Topics in Fourier and Complex Analysis
Part III, Autumn 2009

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Small print This is just a first draft of the first part of the course. I suspect these notes will cover the first 16 hours but I will not be unduly surprised if it takes the entire course to cover the material. The content of the course will be what I say, not what these notes say. Experience shows that skeleton notes (at least when I write them) are very error prone so use these notes with care. I should **very much** appreciate being told of any corrections or possible improvements and might even part with a small reward to the first finder of particular errors.

This course definitely requires a first course in complex variable and enough analysis to be happy with terms like norm, complete metric space and compact. I am happy to give classes on any topics that people request. At at least one point, the course requires measure theory, but you need only quote the required results in examination.

Contents

1 Non-existence of functions of several variables

Theorem 1. *Let λ be irrational We can find increasing continuous functions $\phi_j : [0,1] \to \mathbb{R}$ $[1 \leq j \leq 5]$ with the following property. Given any continuous function $f : [0,1]^2 \to \mathbb{R}$ we can find a function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x,y) = \sum_{j=1}^{5} g(\phi_j(x) + \lambda\phi_j(y)).$$

The main point of Theorem 1 may be expressed as follows.

Theorem 2. *Any continuous function of two variables can be written in terms of continuous functions of one variable and addition.*

That is, there are no true functions of two variables!

For the moment we merely observe that the result is due in successively more exact forms to Kolmogorov, Arnol'd and a succession of mathematicians ending with Kahane whose proof we use here. It is, of course, much easier to prove a specific result like Theorem 1 than one like Theorem 2.

Our first step is to observe that Theorem 1 follows from the apparently simpler result that follows.

Lemma 3. *Let λ be irrational. We can find increasing continuous functions $\phi_j : [0,1] \to \mathbb{R}$ $[1 \leq j \leq 5]$ with the following property. Given any continuous function $F : [0,1]^2 \to \mathbb{R}$ we can find a function $G : \mathbb{R} \to \mathbb{R}$ such that $\|G\|_\infty \leq \|F\|_\infty$ and*

$$\sup_{(x,y)\in[0,1]^2} \left| F(x,y) - \sum_{j=1}^{5} G(\phi_j(x) + \lambda\phi_j(y)) \right| \leq \frac{999}{1000}\|F\|_\infty.$$

(The choice of the constant 999/1000 is, of course, pretty arbitrary.)

Next we make the following observation.

Lemma 4. *We can find a sequence of functions $f_n : [0,1]^2 \to \mathbb{R}$ which are uniformly dense in $C([0,1])^2$.*

This enables us to obtain Lemma 3 from a much more specific result.

Lemma 5. *Let λ be irrational and let the f_n be as in Lemma 4. We can find increasing continuous functions $\phi_j : [0,1] \to \mathbb{R}$ $[1 \leq j \leq 5]$ with the following property. We can find functions $g_n : \mathbb{R} \to \mathbb{R}$ such that $\|g_n\|_\infty \leq \|f_n\|_\infty$ and*

$$\sup_{(x,y)\in[0,1]^2} \left| f_n(x,y) - \sum_{j=1}^{5} g_n(\phi_j(x) + \lambda\phi_j(y)) \right| \leq \frac{998}{1000}\|f_n\|_\infty.$$

One of Kahane's contributions is the observation that the proof Theorem 5 is made easier by the use of Baire category. Although most of the audience is familiar with Baire's category theorem, we shall reprove it here.

Theorem 6. [Baire's category theorem] *If (X,d) is a complete metric space then X can not be written as the union of a countable collection of closed sets with empty interior.*

One way of thinking of a closed set E with empty interior is the following. The property of belonging to E is *unstable* since arbitrarily small changes take one outside E but the the property of not belonging to E is *stable* since, if we are at a point outside E all sufficiently small changes keep us outside E.

We shall prove a slightly stronger version of Baire's theorem.

Theorem 7. *Let (X,d) be a complete metric space. If E_1, E_2, ... are closed sets with empty interiors then $X \setminus \bigcup_{j=1}^\infty E_j$ is dense in X.*

Exercise 8. *(If you are happy with general topology.) Show that a result along the same lines holds true for compact Hausdorff spaces.*

For historical reasons Baire's category theorem is associated with some rather peculiar nomenclature.

Definition 9. *Let (X,d) be a metric space. We say that a a subset A of X is* of the first category *if it is a subset of the union of a countable collection of closed sets with empty interior*[1] *We say that* quasi-all *points of X belong to the complement $X \setminus A$ of X.*

The following observations are trivial but useful.

[1]This usage is not universal. Some authors use the older definition which says that a subset A of X is of the first category if it is the union of a countable collection of closed sets with empty interior.However, so far as I know, all authors who use 'quasi-all' use it in he same way.

Lemma 10. *(i) The countable union of first category sets is itself of first category.*

(ii) If (X, d) is a complete metric space, then Baire's theorem asserts that X is not of first category.

Since Lemma 4 only involve a *countable* set of conditions we can use a Baire category argument. provided that we can find the correct metric space.

Lemma 11. *The space Y of continuous functions $\boldsymbol{\phi} : [0,1] \to \mathbb{R}^5$ with norm*

$$\|\boldsymbol{\phi}\|_\infty = \sup_{t\in[0,1]} \|\boldsymbol{\phi}(t)\|$$

is complete. The subset X of Y consisting of those $\boldsymbol{\phi}$ such that each ϕ_j is increasing is a closed subset of Y. Thus if d is the metric on X obtained by restricting the metric on Y derived from $\| \ \|_\infty$ we have (X, d) complete.

Exercise 12. *Prove Lemma 11*

Lemma 13. *Let $f : [0,1]^2 \to \mathbb{R}$ be continuous and let λ be irrational. Consider the set E of $\boldsymbol{\phi} \in X$ such that there exists a $g : \mathbb{R} \to \mathbb{R}$ such that $\|g\|_\infty \leq \|f\|_\infty$*

$$\sup_{(x,y)\in[0,1]^2} \left| f(x,y) - \sum_{j=1}^{5} g(\phi_j(x) + \lambda\phi_j(y)) \right| < \frac{998}{1000}\|f\|_\infty.$$

Then $X \setminus E$ is a closed set with dense complement in (X, d).

(Notice that it is important to take '$<$' rather than '$\leq$' in the displayed formula of Lemma 13.) Lemma 13 is the heart of the proof and once it is proved we can easily retrace our steps and obtain Theorem 1.

By using appropriate notions of information Vistuškin[2] was able to show that we can not replace continuous by continuously differentiable in Theorem 2. Thus Theorem 1 is an 'exotic' rather than a 'central' result. The next two sections are devoted to the proof of Vistuškin's simplest result.

We conclude the section with some exercises intended to help the reader understand the proof of Theorem 1.

Exercise 14. *Let (X, d) be as in Exercise 11. Show that quasi-all $\phi \in X$ have the property that ϕ_j is strictly increasing (that is to say $\phi_j(s) < \phi_j(t)$ for $0 \leq s < t \leq 1$) for each $1 \leq j \leq 5$. Why does this immediately tell us that we can replace the word 'increasing' by the words 'strictly increasing' in Theorem 1.*

[2]For those who wish to dispense with accents Vitushkin.

Exercise 15. *We say that real numbers λ_1, λ_2, …, λ_n are independent over $\mathbb{Q}$ if the equation*

$$\sum_{j=1}^{n} q_j\lambda_j = 0$$

has no solution with $q_j \in \mathbb{Q}$ $[1 \leq j \leq n]$ apart from the trivial solution with all $q_j = 0$.

By using the fact that the real numbers are uncountable, or otherwise, show that, if λ_1, λ_2, …, λ_n are independent over $\mathbb{Q}$ we can find λ_{n+1} such that λ_1, λ_2, …, λ_{n+1} are independent over $\mathbb{Q}$. Deduce that we can find λ_1, λ_2, … such that λ_1, λ_2, …, λ_n are independent over $\mathbb{Q}$ for all n.

Exercise 16. *Prove that, if λ_1, λ_2, λ_3. are independent over $\mathbb{Q}$ we can find increasing continuous functions $\phi_j : [0,1] \to \mathbb{R}$ $[0 \leq j \leq 6]$, with the following property. Given any continuous function $f : [0,1]^3 \to \mathbb{R}$ we can find a function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x_1, x_2, x_3) = \sum_{j=0}^{6} g\big(\lambda_1\phi_j(x_1) + \lambda_2\phi_j(x_2) + \lambda_3\phi(x_3)\big).$$

[You may prefer to identify the only part of the proof which might fail and concentrate on that.]

Exercise 17. *Investigate to what extent the proof would be simplified if we replaced Theorem 1 by the following slightly less demanding result.*

Given any continuous function $f : [0,1]^2 \to \mathbb{R}$ we can find continuous functions $\phi_r : [0,1] \to \mathbb{R}$ $[0 \leq r \leq 9]$ and continuous functions $g_j : \mathbb{R} \to \mathbb{R}$ $[0 \leq j \leq 4]$ such that

$$f(x,y) = \sum_{j=0}^{4} g_j\big(\phi_{2j}(x) + \phi_{2j+1}(y)\big).$$

Exercise 18. *Show that the result corresponding to Theorem 1 is false if $\lambda = 1$ or $\lambda = -1$. Show that it is true for all $\lambda \neq 1, -1$.*

2 Fourier series on the circle

We work on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ (that is on the interval $[0, 2\pi]$ with the two ends 0 and 2π identified). If $f : \mathbb{T} \to \mathbb{C}$ is integrable[3] we write

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\exp -int\,dt.$$

[3]That is to say Lebesgue integrable or Riemann integrable according to the reader's background.

We shall see (Lemma 28) that f is uniquely determined by its Fourier coefficients $\hat{f}(n)$. Indeed it is clear that there is a 'natural identification' (where natural is deliberately used in a vague sense)

$$f(t) \sim \sum_{r=-\infty}^{\infty} \hat{f}(r) \exp irt.$$

However, we shall also see that, even when f is continuous, $\sum_{r=-\infty}^{\infty} \hat{f}(r) \exp irt$ may fail to converge at some points t.

Fejér discovered that, although

$$S_n(f,t) = \sum_{r=-n}^{n} \hat{f}(r) \exp irt$$

may behave badly as $n \to \infty$, the average

$$\sigma_n(f,t) = (n+1)^{-1} \sum_{m=0}^{n} S_m(f,t) = \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} \hat{f}(r) \exp irt$$

behaves much better. (We call $\sigma_n(f,t)$ the Fejér sum. We also write $S_n(f,t) = S_n(f)(t)$ and $\sigma_n(f,t) = \sigma_n(f)(t)$.)

Exercise 19. *Let a_1, a_2, ... be a sequence of complex numbers.*

(i) Show that, if $a_n \to a$, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \to a$$

as $n \to \infty$.

(ii) By taking an appropriate sequence of 0s and 1s or otherwise find a sequence a_n such that a_n does not tend to a limit as $n \to \infty$ but $(a_1 + a_2 + \cdots + a_n)/n$ does.

(iii) By taking an appropriate sequence of 0s and 1s or otherwise find a bounded sequence a_n such that $(a_1 + a_2 + \cdots + a_n)/n$ does not tend to a limit as $n \to \infty$.

In what follows we define

$$f * g(t) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t-s)g(s)\, ds$$

(for appropriate f and g).

Lemma 20. *If f is integrable we have*

$$S_n(f) = f * D_n$$
$$\sigma_n(f) = f * K_n.$$

where

$$D_n(t) = \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)}$$
$$K_n(t) = \frac{1}{n+1}\left(\frac{\sin(\frac{n+1}{2}t)}{\sin(\frac{1}{2}t)}\right)^2$$

for $t \neq 0$.

The key differences between the Dirichlet kernel D_n and the Fejér kernel K_n are illustrated by the next two lemmas.

Lemma 21. *(i)* $\frac{1}{2\pi}\int_{\mathbb{T}} D_n(t)\,dt = 1.$

(ii) If $t \neq \pi$, then $D_n(t)$ does not tend to a limit as $n \to \infty$.

(iii) There is a constant $A > 0$ such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} |D_n(t)|\,dt \geq A \log n$$

for $n \geq 1$.

Lemma 22. *(i)* $\frac{1}{2\pi}\int_{\mathbb{T}} K_n(t)\,dt = 1.$

(ii) If $\eta > 0$, then $K_n \to 0$ uniformly for $|t| \geq \eta$ as $n \to \infty$.

(iii) $K_n(t) \geq 0$ for all t.

The properties set out in Lemma 22 show why Fejér sums work so well.

Theorem 23. *(i) If $f : \mathbb{T} \to \mathbb{C}$ is integrable and f is continuous at t, then*

$$\sigma_n(f, t) \to f(t)$$

as $n \to \infty$.

(ii) If $f : \mathbb{T} \to \mathbb{C}$ is continuous, then

$$\sigma_n(f) \to f$$

uniformly as $n \to \infty$.

Exercise 24. *Suppose that $L_n : \mathbb{T} \to \mathbb{R}$ is continuous (if you know Lebesgue theory you merely need integrable) and*

(A) $\displaystyle\frac{1}{2\pi}\int_{\mathbb{T}} L_n(t)\,dt = 1$,

(B) If $\eta > 0$, then $L_n \to 0$ uniformly for $|t| \geq \eta$ as $n \to \infty$,

(C) $L_n(t) \geq 0$ for all t.

(i) Show that, if $f : \mathbb{T} \to \mathbb{C}$ is integrable and f is continuous at t, then

$$L_n * f(t) \to f(t)$$

as $n \to \infty$.

(ii) Show that, if $f : \mathbb{T} \to \mathbb{C}$ is continuous, then

$$L_n * f \to f$$

uniformly as $n \to \infty$.

(iii) Show that condition (C) can be replaced by

(C') There exists a constant $A > 0$ such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} |L_n(t)|\,dt \leq A$$

in parts (i) and (ii). [You need only give the proof in one case and say that the other is 'similar'.]

Exercise 25. *Suppose that $L_n : \mathbb{T} \to \mathbb{R}$ is continuous but that*

$$\sup_n \frac{1}{2\pi}\int_{\mathbb{T}} |L_n(t)|\,dt = \infty.$$

Show that we can find a sequence of continuous functions $g_n : \mathbb{T} \to \mathbb{R}$ with $|g_n(t)| \leq 1$ for all t, $L_n * g_n(0) \geq 0$ for all n and

$$\sup_n L_n * g_n(0) = \infty.$$

(i) If you know some functional analysis deduce the existence of a continuous function f such that

$$\sup_n L_n * f(0) = \infty.$$

(ii) Even if you can obtain the result of (i) by slick functional analysis there is some point in obtaining the result directly.

(a) Suppose that we have defined positive integers $n(1) < n(2) < \cdots < n(k)$, a continuous function g_k and a real number $\epsilon(k)$ with $2^{-k} > \epsilon(k) >$

0. *Show that there is an* $\epsilon(k+1)$ *with* $\epsilon(k)/2 > \epsilon(k+1) > 0$ *such that whenever* g *is a continuous function with* $\|g - g_k\|_\infty < 2\epsilon(k+1)$ *we have* $|L_{n(j)} * g(0) - L_{n(j)} * g_k(0)| \leq 1$. *for* $1 \leq j \leq k$.

(b) Continuing with the notation of (a), show that there exists an $n(k+1) > n(k)$ *and a continuous function* g_{k+1} *with* $\|g_{k+1} - g_k\|_\infty \leq \epsilon(k+1)$ *such that* $|L_{n(k+1)} * g_{k+1}(0)| > 2^{k+1}$.

(c) By carrying out the appropriate induction and considering the uniform limit of g_k *obtain (i).*

(iii) Show that there exists a continuous function f *such that* $S_n(f, 0)$ *fails to converge as* $n \to \infty$.

Theorem 23 has several very useful consequences.

Theorem 26 (Density of trigonometric polynomials)**.** *The trigonometric polynomials are uniformly dense in the continuous functions on* $\mathbb{T}$.

Lemma 27 (Riemann-Lebesgue lemma)**.** *If* f *is an integrable function on* $\mathbb{T}$, *then* $\hat{f}(n) \to 0$ *as* $|n| \to \infty$.

Theorem 28 (Uniqueness)**.** *If* f *and* g *are integrable functions on* $\mathbb{T}$ *with* $\hat{f}(n) = \hat{g}(n)$ *for all* n, *then* $f = g$.

Lemma 29. *If* f *is an integrable function on* $\mathbb{T}$ *and* $\sum_j |\hat{f}(j)|$ *converges, then* f *is continuous and* $f(t) = \sum_j \hat{f}(j) \exp ijt$.

As a preliminary to the next couple of results we need the following temporary lemma (which will be immediately superseded by Theorem 32).

Lemma 30 (Bessel's inequality)**.** *If* f *is a continuous function on* $\mathbb{T}$, *then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|^2 \, dt.$$

Theorem 31 (Mean square convergence)**.** *If* f *is a continuous function on* $\mathbb{T}$, *then*

$$\frac{1}{2\pi} \int_{\mathbb{T}} |f(t) - S_n(f,t)|^2 \, dt \to 0$$

as $n \to \infty$.

Theorem 32 (Parseval's Theorem)**.** *If* f *is a continuous function on* $\mathbb{T}$, *then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|^2 \, dt.$$

More generally, if f and g are continuous

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)^* = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(t)^*\,dt.$$

(The extension to all L^2 functions of Theorems 31 and 32 uses easy measure theory.)

Exercise 33. *If you use Lebesgue integration, state and prove Theorems 31 and 32 for $(L^2(\mathbb{T}), \|\ \|_2)$.*

If you use Riemann integration, extend and prove Theorems 31 and 32 for all Riemann integrable function.

Note the following complement to the Riemann-Lebesgue lemma.

Lemma 34. *If $\kappa(n) \to \infty$ as $n \to \infty$, then we can find a continuous function f such that $\limsup_{n\to\infty} \kappa(n)\hat{f}(n) = \infty$.*

The proof of the next result is perhaps more interesting than the result itself.

Lemma 35. *Suppose that f is an integrable function on $\mathbb{T}$ such that there exists an A with $|\hat{f}(n)| \leq A|n|^{-1}$ for all $n \neq 0$. If f is continuous at t, then $S_n(f,t) \to f(t)$ as $n \to \infty$.*

Exercise 36. *Suppose that $a_n \in \mathbb{C}$ and there exists an A with $|a_n| \leq A|n|^{-1}$ for all $n \geq 1$. Write*

$$s_n = \sum_{r=0}^{n} a_r.$$

Show that, if

$$\frac{s_0 + s_1 + \cdots + s_n}{n+1} \to s$$

as $n \to \infty$, then $s_n \to s$ as $n \to \infty$. (Results like this are called Tauberian theorems.)

Exercise 37. *(i) Suppose that $f : [-\pi, \pi) \to \mathbb{R}$ is increasing and bounded. Write $f(\pi) = \lim_{t\to 0} f(\pi - t)$. Show that*

$$\int_{-\pi}^{\pi} f(t)\exp it\,dt = \int_0^{\pi} (f(t) - f(t-\pi))\exp it\,dt$$

and deduce that $|\hat{f}(1)| \leq (f(\pi) - f(-\pi))/2 \leq (f(\pi) - f(-\pi))$.

(ii) Under the assumptions of (i) show that

$$|\hat{f}(n)| \leq (f(\pi) - f(-\pi))/|n|$$

for all $n \neq 0$.

(iii) (Dirichlet's theorem) Suppose that $g = f_1 - f_2$ where $f_k : [-\pi, \pi) \to \mathbb{R}$ is increasing and bounded $[k = 1, 2]$. (It can be shown that functions g of this form are the, so called, functions of bounded variation.) Show that if g is continuous at t, then $S_n(g, t) \to f(t)$ as $n \to \infty$.

Most readers will already be aware of the next fact.

Lemma 38. *If $f : \mathbb{T} \to \mathbb{C}$ is continuously differentiable, then*

$$(f')\hat{}(n) = in\hat{f}(n).$$

This means that Lemma 35 applies, but we can do better.

Lemma 39. *If $f : \mathbb{T} \to \mathbb{C}$ is continuously differentiable, then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Here is a beautiful application due to Weyl of Theorem 26. If x is real, let us write $\langle x \rangle$ for the fractional part of x, that is, let us write

$$\langle x \rangle = x - [x].$$

Theorem 40. *If α is an irrational number and $0 \leq a \leq b \leq 1$, then*

$$\frac{\operatorname{card}\{1 \leq n \leq N \mid \langle n\alpha \rangle \in [a, b]\}}{N} \to b - a$$

as $N \to \infty$. The result is false if α is rational.

(Of course this result may be deduced from the ergodic theorem and Theorem 26 itself can be deduced from the Stone-Weierstrass theorem but the techniques used can be extended in directions not covered by the more general theorems.)

3 Jackson's theorems

Once we have the idea of using different kernels such as Dirichlet's kernel and Féjer's kernel we can try our hand at designing kernels for a particular purpose. The proof of the next theorem provides an excellent example.

Theorem 41. [Jackson's first theorem] *There exists a constant C with the following property. If $f : \mathbb{T} \to \mathbb{R}$ is once continuously differentiable then given $n \geq 1$ we can find a real trigonometric polynomial P_n of degree at most n such that*

$$\|P_n - f\|_\infty \leq Cn^{-1}\|f'\|_\infty$$

Jackson's theorem provides a quantitative statement of the idea that well behaved functions are easier to approximate.

Exercise 42. *It is easy to obtain a weak quantitative statement of the idea well behaved functions are easier to approximate. Show by integrating by parts that if $f : \mathbb{T} \to \mathbb{R}$ is k times continuously differentiable*

$$|\hat{f}(r)| \leq A_k|r|^{-k}\|f^{(r)}\|_\infty$$

for all $r \neq 0$ and some constant A_k independent of f.

Deduce that, if $k \geq 2$

$$\|S_n(f) - f\|_\infty \leq B_k|n|^{2-k}$$

for all $n \neq 0$ and some constant B_k independent of f.

Our proof of Theorem 41 depends on properties of the Jackson kernel J_n defined by

$$J_n(t) = \gamma_n^{-1}K_n(t)^2 = \lambda_n^{-1}\left(\frac{\sin\left(nt/2\right)}{\sin t/2}\right)^4$$

for $t \neq 0$, $J_n(0) = \lambda_n^{-1}n^2$ where γ_n and λ_n are chosen so that

$$\frac{1}{2\pi}\int_{\mathbb{T}} J_n(t)\,dt = 1.$$

Exercise 43. *By using convexity, or otherwise, show that*

$$t \geq \sin t \geq \frac{2t}{\pi}$$

for all $t \in [0, \pi/2]$

Lemma 44. *There exist strictly positive constants A, A' and B such that*

$$An^3 \geq \lambda_n \geq A'n^3$$

and

$$\frac{1}{2\pi}\int_{\mathbb{T}} |t| J_n(t)\,dt \leq Bn^{-1}.$$

We can now prove a version of Theorem 41.

Theorem 45. *There exists a constant C' with the following property. If $f : \mathbb{T} \to \mathbb{R}$ is once continuously differentiable then given $n \geq 1$ we can find a real trigonometric polynomial Q_n of degree at most $2(n-1)$ such that*

$$\|Q_n - f\|_\infty \leq C'n^{-1}$$

Exercise 46. *Deduce Theorem 41 from Theorem 45.*

It is easy to guess the generalisation to higher derivatives.

Theorem 47. [Jackson's second theorem] *There exists a constant C_k with the following property. If $f : \mathbb{T} \to \mathbb{R}$ is k times continuously differentiable then given $n \geq 1$ we can find a real trigonometric polynomial P_n of degree at most n such that*

$$\|P_n - f\|_\infty \leq C_k n^{-k} \|f^{(k)}\|_\infty$$

It is also easy to guess one of the tools used.

Exercise 48. *(i) Suppose that we set*

$$J_{n,r}(t) = \gamma_n^{-1} K_n(t)^{2r} = \lambda_{n,r}^{-1}\left(\frac{\sin\left(nt/2\right)}{\sin t/2}\right)^{2r}$$

for $t \neq 0$, $J_n(0) = \lambda_n^{-1} n^2$ where $\gamma_{n,r}$ and $\lambda_{n,r}$ are chosen so that

$$\frac{1}{2\pi}\int_{\mathbb{T}} J_{n,r}(t)\,dt = 1.$$

Show that there exist constants $B_{n,r,j}$ such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} |t|^j J_n(t)\,dt \leq B_{n,r,j} n^{-j}$$

for all $0 \leq j \leq 2r - 2$.

(ii) It is instructive (though not necessary) to see that our particular choice of kernel is not unique. Construct another trigonometric polynomial $\tilde{J}_{n,r}$ of degree at most Arn (for some suitable constant A) which is everywhere real and positive and satisfies

$$\frac{1}{2\pi}\int_{\mathbb{T}} |t|^j \tilde{J}_{n,r}(t)\,dt \leq \tilde{B}_{n,r,j} n^{-j}$$

for all $0 \leq j \leq 2r-2$.

Our proof of Jackson's first theorem depended on the mean value inequality

$$|f(s) - f(t)| \leq \|f'\|_\infty |t - s|.$$

We extend this to higher derivatives by using the difference operator $\triangle_h$ defined by

$$(\triangle_h f)(x) = f(x+h) - f(x).$$

We write $\triangle_h^1 f = \triangle_h f$ and $\triangle_h^n f = \triangle_h(\triangle^{n-1} f)$

Exercise 49. *Let $f \in C_{\mathbb{R}}(\mathbb{T})$.*

(i) Using induction, or otherwise, show that, if f is k times continuously differentiable, then

$$\|\triangle_h^k f\|_\infty \leq k!|h|^k \|f^{(k)}\|_\infty.$$

(ii) Using induction, or otherwise, show that

$$\triangle_h^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh).$$

As before we prove a slight variant of the theorem as stated.

Theorem 50. *There exist constants C'_k with the following property. If $f : \mathbb{T} \to \mathbb{R}$ is k times continuously differentiable then given $n \geq 1$ we can find a real trigonometric polynomial Q_n of degree at most $2(n-1)k$ such that*

$$\|Q_n - f\|_\infty \leq C'_k n^{-k} \|f^{(k)}\|_\infty$$

Exercise 51. *Here is a another proof of Jackson's second theorem. We did not use it because we want to extend the proof to n dimensions.*

(i) Suppose that $f : \mathbb{T} \to \mathbb{R}$ is once continuously differentiable and in addition $\int_{\mathbb{T}} f(t)\,dt = 0$. If P is a real trigonometric polynomial of such that $\|f - P\|_\infty \leq \epsilon$ show that $|\hat{Q}(0)| \leq \epsilon$.

(ii) Suppose that $f : \mathbb{T} \rightarrow \mathbb{R}$ is once continuously differentiable and in addition $\int_{\mathbb{T}} f(t)\,dt = 0$. Show that there exists a real trigonometric polynomial Q_n of degree at most n with $\hat{Q}_n(0) = 0$ and

$$\|Q_n - f\|_\infty \leq 2C_1 n^{-1} \|f'\|_\infty$$

where C_1 is the constant that occurs in Theorem 41.

(iii) Suppose that $f : \mathbb{T} \rightarrow \mathbb{R}$ is twice continuously differentiable. By using (ii), show that there exists a real trigonometric polynomial Q_n of degree at most n with $\hat{Q}_n(0) = 0$ and

$$\|Q_n - f'\|_\infty \leq 2C_1 n^{-1} \|f''\|_\infty.$$

Hence show that there is real trigonometric polynomial R_n of degree at most n with

$$\|R_n - f\|_\infty \leq 2C_1^2 n^{-2} \|f''\|_\infty.$$

(iv) Prove Theorem 47

Exercise 52. *(i) If $f \in C(\mathbb{T}$ and P is trigonometric polynomial of degree at most n show that*

$$\|f - P\|_\infty \geq \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f(t) - P(t)|^2 \, dt \right)^{1/2} \geq |\hat{f}(m)|$$

for any $|m| > n$.

(ii) Let $0 < n(1) < n(2) < \ldots$. If

$$f(t) = \sum_{j=0}^{\infty} 2^{-j} n(j)^{-1} \cos n(j)t$$

show that f is a well defined once continuously differentiable function with $\|f'\|_\infty \leq 1$ and

$$\inf\{\|f - P\|_\infty : P \text{ a trigonometric polynomial of degree at most } n(j) - 1\} \geq 2^{-j} n(j)^{-1}.$$

Deduce that, if κ_n is a sequence of strictly positive numbers with $\kappa_n n \rightarrow \infty$ there exists a continuously differentiable function g with

$$\limsup \kappa_n \inf\{\|f - P\|_\infty : P \text{ a trigonometric polynomial of degree at most } n\} = \infty.$$

In this sense, Jackson's first theorem is best possible.

(iii) Show that Jackson's second theorem is best possible in a similar sense.

We shall need a multidimensional version of Jackson's second theorem. The proof is a natural induction but notationally a bit tedious.

We use the norm $\| \ \|_p$ on $C^p(\mathbb{T}^n)$ (the space of real p times continuously differentiable functions on $C^p(\mathbb{T}^n)$ given by

$$\|f\|_p = \|f\| + \frac{1}{n} \sum_{1 \leq p_1+p_2+\ldots+p_n \leq p} \sup_{\mathbf{t}\in\mathbb{T}^n} \left| \frac{\partial^{p_1+p_2+\ldots+p_n} f}{\partial^{p_1}x_1 \partial^{p_2}x_2 \ldots \partial^{p_n}x_n}(\mathbf{t}) \right|.$$

Theorem 53. *There exist constants $A_{p,n}$ with the following property. If $f \in C^p(\mathbb{T}^n)$, then, given $N \geq 1$ we can find a real trigonometric polynomial*

$$P_N(\mathbf{t}) = \sum_{|j(u)| \leq N} a_{j(1)j(2)\ldots j(n)} \exp\left(i \sum_{v=1}^{n} j(v) t_v\right)$$

such that

$$\|P_N - f\|_\infty \leq A_{p,n} N^{-p} \|f\|_p.$$

4 Vistuškin's theorem

We now prove a theorem of Vistuškin which shows that, if we demand continuous differentiability, there are genuine functions of several variables. We need a definition to make things precise.

Definition 54. *Let $n > m \geq 1$. Define c_j, $s_j \in C(\mathbb{T}^n)$ by*

$$s_j(\mathbf{t}) = \sin t_j,\ \ c_j(\mathbf{t}) = \cos t_j\ [1 \leq j \leq n].$$

and write

$$\mathcal{E}_0 = \{s_j\,:\,1 \leq j \leq n\} \cup \{c_j\,:\,1 \leq j \leq n\}.$$

If E is a subset of $C(\mathbb{T}^m$ define $\mathcal{E}_r(E)$, inductively by setting $\mathcal{E}_0(E) = \mathcal{E}_0$ and taking $\mathcal{E}_r(E)$ to be the set of all functions $f \in C(\mathbb{T}^n)$ given by

$$f(\mathbf{t}) = g(u_1(\mathbf{t}), u_2(\mathbf{t}), \ldots, u_m(\mathbf{t})$$

with $u_l \in \mathcal{E}_{r-1}(E)$ $[1 \leq l \leq m]$ and $g \in E$. We say that an $f \in C_{\mathbb{R}}(\mathbb{T}^n)$ is written in terms of functions in E if $f \in \mathcal{E}_r(E)$ for some $r \geq 1$.

We can now state our theorem.

Theorem 55. *If $n > m \geq 1$, $p \geq q \geq 1$ and $n/p > m/q$ there exists an $f \in C^p(\mathbb{T}^n)$ which cannot be written in terms of functions in $C^q(\mathbb{T}^m)$.*

Our proof depends on the notion of ϵ-entropy introduced by Kolmogorov.

Definition 56. *We work in $C_{\mathbb{R}}(\mathbb{T}^n)$ equipped with the uniform norm. Let E be a subset of $C_{\mathbb{R}}(\mathbb{T}^n)$ and $\epsilon > 0$. If E cannot be covered by a finite set of closed balls*

$$\tilde{B}(f, \epsilon) = \{g \in C_{\mathbb{R}}(\mathbb{T}^n) \,:\, \|f - g\|_\infty \leq \epsilon\}$$

we take $H(\epsilon, E) = \infty$. If E can be covered by a finite set of such balls, we write $N(\epsilon, E)$ for the least number of balls required and define $H(\epsilon, K)$, the ϵ-entropy *of K by*

$$H(\epsilon, K) = \log N(\epsilon, K).$$

Suppose that we are using the functions $f \in E$ as messages but we cannot distinguish two messages f_1 and f_2 if their uniform distance is less than about ϵ. Then *very roughly speaking* we can only distinguish about $N(\epsilon, E)$ messages and the amount of information we can send (defined *roughly speaking* as the logarithm of the number of possible distinct messages) is about $H(\epsilon, K)$.

We need the following simple observation. (Here $\mathrm{Cl}_\infty E$) denotes the closure in the uniform norm.)

Exercise 57. *Let E be a subset of $C_{\mathbb{R}}(\mathbb{T}^n)$ and $\epsilon > 0$. Then*

$$H(\epsilon, E) = \log N(\epsilon, \mathrm{Cl}_\infty E).$$

Since we are interested in the behaviour of $H(\epsilon, E)$ as $\epsilon \to 0$, we shall only be interested in those E whose uniform closure is compact in $(C_{\mathbb{R}}(\mathbb{T}^n, \| \,\|)$, that is to say those E which are bounded and uniformly equicontinuous. (If you have not met uniform equicontinuity before will will talk about it later.)

The sets E we shall consider are balls in $C^p(\mathbb{T}^n)$ with an the norm $\| \,\|_{(p)}$. defined when we introduced Theorem 53.

The key inequality is given by the next theorem.

Theorem 58. *Let $B_{p,n}$ be the closed unit ball in $(C^p(\mathbb{T}^n), \| \,\|_{(p)})$. Then there exist constants $C_{p,n}$ and $C'_{p,n}$ such that*

$$C_n \epsilon^{-n/p} \leq H(\epsilon, B_{p,n}) \leq C'_n \epsilon^{-n/p} \log \epsilon^{-1}$$

for $0 < \epsilon < 1/2$.

Notice that this theorem is a *quantitative* version of the much simpler observation that B_n is uniformly equicontinuous. Notice also that we will be considering two sorts of balls:- *uniform balls*, that is to say balls in $(C_{\mathbb{R}}(\mathbb{T}^n, \| \,\|)$, and C^p-*balls*, that is to say balls in $(C_{\mathbb{R}}^{(p)}(\mathbb{T}^n, \| \,\|_{(p)})$.

The next lemma brings us closer to Vistuškin's theorem

Lemma 59. *Let $n > m \geq 1$. Let $B_{q,m}$ be the unit ball in $(C^q(\mathbb{T}^m), \| \ \|_{(1)})$. If r and u are strictly positive integers then, using the notation of Definitions 54 and 56, we know that there is a constant $C(q, r, u, m, n)$ such that*

$$H(\epsilon, \mathcal{E}_r(uB_{q,m}) \leq C(q, r, um, n)\epsilon^{-m} \log \epsilon^{-1}$$

for all $0 < \epsilon < 1/2$.

We can now prove a Baire category version of Theorem 55.

Theorem 60. *If $n > m \geq 1$, $p, q \geq 1$ $p/n \geq q/m$ and we work in $(C^p(\mathbb{T}^n), \| \ \|_{(p)})$, then quasi-all functions in $(C^p(\mathbb{T}^n), \| \ \|_{(p)})$ cannot be written in terms of functions in $C^q(\mathbb{T}^m)$.*

Exercise 61. *Modify the discussion above to show that there exists an $f \in C^1(\mathbb{T}^2)$ which cannot be written in terms of functions in $C^1(\mathbb{T})$ and the addition function function $(x, y) \mapsto x = y$.*

5 Simple connectedness and the logarithm

The rest of these lectures form a short second course in complex variable theory with an emphasis on technique rather than theory. None the less I intend to be rigorous and you should feel free to question any 'hand waving' that I indulge in.

But where should rigour start? It is neither necessary nor desirable to start by reproving all the results of a first course. Instead I shall proceed on the assumption that all the standard theorems (Cauchy's theorem, Taylor's theorem, Laurent's theorem and so on) have been proved rigorously for analytic functions[4] on an open disc and extend them as necessary.

Almost all the members of the audience are already familiar with one sort of extension.

Definition 62. *An open set U in $\mathbb{C}$ is called* disconnected *if we can find open sets U_1 and U_2 such that*
(i) $U_1 \cup U_2 = U$,
(ii) $U_1 \cap U_2 = \emptyset$,
(iii) $U_1, U_2 \neq \emptyset$.
An open set which is not disconnected is called connected.

[4] Analytic functions are sometimes called 'holomorphic functions'. We shall call a function which is 'analytic except for poles' a 'meromorphic function'.

Theorem 63. *If U is an open connected set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ is analytic and not identically zero then all the zeros of f are isolated that is, given $w \in U$ with $f(w) = 0$ we can find a $\delta > 0$ such that $D(w, \delta) \subseteq U$ and $f(z) \neq 0$ whenever $z \in D(w, \delta)$ and $z \neq w$.*

Here and elsewhere

$$D(w, \delta) = \{z : |w - z| < \delta\}.$$

The hypothesis of connectedness is exactly what we need in Theorem 63.

Theorem 64. *If U is an open set then U is connected if and only if the zeros of every non-constant analytic function on U are isolated.*

If necessary, I shall quote results along the lines of Theorem 63 without proof but I will be happy to give proofs in supplementary lectures if requested.

Exercise 65 (Maximum principle)**.** *(i) Suppose that $a, b \in \mathbb{C}$ with $b \neq 0$ and N is an integer with $N \geq 1$. Show that there is a $\theta \in \mathbb{R}$ such that*

$$|a + b(\delta \exp i\theta)^N| = |a| + |b|\delta^N$$

for all real δ with $\delta \geq 0$.

(ii) Suppose that $f : D(0, 1) \to \mathbb{C}$ is analytic. Show that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where there is some constant M such that $|a_n| \leq M2^n$ (we can make much better estimates). Deduce that either f is constant or we can find $N \geq 1$ and $a_N \neq 0$ such that

$$f(z) = a_0 + (a_N + \eta(z))z^N$$

with $\eta_z \to 0$ as $z \to 0$.

(iii) If U is a connected open subset of $\mathbb{C}$ and f is a non-constant analytic function on U, show that $|f|$ has no maxima.

(iv) Does the result of (iii) mean that f is unbounded on U? Give reasons.

(v) Show that if is an open set which is not connected then there exists a non-constant analytic function f on U such that $|f|$ has a maximum.

Exercise 66. *(i) Suppose $f : D(0, 1) \to \mathbb{C}$ is a non-constant analytic function with $f(0) = 0$. Show that we can find a δ with $0 < \delta < 1$ such that $f(z) \neq 0$ for all $|z| = \delta$ and an $\epsilon > 0$ such that $|f(z)| \geq \epsilon$ for all $|z| = \delta$. Use Rouché's theorem to deduce that $f(D(0, 1)) \supseteq D(0, \epsilon)$.*

(ii) **(Open mapping theorem)** *If U is a connected open subset of $\mathbb{C}$ and f is a non-constant analytic function on U show that $f(U)$ is open.*

(iii) Deduce the result of Exercise 65. (Thus the maximum principle follows from the open mapping theorem.)

Exercise 67. *Let D be the open unit disc. Suppose $f : D \to \mathbb{C}$ is analytic and $f(0) = f'(0) = 0$ but f is not identically zero. Use Rouché's theorem (and the fact that the zeros of f' are isolated) to show that there exists $\eta_1, \eta_2 > 0$ such that if $0 < |w| < \eta_1$ the equation $f(z) = w$ has at least two distinct solutions with $|z| < \eta_2$.*

Deduce that if Ω_1 and Ω_2 are open subsets of $\mathbb{C}$ and f is a conformal mapping *of Ω_1 into Ω_2 then the inverse map f^{-1} is analytic and so a conformal map of Ω_2 into Ω_1.*

It can be argued that much of complex analysis reduces to the study of the logarithm and this course is no exception. We need a general condition on an open set which allows us to define a logarithm. Recall that we write $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Definition 68. *An open set U in $\mathbb{C}$ is said to be* simply connected *if it is connected and given any continuous function $\gamma : \mathbb{T} \to U$ we can find a continuous function $G : [0,1] \times \mathbb{T} \to U$ such that*

$$G(0,t) = \gamma(t)$$
$$G(1,t) = G(1,0)$$

for all $t \in \mathbb{T}$.

In the language of elementary algebraic topology a connected open set is simply connected if every loop can be homotoped to a point.

Theorem 69. *If U is an open simply connected set in $\mathbb{C}$ that does not contain 0 we can find an analytic function $\log : U \to \mathbb{C}$ such that $\exp(\log z) = z$ for all $z \in U$. The function $\log$ is unique up to the addition of integer multiple of $2\pi i$.*

From an elementary viewpoint, the most direct way of proving Theorem 69 is to show that any piece wise smooth loop can be homotoped *through piecewise smooth loops* to a point and then use the integral definition of the logarithm. However, the proof is a little messy and we shall use a different approach which is longer but introduces some useful ideas.

Theorem 70. *(i) If* $0 < r < |w|$ *we can find an analytic function* $\log : D(w,r) \to \mathbb{C}$ *such that* $\exp(\log z) = z$ *for all* $z \in D(w,r)$*. The function* $\log$ *is unique up to the addition of integer multiple of* $2\pi i$*.*

(ii) If $\gamma : [a,b] \to \mathbb{C} \setminus \{0\}$ *is continuous we can find a continuous function* $\tilde{\gamma} : [a,b] \to \mathbb{C}$ *such that* $\exp \circ \tilde{\gamma} = \gamma$*.*

(iii) Under the hypotheses of (ii), if $\tilde{\tilde{\gamma}} : [a,b] \to \mathbb{C}$ *is a continuous function such that* $\exp \circ \tilde{\tilde{\gamma}} = \gamma$ *then we can find an integer* n *such that* $\tilde{\tilde{\gamma}} = \tilde{\gamma} + 2\pi i n$*.*

(iv) If U *is a simply connected open set not containing* 0 *then, if* $\gamma : [a,b] \to U$ *is continuous,* $\gamma(a) = \gamma(b)$*, and* $\tilde{\gamma} : [a,b] \to \mathbb{C}$ *is a continuous function such that* $\exp \circ \tilde{\gamma} = \gamma$ *then* $\tilde{\gamma}(a) = \tilde{\gamma}(b)$*.*

Theorem 69 is now relatively easy to prove.

It would be nice to show that simple connectedness is the correct condition here. The following result, although not the best possible, is hard enough and shows that this is effectively the case.

Lemma 71. *Suppose that* U *is a non-empty open connected set in* $\mathbb{C}$ *with non-empty complement. The following two conditions are equivalent.*

(i) The set U *is simply connected.*

(ii) If $f : U \to \mathbb{C}$ *is a non-constant analytic function with no zeros then we can find an analytic function* $\log : f(U) \to \mathbb{C}$ *with* $\exp(\log f(z)) = f(z)$ *for all* $z \in U$

(In looking at condition (ii), recall that the open mapping theorem given in Exercise 66 tells us that $f(U)$ is open.) The reader is invited to try and prove this result directly but we shall obtain it only after a long chain of arguments leading to the Riemann mapping theorem.

The following result is trivial but worth noting.

Lemma 72. *If* U *and* V *are open subsets of* $\mathbb{C}$ *such that there exists a homeomorphism* $f : U \to V$ *then if* U *is simply connected so is* V*.*

Exercise 73. *In the next two exercises we develop an alternative approach to Theorem 69 along the lines suggested above.*

(i) Suppose that U *is an open set in* $\mathbb{C}$ *and that* $G : [0,1] \times \mathbb{T} \to U$ *is a continuous function. Show, by using compactness arguments or otherwise, that there exists an* $\epsilon > 0$ *such that* $N(G(s,t), \epsilon) \subseteq U$ *for all* $(s,t) \in [0,1] \times \mathbb{T}$*, and that we can find an integer* $N \geq 1$ *such that if*

$$(s_1, t_1), (s_2, t_2) \in [0,1] \times \mathbb{T} \text{ and } |s_1 - s_2| < 4N^{-1},\ |t_1 - t_2| < 8\pi N^{-1}$$

then $|G(s_1, t_1) - G(s_2, t_2)| < \epsilon/4$*.*

(ii) Continuing with the notation and hypotheses of of (i) show that if $\gamma_1, \gamma_2 : \mathbb{T} \to \mathbb{C}$ are the piecewise linear functions[5] *with*

$$\begin{aligned}\gamma_0(2\pi r/N) &= G(0, 2\pi r/N)\\ \gamma_1(2\pi r/N) &= G(1, 2\pi r/N)\end{aligned}$$

for all integers r with $0 \leq r \leq N$ then there exists a constant λ and a continuous function $H : [0,1] \times \mathbb{T} \to U$ with

$$\begin{aligned}H(0,t) &= \gamma_0(0,t)\\ H(1,t) &= \gamma_1(1,t)\end{aligned}$$

for all $t \in [0,1]$, such that, for each fixed t, $H(s,t)$ is a piecewise linear function of s and the curve $H(\ ,t) : \mathbb{T} \to U$ is of length less than λ.

(iii) Continuing with the notation and hypotheses of of (i) show that if $G(s,1)$ and $G(s,0)$ are piecewise smooth functions of s then there exists a constant λ and a continuous function $F : [0,1] \times \mathbb{T} \to U$ with

$$\begin{aligned}F(0,t) &= \gamma_0(0,t)\\ F(1,t) &= \gamma_1(1,t)\end{aligned}$$

for all $t \in [0,1]$, such that, for each fixed t, $F(s,t)$ is a piecewise smooth function of s and the curve $F(\ ,t) : \mathbb{T} \to U$ is of length less than λ.

(iv) Show that in any simply connected open set any piece wise smooth loop can be homotoped through piecewise smooth loops of bounded length to a point.

Exercise 74. *(i) Suppose that U is an open set in $\mathbb{C}$ and $F : [0,1] \times \mathbb{T} \to U$ is a continuous function such that, for each fixed t, $F(s,t)$ is a piecewise smooth function of s and the curve $F(\ ,t) : \mathbb{T} \to U$ is of length less than λ. We write Γ_s for the contour defined by $F(\ ,t)$. Show by a compactness argument, or otherwise, that if $f : U \to \mathbb{C}$ is continuous then $\int_{\Gamma_s} f(z)\,dz$ is a continuous function of s.*

(ii) If $0 < \delta < |w|$ show that if Γ is a contour lying entirely within $N(w,\delta)$ joining $z_1 = r_1 e^{i\theta_1}$ to $z_2 = r_2 e^{i\theta_2}$ [$r_1, r_2 > 0$, $\theta_1, \theta_2 \in \mathbb{R}$] show that

$$\int_\Gamma \frac{1}{z}\,dz = (\log r_2 - \log r_1) + i(\theta_1 - \theta_2) + 2n\pi i$$

for some integer n.

[5] Strictly speaking the simplest piecewise linear functions.

(iii) By using compactness arguments to split Γ into suitable bits, or otherwise, show that if U is any open set not containing 0 and Γ is any closed contour (i.e. loop) lying entirely within U then

$$\int_\Gamma \frac{1}{z}\,dz = 2N\pi i$$

for some integer N.

(iv) Use results from this exercise and its predecessor to show that if U is any simply connected open set not containing 0 and Γ is any closed contour lying entirely within U then

$$\int_\Gamma \frac{1}{z}\,dz = 0.$$

Hence, prove Theorem 69.

Exercise 75. *Let us say that two open subsets of $\mathbb{C}$, Ω_1 and Ω_2 are* conformally equivalent *if there is a conformal mapping of Ω_1 into Ω_2. Show that conformal equivalence is indeed an equivalence relation.*

6 The Riemann mapping theorem

By using a very beautiful physical argument, Riemann obtained the following result.

Theorem 76 (Riemann mapping theorem)**.** *If Ω is an non-empty, open, simply connected subset of $\mathbb{C}$ with non-empty complement then there exists a conformal map of Ω to the unit disc $D(0,1)$.*

Notice that we can reduce this result to a version which is easier to think about.

Theorem 77. *If Ω is an open simply connected subset of D and $0 \in \Omega$, then there exists a conformal map $f : \Omega \to D$.*

Unfortunately his argument depended on the assumption of the existence of a function which minimises a certain energy. Since Riemann was an intellectual giant and his result is correct it is often suggested that all that was needed was a little rigour to be produced by pygmies. However, Riemann's argument actually fails in the related three dimensional case so (in the lecturer's opinion) although Riemann's argument certainly showed that a very wide class of sets could be conformally transformed into the unit disc the extreme generality of the final result could not reasonably have been expected from his argument alone.

In order to rescue the Riemann mapping theorem mathematicians embarked on two separate programmes. The first was to study conformal mapping in more detail and the second to find abstract principles to guarantee the existence of minima in a wide range of general circumstances (in modern terms, to find appropriate compact spaces). The contents of this section come from the first of these programmes, the contents of the next (on equicontinuity) come from the second. (As a point of history, the first complete proof of the Riemann mapping theorem was given by Poincaré.)

Theorem 78 (Schwarz's inequality)**.** *If $f : D(0,1) \to D(0,1)$ is analytic and $f(0) = 0$ then*

(i) $|f(z)| \leq |z|$ for all $|z| < 1$ and $|f'(0)| \leq 1$.

(ii) If $|f(w)| = |w|$ for some $|w| < 1$ with $w \neq 0$, or if $|f'(0)| = 1$, then we can find a $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$ for all $|z| < 1$.

Schwarz's inequality enables us to classify the conformal maps of the unit disc into itself. If $a \in D(0,1)$ and $\theta \in \mathbb{R}$ let us write

$$\begin{aligned} T_a(z) &= \frac{z-a}{1-a^*z} \\ R_\theta(z) &= e^{i\theta} z \end{aligned}$$

Lemma 79. *If $a \in D(0,1)$ and $\theta \in \mathbb{R}$ then T_a and R_θ map $D(0,1)$ conformally into itself. Further $T_a^{-1} = T_a$.*

Theorem 80. *(i) If S maps $D(0,1)$ conformally into itself then we can find $a \in D(0,1)$ and $\theta \in \mathbb{R}$ such that $S = R_\theta T_a$. If $S = R_{\theta'} T_{a'}$ with $a' \in D(0,1)$ and $\theta' \in \mathbb{R}$ then $a = a'$ and $\theta - \theta' \in 2\pi\mathbb{Z}$.*

(ii) Let U be a simply connected open set and $a \in U$. If there exists a conformal map $g : U \to D(0,1)$ then there exists precisely one conformal map $f : U \to D(0,1)$ with $f(a) = 0$ and $f'(a)$ real and positive.

We shall prove the following version of Theorem 77.

Theorem 81. *If Ω is an open simply connected subset of D and $0 \in \Omega$, then there exists a conformal map $f : \Omega \to D$ with $f(0) = 0$ and $f'(0)$ real and positive.*

Exercise 82. **[Pick's inequality]** *Let a, $b \in \mathbb{C}$, R, $S > 0$. Set*

$$D_1 = \{z \in \mathbb{C} \,:\, |z-a| < R\}, \text{ and } D_2 = \{z \in \mathbb{C} \,:\, |z-b| < S\}.$$

If $f : D_1 \to \mathbb{C}$ is analytic and $f(D_1) \subseteq D_2$ show that

$$\left| S\frac{f(z) - f(w)}{S^2 - f(z)^* f(w)} \right| \leq \left| R\frac{z - w}{R^2 - z^* w} \right|$$

for $z \neq w$, $z, w \in D_1$ and

$$|f'(w)| \leq \frac{R}{S}\frac{S^2 - |f(w)|^2}{R^2 - |w|^2}$$

for $w \in D_1$ Show that if we have equality in the first inequality for any z and w or in the second for any w, then f is a Möbius map.

We conclude this section with some results which are not needed for the proof of the Riemann mapping theorem but which show that the 'surrounding scenery'.

There is one remark that needs to be made. to add one remark. In elementary complex variable theory we use the heuristic 'see how the boundaries transform'. This worked in Lemma 79 because Möbius maps are defined on the whole plane $\mathbb{C}$ (apart, perhaps, from one point). If we look at the more general conformal maps considered in Riemann's theorem, we run into two linked problems. The first is that the boundary of an open simply connected set may be rather complicated and the second is that the conformal maps may not extend to 'nice' maps on the closure of the sets considered. I do not wish to spend time showing precisely how nasty things can become, but the following simple example should convince you that it would be ill advised to try to include boundaries in our discussion.

Exercise 83. *Find an explicit conformal map T taking*

$$\Omega = D \setminus \{x \in \mathbb{R}\, 0 \leq x \leq 1\}$$

to D. Show that there is no continuous function $S : \operatorname{Cl}\Omega \to \operatorname{Cl} D$ with $Sz = Tz$ for all $z \in \Omega$.

Explain why if $\tilde{T}$ is any conformal map of Ω to D there is no continuous function $\tilde{S} : \operatorname{Cl}\Omega \to \operatorname{Cl} D$ with $\tilde{S}z = \tilde{T}z$ for all $z \in \Omega$.

Example 84. *If $a, b \in D(0,1)$ then there exists a conformal map*

$$f : D(0,1) \setminus \{a\} \to D(0,1) \setminus \{b\}.$$

Example 85. *If $a_1, a_2, b_1, b_2 \in D(0,1)$ then there exists a conformal map*

$$f : D(0,1) \setminus \{a_1, a_2\} \to D(0,1) \setminus \{b_1, b_2\}$$

if and only if

$$\left| \frac{a_2 - a_1}{a_1^* a_2 - 1} \right| = \left| \frac{b_2 - b_1}{b_1^* b_2 - 1} \right|.$$

In Example 84 we see the the 'natural rigidity' of complex analysis reassert itself.

We now give a more complicated example of this rigidity. The ideas will be reused later in the proof of Picard's Little Theorem. I assume the reader has already met.

Definition 86. *(i) Let* $\mathbf{p}$ *and* $\mathbf{q}$ *are orthonormal vectors in* $\mathbb{R}^2$. *If* $\mathbf{a}$ *is a vector in* $\mathbb{R}^2$ *and* $x, y \in \mathbb{R}$ *the reflection of* $\mathbf{a} + x\mathbf{p} + y\mathbf{q}$ *in the line through* $\mathbf{a}$ *parallel to* $\mathbf{p}$ *is* $\mathbf{a} + x\mathbf{p} - y\mathbf{q}$.

(ii) If $\mathbf{a}$ *and are vectors in* $\mathbb{R}^2$ *and* $R, r > 0$ *then the reflection of* $\mathbf{a} + r\mathbf{b}$ *in the circle centre* $\mathbf{a}$ *and radius* R *is* $\mathbf{a} + r^{-1}R^2\mathbf{b}$.

Lemma 87 (Schwarz reflection principle). *Let* Σ_1 *and* Σ_2 *be two circles (or straight lines). Suppose* G *is an open set which is taken to itself by reflection in* Σ_1. *Write* G_+ *for that part of* G *on one side*[6] *of* Σ_1 *and* $G_0 = G \cap \Sigma_1$. *If* $f : G_+ \cup G_0$ *is a continuous function, analytic on* G_+ *with* $f(G_0) \subseteq \Sigma_2$ *then we can find an analytic function* $\tilde{f} : G \to \mathbb{C}$ *with* $\tilde{f}(z) = f(z)$ *for all* $z \in G_+ \cup G_0$. *If* $f(G_+)$ *lies on one side of* Σ_2 *then we can ensure that* $\tilde{f}(G_-)$ *lies on the other.*

We first prove the result when Σ_1 and Σ_2 are the real axis and then use Möbius transforms to get the full result.

The next theorem is less powerful than it appears (The theorem we would wish for is true, but will not be proved here.)

Theorem 88. *There exists a homeomorphism*

$$f : \{z : a \leq |z| \leq b\} \to \{z : A \leq |z| \leq B\}$$

whose restriction

$$\tilde{f} : \{z : a < |z| < b\} \to \{z : A < |z| < B\}$$

is conformal if and only if $a/b = A/B$.

Exercise 89. *If* $a/b = A/B$ *find all the maps of the type described in the previous theorem.*

[6] There are no topological difficulties here. The two sides of $|z-a| = r$ are $\{z : |z-a| < r\}$ and $\{z : |z-a| > r\}$.

7 Equicontinuity

(The reader is welcome to take a different approach but if she derives the results from more general theorems she must be able to prove those theorems.) We start with a general remark.

Lemma 90. *If Ω is an open subset of $\mathbb{R}^n$ we can find a sequence K_1, K_2, ... of compact subsets of Ω such that*

$$\bigcup n = 1^{\infty} K_n = \Omega.$$

Definition 91. *Let K be a compact subset of $\mathbb{R}^n$ with the standard metric d. A subset $\mathcal{F} \subseteq C(K)$ is said to be* uniformly equicontinuous[7] *if, given $\epsilon > 0$, we can find a $\delta(\epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $d(x, y) < \delta(\epsilon)$ and $f \in \mathcal{F}$.*

Exercise 92. *We use the notation and hypotheses of A subset $\mathcal{F} \subseteq C(K)$ is said to be* equicontinuous at the point x *if, given $\epsilon > 0$, we can find a $\delta(\epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $d(x, y) < \delta(\epsilon)$ and $f \in \mathcal{F}$. Show that, if $\mathcal{F}$ is eqicontinuous at every point of K, then $\mathcal{F}$ is uniformly eqicontinuous.*

Definition 93. *Let X be a compact subset of $\mathbb{R}^n$. A subset $\mathcal{F} \subseteq C(X)$ is said to be* uniformly bounded *if we can find a C such that $\|f\|_\infty \leq C$ whenever $f \in \mathcal{F}$.*

Theorem 94. [The Arzelá–Ascoli theorem] *Let K be a compact subset of $\mathbb{R}^n$. Then $\mathcal{F} \subseteq C(K)$ is compact if and only if $\mathcal{F}$ is closed, uniformly bounded and uniformly equicontinuous.*

The natural mode of convergence for analytic functions on an open set is 'converging uniformly on compacta'.

Definition 95. *Let Ω be an open set in $\mathbb{C}$. Consider a sequence of $f_n : \Omega \to \mathbb{C}$ and an $f : \Omega \to \mathbb{C}$. We say that $f_n \to f$ uniformly on compacta if whenever K is a compact subset of Ω $f_n \to f$ uniformly on K.*

Exercise 96. *We use the notation and hypotheses of Definition 95.*

(i) If $f_n \to f$ uniformly on compacta and each f_n is continuous on Ω, show that f is continuous on Ω.

(ii) If $f_n \to f$ uniformly on compacta and each f_n is analytic on Ω, show that f is analytic on Ω.

[7] Traditionally the word 'equicontinuous' was used instead of the phrase 'uniformly equicontinuous' and the majority of mathematicians follow this older usage. See Exercise 92.

Theorem 97. [**Montel's theorem**] *Let Ω be an open set in $\mathbb{C}$ and $\mathcal{F}$ a set of analytic function $f : \Omega \to \mathbb{C}$. Then every sequence of functions in $\mathcal{F}$ contains a subsequence which is uniformly convergent on compacta if and only if $\mathcal{F}$ is uniformly bounded on compacta.*

The next exercise is not needed, but may help put things in perspective.

Exercise 98. *Let Ω be an open set in $\mathbb{C}$ and suppose that K_1, K_2, ... are compact sets such that $K_m \subseteq \Omega$ and $\bigcap_{m=1}^{\infty} \operatorname{Int} K_j = \Omega$.*

(i) Show that the equation

$$d_K(f,g) = \sum_{m=1}^{\infty} 2^{-m} \max\{1, \sup_{z \in K_m} |f(z) - g(z)|\}$$

defines a metric on $A(\Omega)$, the space of analytic functions $f : \Omega \to \mathbb{C}$. Show that d_K is complete.

(ii) If f, $f_n \in A(\Omega)$ show that $d_K(f, f_n) \to 0$ if and only if $f_n \to f$ uniformly on compacta. If $g_n \in A(\Omega)$ show that g_n is Cauchy for d if and only if $g_n - g_m \to 0$ uniformly on compacta.

Observe that Montel's theorem may be restated as saying that a closed subset $\mathcal{F}$ of $\big(A(\Omega), d_K\big)$ is compact if and only if it is uniformly bounded on compacta.

(iii) If L_1, L_2, ... are compact sets such that $L_m \subseteq \Omega$ and $\bigcap_{m=1}^{\infty} \operatorname{Int} L_j = \Omega$ and we set

$$d_L(f,g) = \sum_{m=1}^{\infty} 2^{-m} \max\{1, \sup_{z \in L_m} |f(z) - g(z)|\}$$

show that the identity map $\iota : \big(A(\Omega), d_L\big) \to \big(A(\Omega), d_K\big)$ is a homeomorphism which preserves Cauchy sequences. Is it true that we can always find a $C > 1$ such that

$$C d_L(f,g) \geq d_K(f,g) \geq C^{-1} d_L(f,g)?$$

We shall be dealing with the limit of of injective analytic functions and will make use of the following result.

Theorem 99. [**Hurwitz's theorem**] *Suppose that Ω is a pathwise connected open set. If $f_n : \Omega \to \mathbb{C}$ is an injective analytic function and $f_n \to f$ uniformly on compacta, then either f is a constant function or f is injective.*

We are now in a position to embark on a proof of the Riemann mapping theorem.

Lemma 100. *If Ω is an open simply connected subset of D and $0 \in \Omega$, then there exists an injective analytic function $f : \Omega \to D$ with $f(0) = 0$, $f'(0)$ real and positive such that, if $g : \Omega \to D$ is an injective analytic function with $g(0) = 0$ and $g'(0)$ real and positive, then $f'(0) \geq g'(0)$.*

Lemma 101. *Suppose that Ω is an open simply connected subset of D and $f : \Omega \to D$ is an injective analytic function with $f(0) = 0$ and $f'(0)$ real and positive. If f is not surjective, we can find an injective analytic function $g : \Omega \to D$ with $g(0) = 0$, $g'(0)$ real and $g'(0) > f'(0)$*

Lemmas 100 and 101 yield Theorem 81 and by the earlier this gives Theorem 76

Exercise 102. *Reprove Lemma 101 using nth roots rather than square roots.*

8 Boundary behaviour of conformal maps

We now return to the boundary behaviour of the Riemann mapping. (Strictly speaking we should say, a Riemann mapping but we have seen that it is 'essentially unique'. We saw in Exercise 83 that there is no general theorem but the following result is very satisfactory.

Theorem 103. *If Ω is a simply connected open set in $\mathbb{C}$ with boundary a Jordan curve then any bijective analytic map $f : D(0,1) \to \Omega$ can be extended to a bijective continuous map from $\overline{D(0,1)} \to \overline{\Omega}$.*

Recall[8] that a *Jordan curve* is a continuous injective map $\gamma : \mathbb{T} \to \mathbb{C}$. We say that γ is the boundary of Ω if the image of γ is $\overline{\Omega} \setminus \Omega$.

I shall use the proof in Zygmund's magnificent treatise [9] (see Theorem 10.9 of Chapter VII) which has the advantage of minimising the topology but the minor disadvantage of using measure theory (students who do not know measure theory may take the results on trust) and the slightly greater disadvantage of using an idea from Fourier analysis (the *conjugate* trigonometric sum $\tilde{S}_N(f,t)$) which can not be properly placed in context here.

We shall use the following simple consequence of Fejér's theorem (Theorem 23).

Lemma 104. *If $f : \mathbb{T} \to \mathbb{R}$ is integrable but $S_N(f,x) \to \infty$ as $N \to \infty$ then f can not be continuous at x.*

[8] In the normal weasel-worded mathematical sense.

Exercise 105. *If $f : \mathbb{T} \to \mathbb{R}$ is integrable and there exist $\delta > 0$ and $M > 0$ such that $|f(t)| \leq M$ for all $|t| < \delta$ show that it is not possible to have $S_N(f, 0) \to \infty$ as $N \to \infty$.*

Lemma 106. *(i) If $f : \mathbb{T} \to \mathbb{C}$ is integrable and f is continuous at x then*

$$\frac{\tilde{S}_N(f, x)}{\log N} \to 0$$

as $N \to 0$.

(ii) If $h(t) = \operatorname{sgn}(t) - t/\pi$ then there is a non-zero constant L such that

$$\frac{\tilde{S}_N(h, 0)}{\log N} \to L$$

as $N \to \infty$.

(iii) If $f : \mathbb{T} \to \mathbb{C}$ is integrable and $f(x + \eta) \to f(x+)$, $f(x - \eta) \to f(x-)$ as $\eta \to 0$ through positive values then

$$\frac{\tilde{S}_N(f, x)}{\log N} \to \frac{L(f(x+) - f(x-))}{2}$$

as $N \to 0$.

We now come to the object of our Fourier analysis.

Lemma 107. *If $f : \mathbb{T} \to \mathbb{C}$ is integrable with $\hat{f}(n) = 0$ for $n < 0$. If $f(x + \eta) \to f(x+)$, $f(x - \eta) \to f(x-)$ as $\eta \to 0$ through positive values then $f(x+) = f(x-)$.*

In other words, power series cannot have 'discontinuities of the first kind'.

Exercise 108. *Give an example of a discontinuous function with no discontinuities of the first kind.*

Once Lemma 107 has been got out of the way we can return to the proof of Theorem 103 on the boundary behaviour of the Riemann mapping. The proof turns out to be long but reasonably clear. We start with a very general result.

Lemma 109. *If Ω is a simply connected open set in $\mathbb{C}$ and $f : D(0, 1) \to \Omega$ is a bijective bicontinuous map then given any compact subset K of Ω we can find an $1 > r_K > 0$ such that, whenever $1 > |z| > r_K$, $f(z) \notin K$.*

Any bounded open set Ω has an area $|\Omega|$ and a simple application of the Cauchy-Riemann equations yields the following result.

Lemma 110. *Suppose that Ω is a simply connected bounded open set in $\mathbb{C}$ and $f : D(0,1) \to \Omega$ is a bijective analytic map. Then*

$$|\Omega| = \int_{0 \leq r < 1} \int_0^{2\pi} |f'(re^{i\theta})|^2 r \, d\theta \, dr.$$

Lemma 111. *Suppose that Ω is a simply connected bounded open set in $\mathbb{C}$ and $f : D(0,1) \to \Omega$ is a bijective analytic map. The set X of $\theta[0, 2\pi)$ such that $f(re^{i\theta})$ tends to a limit as $r \to 1$ from below has complement of Lebesgue measure* 0.

From now on until the end of the section we operate under the standing hypothesis that Ω is a simply connected open set in $\mathbb{C}$ with boundary a Jordan curve. This means that Ω is bounded (we shall accept this as a topological fact). We take X as in Lemma 111 and write $f(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta})$ whenever $\theta \in X$. We shall assume (as we may without loss of generality) that $0 \in X$.

Lemma 112. *Under our standing hypotheses we can find a continuous bijective map $g : \mathbb{T} \to \mathbb{C}$ such that $g(0) = f(1)$ and such that, if $x_1, x_2 \in X$ with $0 \leq x_1 \leq x_2 < 2\pi$ and t_1, t_2 satisfy $g(t_1) = x_1$, $g(t_2) = x_2$ and $0 \leq t_1, t_2 < 2\pi$ then $t_1 \leq t_2$.*

(The reader will, I hope, either excuse or correct the slight abuse of notation.)

We now need a simple lemma.

Lemma 113. *Suppose $G : D(0,1) \to \mathbb{C}$ is a bounded analytic function such that $G(re^{i\theta}) \to 0$ as $r \to 1-$ for all $|\theta| < \delta$ and some $\delta > 0$. Then $G = 0$.*

Using this we can strengthen Lemma 112

Lemma 114. *Under our standing hypotheses we can find a continuous bijective map $\gamma : \mathbb{T} \to \mathbb{C}$ such that $\gamma(0) = f(1)$ and such that, if $x_1, x_2 \in X$ with $0 \leq x_1 < x_2 < 2\pi$ and t_1, t_2 satisfy $g(t_1) = x_1$, $g(t_2) = x_2$ and $0 \leq t_1, t_2 < 2\pi$ then $t_1 < t_2$.*

From now on we add to our standing hypotheses the condition that γ satisfies the conclusions of Lemma 114.

We now 'fill in the gaps'.

Lemma 115. *We can find a strictly increasing function $w : [0, 2\pi] \to [0, 2\pi]$ with $w(0) = 0$ and $w(2\pi) = 2\pi$, such that $\gamma(w(\theta)) = f(e^{i\theta})$ for all $\theta \in X$.*

We now set $f(e^{i\theta}) = \gamma(w(\theta))$ and $F(\theta) = f(e^{i\theta})$ for all θ. A simple use of dominated convergence gives us the next lemma.

Lemma 116. *If $f(z) = \sum_{n=1}^{\infty} c_n z^n$ for $|z| < 1$ then, we have $\hat{F}(n) = c_n$ for $n \geq 0$ and $\hat{\gamma}(n) = 0$ for $n < 0$.*

However increasing functions can only have discontinuities of the first kind. Thus w and so F can only have discontinuities of the first kind. But, using our investment in Fourier analysis (Lemma 107) we see that F can have no discontinuities of the first kind..

Lemma 117. *The function $F : \mathbb{T} \to \mathbb{C}$ is continuous.*

Using the density of X in $\mathbb{T}$ we have the required result.

Lemma 118. *The function $f : \overline{D(0,1)} \to \overline{\Omega}$ is continuous and bijective.*

This completes the proof of Theorem103.

Using a little analytic topology we may restate Theorem103 as follows.

Theorem 119. *If Ω is a simply connected open set in $\mathbb{C}$ with boundary a Jordan curve then any bijective analytic map $f : D(0,1) \to \Omega$ can be extended to a bijective continuous map from $\overline{D(0,1)} \to \overline{\Omega}$. The map $f^{-1}\overline{\Omega} \to \overline{D(0,1)}$ is continuous on $\overline{\Omega}$.*

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9 Picard's little theorem

The object of this section is to prove the following remarkable result.

Theorem 120 (Picard's little theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is analytic and non-constant, then $\mathbb{C} \setminus f(\mathbb{C})$ contains at most one point.*

The example of exp shows that $\mathbb{C} \setminus f(\mathbb{C})$ may contain one point.

The key to Picard's theorem is the following result.

Theorem 121. *There exists an analytic map $\lambda : D(0,1) \to \mathbb{C} \setminus \{0,1\}$ with the property that given $z_0 \in \mathbb{C} \setminus \{0,1\}$, $w_0 \in D(0,1)$ and $\delta > 0$ such that $\lambda(w_0) = z_0$ and $D(z_0, \delta) \subseteq \mathbb{C} \setminus \{0,1\}$ we can find an analytic function $g : D(z_0, \delta) \to D(0,1)$ such that $\lambda(g(z)) = z$ for all $z \in D(z_0, \delta)$.*

We combine this with a result whose proof differs hardly at all from that of Theorem 69.

Lemma 122. *Suppose that U and V are open sets and that $\tau : U \to V$ is a analytic map with the following property. Given $u_0 \in U$ and $v_0 \in V$ such that $\tau(u_0) = v_0$ then, given any $\delta > 0$ with $D(v_0, \delta) \subseteq V$, we can find an analytic function $g : D(v_0, \delta) \to U$ such that $\lambda(g(z)) = z$ for all $z \in D(v_0, \delta)$. Then if W is an open simply connected set and $f : W \to U$ is analytic we can find an analytic function $F : W \to U$ such that $\tau(F(z)) = f(z)$ for all $zinW$.*

(The key words here are 'lifting' and 'monodromy'. It is at points like this that the resolutely 'practical' nature of the presentation shows its weaknesses. A little more theory about analytic continuation for its own sake would turn a 'technique' into a theorem.)

We now combine the Schwarz reflection principle (given in Lemma 87) with the work of section 8 on boundary behaviour. By repeated use of the Schwarz reflection principle we continue f analytically to the whole of $\mathcal{H}$.

Lemma 123. *Let $\mathcal{H}$ be the upper half plane. There exists an analytic map $\tau : \mathcal{H} \to \mathbb{C} \setminus \{0, 1\}$ with the property that given $z_0 \in \mathbb{C} \setminus \{0, 1\}$ and $w_0 \in \mathcal{H}$ such that $\tau(w_0) = z_0$ we can find $\delta > 0$ with $D(z_0, \delta) \subseteq \mathbb{C} \setminus \{0, 1\}$ and an analytic function $g : D(z_0, \delta) \to \mathcal{H}$ such that $\tau(g(z)) = z$ for all $z \in D(z_0, \delta)$.*

Since $\mathcal{H}$ can be mapped conformally to $D(0, 1)$ Theorem 121 follows at once and we have proved Picard's little theorem.

10 Picard's great theorem

The object of this section is to prove the following remarkable generalisation of the Casorati–Weierstrass theorem.

Theorem 124. Picard's great theorem *Let Ω be an open subset of $\mathbb{C}$ and let $w_0 \in \Omega$. If $f : \Omega \setminus \{w_0\}$ is analytic with w as an essential singularity, then we can find an $\omega_0 \in \mathbb{C}$ such that, given any $\delta > 0$ and any $\omega \neq \omega_0$ we can find a $w \in \Omega \setminus \{w_0\}$ with $|w - w_0| < \delta$ and $f(w) = \omega$.*

Exercise 125. *(i) If $f : \mathbb{C} \to \mathbb{C}$ is analytic and $|z^{-n-1} f(z)| \to 0$ as $|z| \to 0$ show, by looking at the coefficients of the Taylor expansion,or otherwise, that f is a polynomial of degree at most n.*

(ii) Continuing with the notation of (i), define $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $g(z) = f(1/z)$. Show that either f is a polynomial or 0 is an essential singularity.

(iii) Deduce Picard's little theorem from Picard's great theorem.

We introduce a couple of definitions. We could avoid using them, but the reader may find them helpful in later work.

Definition 126. *Let Ω be an open set in $\mathbb{C}$ and $f_n : \Omega \to \mathbb{C}$ a sequence of analytic functions. We say that f_n* diverges to infinity uniformly on compacta *if, given K a compact subset of Ω and $C > 0$, we can find an N such that $|f_n(z)| \geq C$ for all $z \in K$ and $n \geq N$.*

Definition 127. *Let Ω be an open set in $\mathbb{C}$ and $\mathcal{F}$ a set of analytic function $f : \Omega \to \mathbb{C}$. We say that $\mathcal{F}$ is* normal *if every sequence of functions in $\mathcal{F}$ contains a subsequence which is either uniformly convergent on compacta or diverges to infinity uniformly on compacta.*

In view of our earlier discussion of uniformly convergence on compacta the next result is closer to an exercise than a lemma.

Exercise 128. *Let Ω be a connected open set in $\mathbb{C}$ and $\mathcal{F}$ a set of analytic function $f : \Omega \to \mathbb{C}$. Then $\mathcal{F}$ is normal if and only if, given $w \in \Omega$, we can find a $\delta > 0$ such that $D(w, \delta) \subseteq \Omega$ and*

$$\mathcal{F}_{D(w,\delta)} = \{f|_{D(w,\delta)} \, : \, f \in \mathcal{F}\}$$

is normal. (More briefly a 'locally normal' family is normal.)

Give a simple counterexample to show that we need Ω connected.

If Ω is an open subset of $\mathbb{C}$ and $a \neq b$, let us write $\mathcal{F}_{a,b}(\Omega)$ for the set of analytic functions on Ω which do not take the values a or b. As usual we write D for the open unit disc.

Theorem 129. **[Schottky]** *Given any $\epsilon > 0$ and any $1 > \rho > 0$ we can find a $\delta > 0$ such that the following is true for an $f \in \mathcal{F}_{0,1}(D)$.*

(i) $|f(0)| \leq \delta \Rightarrow |f(z)| \leq \epsilon$ for all $|z| \leq \rho$.

(ii) $|f(0) - 1| \leq \delta \Rightarrow |f(z) - 1| \leq \epsilon$ for all $|z| \leq \rho$.

(iii) $|f(0)| \geq \delta^{-1} \Rightarrow |f(z)| \geq \epsilon^{-1}$ for all $|z| \leq \rho$.

Lemma 130. *Let U be an open disc If $f_n \in \mathcal{F}_{0,1}(U)$ and there exists a $w \in D$ such that $f_n(w) \to 0$ then f_n converges uniformly on compacta to 0. Similar results hold with 0 replaced by 1 and ∞.*

Theorem 131. **[A theorem of Montel]** *If Ω is a connected open set and $a \neq b$ $\mathcal{F}_{a,b}(\Omega)$ is a normal family.*

We can now prove Picard's great theorem.

11 References and further reading

There is not a great deal on Kolmogorov's and Vituškin's theorem. I will hand out some notes intended for different purposes which (if only because they are available) are probably as satisfactory as the references that I used to prepare them ([3], [5], [8]). Jackson's theorems are dealt with in [5] and elsewhere.

There exist many good books on advanced classical complex variable theory which cover what is in this course and much more. I particularly like [7] and [1]. For those who wish to study from the masters there are Hille's two volumes [2] and the elegant text of Nevanlinna [6].

There are also many excellent books on Fourier analysis. I used [9] but those who want a first introduction are recommended to look at [4].

References

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