > |v|$, $f(u) > f(v)$. Thus f has no local maxima (and so no global maximum).

Observation 3. If we restrict our attention to those x with $0 < x < 1$ and set $f(x) = 1/x$, then $f((1 + K)^{-1}) > K$ and $0 < (1 + K)^{-1} < 1$ for $K > 0$, so f is unbounded. If $0 < u < v < 1$, then $f(u) > f(v)$, so f has no local maxima

Observation 4. If we restrict our attention to those x with $0 < x < 1$ and set $f(x) = x$, then $0 \leq f(x) \leq 1$ whenever $0 < x < 1$, but, if $0 < x < 1$ and $x < y < 1$, then $f(y) > f(x)$, so f has no local maxima and so no global maximum.

Exercise 11.5.1

By the fundamental theorem,

$$G'(t) = g'(t) - f(t) = 0$$

so, by Plausible Statement B,

$$G(t) = c$$

for some constant and so

$$g(t) = \int_a^t f(x)\,dx + c.$$

Exercise 11.5.2

If $g'(t) = 0$ for all t, then $|g'(t)| \leq 0$ for all t, whence

$$|g(b) - g(a)| \leq 0|b - a| = 0.$$

Thus $g(a) = g(b)$ for all a and b so $g(x) = g(0)$ for all x.