Analysis I
Course C5

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Small print The syllabus for the course is defined by the Faculty Board Schedules (which are minimal for lecturing and maximal for examining). I should **very much** appreciate being told of any corrections or possible improvements and might even part with a small reward to the first finder of particular errors. This document is written in LaTeX2e and should be available from my home page http://www.dpmms.cam.ac.uk/~twk in latex, dvi and ps formats. My e-mail address is `twk@dpmms`. Some, fairly useless, comments on the exercises are available for supervisors from me or the secretaries in DPMMS.

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1 Why do we bother?

It is surprising how many people think that analysis consists in the difficult proofs of obvious theorems. All we need know, they say, is what a limit is, the definition of continuity and the definition of the derivative. All the rest is 'intuitively clear'.

If pressed they will agree that these definitions apply as much to the rationals $\mathbb{Q}$ as to the real numbers $\mathbb{R}$. They then have to explain the following interesting example.

Example 1.1. *If $f : \mathbb{Q} \to \mathbb{Q}$ is given by*

$$\begin{aligned} f(x) &= -1 \qquad \text{if } x^2 < 2, \\ f(x) &= 1 \qquad \text{otherwise,} \end{aligned}$$

then

(i) f is a continuous function with $f(0) = -1$, $f(2) = 1$ yet there does not exist a c with $f(c) = 0$,

(ii) f is a differentiable function with $f'(x) = 0$ for all x yet f is not constant.

What is the difference between $\mathbb{R}$ and $\mathbb{Q}$ which makes calculus work on one even though it fails on the other? Both are 'ordered fields', that is, both support operations of 'addition' and 'multiplication' together with a relation 'greater than' ('order') with the properties that we expect. If the reader is interested she will find a complete list of the appropriate axioms in texts like the altogether excellent book of Spivak [5] and its many rather less excellent competitors, but, interesting as such things may be, they are irrelevant to our purpose which is not to consider the shared properties of $\mathbb{R}$ and $\mathbb{Q}$ but to identify a *difference* between the two systems which will enable us to exclude the possibility of a function like that of Example 1.1 for functions from $\mathbb{R}$ to $\mathbb{R}$.

To state the difference we need only recall a definition from course C3.

Definition 1.2. *If $a_n \in \mathbb{R}$ for each $n \geq 1$ and $a \in \mathbb{R}$ then we say that $a_n \to a$ if given $\epsilon > 0$ we can find an $n_0(\epsilon)$ such that*

$$|a_n - a| < \epsilon \textit{ for all } n \geq n_0(\epsilon).$$

The key property of the reals, the *fundamental axiom* which makes everything work was also stated in the course C3.

Axiom 1.3. [The fundamental axiom of analysis] *If $a_n \in \mathbb{R}$ for each $n \geq 1$, $A \in \mathbb{R}$ and $a_1 \leq a_2 \leq a_3 \leq \dots$ and $a_n < A$ for each n then there exists an $a \in \mathbb{R}$ such that $a_n \to a$ as $n \to \infty$.*

Less ponderously, and just as rigorously, the fundamental axiom for the real numbers says *every increasing sequence bounded above tends to a limit.*

Everything which depends on the fundamental axiom is analysis, everything else is mere algebra.

2 The axiom of Archimedes

We start by proving the following results on limits, some of which you saw proved in course C3.

Lemma 2.1. *(i) The limit is unique. That is, if $a_n \to a$ and $a_n \to b$ as $n \to \infty$ then $a = b$.*

(ii) If $a_n \to a$ as $n \to \infty$ and $n(1) < n(2) < n(3) \dots$ then $a_{n(j)} \to a$ as $j \to \infty$.

(iii) If $a_n = c$ for all n then $a_n \to c$ as $n \to \infty$.

(iv) If $a_n \to a$ and $b_n \to b$ as $n \to \infty$ then $a_n + b_n \to a + b$.

(v) If $a_n \to a$ and $b_n \to b$ as $n \to \infty$ then $a_n b_n \to ab$.

(vi) If $a_n \to a$ as $n \to \infty$, $a_n \neq 0$ for each n and $a \neq 0$, then $a_n^{-1} \to a^{-1}$.

(vii) If $a_n \leq A$ for each n and $a_n \to a$ as $n \to \infty$ then $a \leq A$.

We need the following variation on the fundamental axiom.

Exercise 2.2. *A decreasing sequence of real numbers bounded below tends to a limit.*

[Hint. If $a \leq b$ then $-b \leq -a$.]

Useful as the results of Lemma 2.1 are, they are also true of sequences in $\mathbb{Q}$. They are therefore mere, if important, algebra. Our first truly 'analysis' result may strike the reader as rather odd.

Theorem 2.3. [Axiom of Archimedes]

$$\frac{1}{n} \to 0 \text{ as } n \to \infty$$

Theorem 2.3 shows that there is no 'exotic' real number $\gimel$ say (to choose an exotic symbol) with the property that $1/n > \gimel$ for all integers $n \geq 1$ yet $\gimel > 0$ (that is, $\gimel$ is strictly positive and yet smaller than all strictly positive rationals). There exist number systems with such exotic numbers (the famous 'non-standard analysis' of Abraham Robinson and the 'surreal numbers' of Conway constitute two such systems) but, just as the rationals are, in some sense, too small a system for the standard theorems of analysis to hold so these non-Archimedean systems are, in some sense, too big. Archimedes and Eudoxus realised the need for an axiom to show that there is no exotic number $\daleth$ bigger than any integer[1] (i.e. $\daleth > n$ for all integers $n \geq 1$; to see the connection with our form of the axiom consider $\gimel = 1/\daleth$). However, in spite of its name, what was an axiom for Archimedes is a theorem for us.

Theorem 2.4. *Given any real number K we can find an integer n with $n > K$.*

3 Series and sums

There is no need to restrict the notion of a limit to real numbers.

Definition 3.1. *If $a_n \in \mathbb{C}$ for each $n \geq 1$ and $a \in \mathbb{C}$ then we say that $a_n \to a$ if given $\epsilon > 0$ we can find an $n_0(\epsilon)$ such that*

$$|a_n - a| < \epsilon \text{ for all } n \geq n_0(\epsilon).$$

Exercise 3.2. *We work in $\mathbb{C}$.*

(i) The limit is unique. That is, if $a_n \to a$ and $a_n \to b$ as $n \to \infty$ then $a = b$.

[1]Footnote for passing historians, this is a course in mathematics.

(ii) If $a_n \to a$ as $n \to \infty$ and $n(1) < n(2) < n(3) \dots$ then $a_{n(j)} \to a$ as $j \to \infty$.

(iii) If $a_n = c$ for all n then $a_n \to c$ as $n \to \infty$.

(iv) If $a_n \to a$ and $b_n \to b$ as $n \to \infty$ then $a_n + b_n \to a + b$.

(v) If $a_n \to a$ and $b_n \to b$ as $n \to \infty$ then $a_n b_n \to ab$.

(vi) If $a_n \to a$ as $n \to \infty$, $a_n \neq 0$ for each n and $a \neq 0$, then $a_n^{-1} \to a^{-1}$.

Exercise 3.3. *Explain why there is no result in Exercise 3.2 corresponding to part (vii) of Lemma 2.1.*

We illustrate some of the ideas introduced by studying infinite sums.

Definition 3.4. *We work in $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. If $a_j \in \mathbb{F}$ we say that $\sum_{j=1}^{\infty} a_j$ converges to s if*

$$\sum_{j=1}^{N} a_j \to s$$

as $N \to \infty$. We write $\sum_{j=1}^{\infty} a_j = s$.

If $\sum_{j=1}^{N} a_j$ does not tend to a limit as $N \to \infty$, we say that the sum $\sum_{j=1}^{\infty} a_j$ diverges.

Lemma 3.5. *We work in $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.*

(i) Suppose a_j, $b_j \in \mathbb{F}$ and λ, $\mu \in \mathbb{F}$. If $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ converge then so does $\sum_{j=1}^{\infty} \lambda a_j + \mu b_j$ and

$$\sum_{j=1}^{\infty} \lambda a_j + \mu b_j = \lambda \sum_{j=1}^{\infty} a_j + \mu \sum_{j=1}^{\infty} b_j.$$

(ii) Suppose a_j, $b_j \in \mathbb{F}$ and there exists an N such that $a_j = b_j$ for all $j \geq N$. Then, either $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ both converge or they both diverge. (In other words, initial terms do not matter.)

Exercise 3.6. *Any problem on sums $\sum_{j=1}^{\infty} a_j$ can be converted into one on sequences by considering the sequence $s_n = \sum_{j=1}^{n} a_j$. Show conversely that a sequence s_n converges if and only if, when we set $a_1 = s_1$ and $a_n = s_n - s_{n-1}$ $[n \geq 2]$ we have $\sum_{j=1}^{\infty} a_j$ convergent. What can you say about $\lim_{n\to\infty} \sum_{j=1}^{n} a_j$ and $\lim_{n\to\infty} s_n$ if both exist?*

The following results are fundamental to the study of sums.

Theorem 3.7. (The comparison test.) *We work in $\mathbb{R}$. Suppose that $0 \leq b_j \leq a_j$ for all j. Then, if $\sum_{j=1}^{\infty} a_j$ converges, so does $\sum_{j=1}^{\infty} b_j$.*

Theorem 3.8. *We work in* $\mathbb{F}$ *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$. *If* $\sum_{j=1}^{\infty} |a_j|$ *converges, then so does* $\sum_{j=1}^{\infty} a_j$.

Theorem 3.8 is often stated using the following definition.

Definition 3.9. *We work in* $\mathbb{F}$ *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$. *If* $\sum_{j=1}^{\infty} |a_j|$ *converges we say that the sum* $\sum_{j=1}^{\infty} a_j$ *is absolutely convergent.*

Theorem 3.8 then becomes the statement that absolute convergence implies convergence.

Here is a trivial but useful consequence of Theorems 3.7 and 3.8.

Lemma 3.10. (Ratio test.) *Suppose* $a_j \in \mathbb{C}$ *and* $|a_{j+1}/a_j| \to l$ *as* $j \to \infty$. *If* $l < 1$, *then* $\sum_{j=1}^{\infty} |a_j|$ *converges. If* $l > 1$, *then* $\sum_{j=1}^{\infty} a_j$ *diverges.*

Of course Lemma 3.10 tells us nothing if $l = 1$ or l does not exist.

Sums which are not absolutely convergent are much harder to deal with in general. It is worth keeping in mind the following trivial observation.

Lemma 3.11. *We work in* $\mathbb{F}$ *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$. *If* $\sum_{j=1}^{\infty} a_j$ *converges, then* $a_j \to 0$ *as* $j \to \infty$.

At a deeper level the following result is sometimes useful.

Lemma 3.12. (Alternating series test.) *We work in* $\mathbb{R}$. *If we have a decreasing sequence of positive numbers* a_n *with* $a_n \to 0$ *as* $n \to \infty$, *then* $\sum_{j=1}^{\infty} (-1)^{j+1} a_j$ *converges.*

Further

$$\left| \sum_{j=1}^{N} (-1)^{j+1} a_j - \sum_{j=1}^{\infty} (-1)^{j+1} a_j \right| \leq |a_{N+1}|$$

for all $N \geq 1$.

The last sentence is sometimes expressed by saying 'the error caused by replacing a convergent infinite alternating sum of decreasing terms by the sum of its first few terms is no greater than the absolute value of the first term neglected'.

Later we will give another test for convergence called the integral test (Lemma 14.4) from which we deduce the result known to many of you that $\sum_{n=1}^{\infty} n^{-1}$ diverges. I will give another proof in the next example.

Example 3.13. *(i)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *is convergent but not absolutely convergent.*

(iii) If $v_{2n} = 1/n$, $v_{2n-1} = -1/(2n)$ *then* $\sum_{n=1}^{\infty} v_n$ *is not convergent.*

4 Least upper bounds

A non-empty bounded set in $\mathbb{R}$ need not have a maximum.

Example 4.1. *The set* $E = \{-1/n : n \geq 1\}$ *is non-empty and any* $e \in E$ *satisfies the inequalities* $-1 \leq e \leq 0$ *but* E *has no largest member.*

However, as we shall see every non-empty bounded set in $\mathbb{R}$ has a least upper bound (or supremum).

Definition 4.2. *Let* E *be a non-empty set in* $\mathbb{R}$. *We say that* α *is a least upper bound for* E *if*

(i) $\alpha \geq e$ *for all* $e \in E$ *[that is,* α *is an upper bound for* E*] and*

(ii) If $\beta \geq e$ *for all* $e \in E$ *then* $\beta \geq \alpha$ *[that is,* α *is the least such upper bound].*

If E has a supremum α we write $\sup_{e\in E} e = \sup E = \alpha$.

Lemma 4.3. *If the least upper bound exists it is unique.*

The following remark is trivial but sometimes helpful.

Lemma 4.4. *Let* E *be a non-empty set in* $\mathbb{R}$. *Then* α *is a least upper bound for* E *if and only if we can find* $e_n \in E$ *with* $e_n \to \alpha$ *and* b_n *such that* $b_n \geq e$ *for all* $e \in E$ *and* $b_n \to \alpha$ *as* $n \to \infty$.

Here is the promised result.

Theorem 4.5. *Any non-empty set in* $\mathbb{R}$ *with an upper bound has a least upper bound.*

We observe that this result is actually equivalent to the fundamental axiom.

Theorem 4.6. *Theorem 4.5 implies the fundamental axiom.*

Of course we have the notion of a greatest lower bound or infimum.

Exercise 4.7. *Define the greatest lower bound in the manner of Definition 4.2, prove its uniqueness in the manner of Lemma 4.3 and state and prove a result corresponding to Lemma 4.4.*

If E has an infimum β we write $\inf_{e\in E} e = \inf E = \beta$. One way of dealing with the infimum is to use the following observation.

Lemma 4.8. *Let E be a non-empty set in $\mathbb{R}$ and write $-E = \{-e : e \in E\}$. Then E has an infimum if and only if $-E$ has a supremum. If E has an infimum* $\inf E = -\sup(-E)$.

Exercise 4.9. *Use Lemma 4.8 and Theorem 4.5 to show that any non-empty set in $\mathbb{R}$ with a lower bound has a greatest lower bound.*

The notion of a supremum will play an important rôle in our proofs of Theorem 5.12 and Theorem 9.2.

The following result is also equivalent to the fundamental axiom (that is, we can deduce it from the fundamental axiom and conversely, if we take it as an axiom, rather than a theorem, then we can deduce the fundamental axiom as a theorem).

Theorem 4.10. [**Bolzano-Weierstrass**] *If $x_n \in \mathbb{R}$ and there exists a K such that $|x_n| \leq K$ for all n, then we can find $n(1) < n(2) < \dots$ and $x \in \mathbb{R}$ such that $x_{n(j)} \to x$ as $j \to \infty$.*

The Bolzano-Weierstrass theorem says that every bounded sequence of reals has a convergent subsequence. Notice that we say nothing about uniqueness; if $x_n = (-1)^n$ then $x_{2n} \to 1$ but $x_{2n+1} \to -1$ as $n \to \infty$.

We shall prove the theorem of Bolzano-Weierstrass by 'lion hunting' but your supervisor may well show you another method. We shall use the Bolzano-Weierstrass theorem to prove that every continuous function on a closed bounded interval is bounded and attains its bounds (Theorem 5.12). The Bolzano-Weierstrass theorem will be much used in the next analysis course because it generalises to many dimensions.

5 Continuity

We make the following definition.

Definition 5.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if given $\epsilon > 0$ we can find a $\delta(\epsilon, x) > 0$ [read 'a delta depending on epsilon and x'] such that*

$$|f(x) - f(y)| < \epsilon$$

for all y with $|x - y| < \delta(\epsilon, x)$.

If f is continuous at each point $x \in \mathbb{R}$ we say that f is a continuous function on $\mathbb{R}$.

I shall do my best to make this seem a reasonable definition but it is important to realise that I am really stating a rule of the game (like a knights move in chess or the definition of offside in football). If you wish to play the game you must accept the rules. Results about continuous functions must be derived from the definition and not stated as 'obvious from the notion of continuity'.

In practice we use a slightly more general definition.

Definition 5.2. *Let E be a subset of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is continuous at $x \in E$ if given $\epsilon > 0$ we can find a $\delta(\epsilon, x) > 0$ [read 'a delta depending on epsilon and x'] such that*

$$|f(x) - f(y)| < \epsilon$$

for all $y \in E$ with $|x - y| < \delta(\epsilon, x)$.

If f is continuous at each point $x \in E$, we say that f is a continuous function on E.

However, it will do no harm and may be positively helpful if, whilst you are getting used to the idea of continuity, you concentrate on the case $E = \mathbb{R}$.

Lemma 5.3. *Suppose that E is a subset of $\mathbb{R}$, that $x \in E$, and that f and g are functions from E to $\mathbb{R}$.*

(i) If $f(x) = c$ for all $x \in E$, then f is continuous on E.

(ii) If f and g are continuous at x, then so is $f + g$.

(iii) Let us define $f \times g : E \to \mathbb{R}$ by $f \times g(t) = f(t)g(t)$ for all $t \in E$. Then if f and g are continuous at x, so is $f \times g$.

(iv) Suppose that $f(t) \neq 0$ for all $t \in E$. If f is continuous at x so is $1/f$.

Lemma 5.4. *Let U and V be subsets of $\mathbb{R}$. Suppose $f : U \to \mathbb{R}$ is such that $f(t) \in V$ for all $t \in U$. If f is continuous at $x \in U$ and $g : V \to \mathbb{R}$ is continuous at $f(x)$, then the composition $g \circ f$ is continuous at x.*

By repeated use of parts (ii) and (iii) of Lemma 5.3 it is easy to show that polynomials $P(t) = \sum_{r=0}^{n} a_r t^r$ are continuous. The details are spelled out in the next exercise.

Exercise 5.5. *Prove the following results.*

(i) Suppose that E is a subset of $\mathbb{R}$ and that $f : E \to \mathbb{R}$ is continuous at $x \in E$. If $x \in E' \subset E$ then the restriction $f|_{E'}$ of f to E' is also continuous at x.

(ii) If $J : \mathbb{R} \to \mathbb{R}$ is defined by $J(x) = x$ for all $x \in \mathbb{R}$, then J is continuous on $\mathbb{R}$.

(iii) Every polynomial P is continuous on $\mathbb{R}$.

(iv) Suppose that P and Q are polynomials and that Q is never zero on some subset E of $\mathbb{R}$. Then the rational function P/Q is continuous on E (or, more precisely, the restriction of P/Q to E is continuous.)

The following result is little more than an observation but will be very useful.

Lemma 5.6. *Suppose that E is a subset of $\mathbb{R}$, that $x \in E$, and that f is continuous at x. If $x_n \in E$ for all n and $x_n \to x$ as $n \to \infty$, then $f(x_n) \to f(x)$ as $n \to \infty$.*

So far in this section we have only done algebra but the next result depends on the fundamental axiom. It is one of the key results of analysis and although my recommendation runs contrary to a century of enlightened pedagogy I can see no objections to students learning the proof as a model. Notice that the theorem resolves the problem posed by Example 1.1 (i).

Theorem 5.7. (The intermediate value theorem). *If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \geq 0 \geq f(b)$ then there exists a $c \in [a, b]$ such that $f(c) = 0$.*

Exercises 5.8 to 5.10 are applications of the intermediate value theorem.

Exercise 5.8. *Show that any real polynomial of odd degree has at least one root. Is the result true for polynomials of even degree? Give a proof or counterexample.*

Exercise 5.9. *Suppose that $g : [0, 1] \to [0, 1]$ is a continuous function. By considering $f(x) = g(x) - x$, or otherwise, show that there exists a $c \in [0, 1]$ with $g(c) = c$. (Thus every continuous map of $[0, 1]$ into itself has a fixed point.) Give an example of a bijective (but, necessarily, non-continuous) function $h : [0, 1] \to [0, 1]$ such that $h(x) \neq x$ for all $x \in [0, 1]$.*
[Hint: First find a function $H : [0, 1] \setminus \{0,\ 1,\ 1/2\} \to [0, 1] \setminus \{0,\ 1,\ 1/2\}$ such that $H(x) \neq x$.]

Exercise 5.10. *Every mid-summer day at six o'clock in the morning, the youngest monk from the monastery of Damt starts to climb the narrow path up Mount Dipmes. At six in the evening he reaches the small temple at the peak where he spends the night in meditation. At six o'clock in the morning on the following day he starts downwards, arriving back at the monastery at six in the evening. Of course, he does not always walk at the same speed. Show that, none the less, there will be some time of day when he will be at the same place on the path on both his upward and downward journeys.*

We proved the intermediate value theorem (Theorem 5.7) by lion hunting. We prove the next two theorems by using the Bolzano-Weierstrass Theorem (Theorem 4.10). Again the results are very important and I can see no objection to learning the proofs as a model.

Theorem 5.11. *If $f : [a, b] \to \mathbb{R}$ is continuous then we can find an M such that $|f(x)| \leq M$ for all $x \in [a, b]$.*

In other words a continuous function on a closed bounded interval is bounded. We improve this result in the next theorem.

Theorem 5.12. *If $f : [a, b] \to \mathbb{R}$ is continuous then we can find x_1, $x_2 \in [a, b]$ such that*

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all $x \in [a, b]$.

In other words a continuous function on a closed bounded interval is bounded and attains its bounds.

6 Differentiation

In this section it will be useful to have another type of limit.

Definition 6.1. *Let E be a subset of $\mathbb{R}$, f be some function from E to $\mathbb{R}$, and x some point of E. If $l \in \mathbb{R}$ we say that $f(y) \to l$ as $y \to x$ [or, if we wish to emphasise the restriction to E that $f(y) \to l$ as $y \to x$ through values $y \in E$] if, given $\epsilon > 0$, we can find a $\delta(\epsilon) > 0$ [read 'a delta depending on epsilon'] such that*

$$|f(y) - l| < \epsilon$$

for all $y \in E$ with $0 < |x - y| < \delta(\epsilon)$.

As before there is no real loss if the reader initially takes $E = \mathbb{R}$.

The following two exercises are easy but useful.

Exercise 6.2. *Let E be a subset of $\mathbb{R}$. Show that a function $f : E \to \mathbb{R}$ is continuous at $x \in E$ if and only if $f(y) \to f(x)$ as $y \to x$.*

Exercise 6.3. *Let E be a subset of $\mathbb{R}$, f, g be some functions from E to $\mathbb{R}$, and x some point of E. (i) The limit is unique. That is, if $f(y) \to l$ and $f(y) \to k$ as $y \to x$ then $l = k$.*

(ii) If $x \in E' \subseteq E$ and $f(y) \to l$ as $y \to x$ through values $y \in E$, then $f(y) \to l$ as $y \to x$ through values $y \in E'$.

(iii) If $f(t) = c$ for all $t \in E$ then $f(y) \to c$ as $y \to x$.
(iv) If $f(y) \to l$ and $g(y) \to k$ as $y \to x$ then $f(y) + g(y) \to l + k$.
(v) If $f(y) \to l$ and $g(y) \to k$ as $y \to x$ then $f(y)g(y) \to lk$.
(vi) If $f(y) \to l$ as $y \to x$, $f(t) \neq 0$ for each $t \in E$ and $l \neq 0$ then $f(t)^{-1} \to l^{-1}$.
(vii) If $f(t) \leq L$ for each $t \in E$ and $f(y) \to l$ as $y \to x$ then $l \leq L$.

We can now define the derivative.

Definition 6.4. *Let E be a subset of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is differentiable at $x \in E$ with derivative $f'(x)$ if*

$$\frac{f(y) - f(x)}{y - x} \to f'(x)$$

as $y \to x$.

If f is differentiable at each point $x \in E$, we say that f is a differentiable function on E.

As usual, no harm will be done if you replace E by $\mathbb{R}$.
Here are some easy consequences of the definition.

Exercise 6.5. *Let E be a subset of $\mathbb{R}$, f some function from E to $\mathbb{R}$, and x some point of E. Show that if f is differentiable at x then f is continuous at x.*

Exercise 6.6. *Let E be a subset of $\mathbb{R}$, f, g be some functions from E to $\mathbb{R}$, and x some point of E. Prove the following results.*
(i) If $f(t) = c$ for all $t \in E$ then f is differentiable at x with $f'(x) = 0$.
(ii) If f and g are differentiable at x then so is their sum $f + g$ and

$$(f + g)'(x) = f'(x) + g'(x).$$

(iii) If f and g are differentiable at x then so is their product $f \times g$ and

$$(f \times g)'(x) = f'(x)g(x) + f(x)g'(x).$$

(iv) If f is differentiable at x and $f(t) \neq 0$ for all $t \in E$ then $1/f$ is differentiable at x and

$$(1/f)'(x) = -f'(x)/f(x)^2.$$

(v) If $f(t) = \sum_{r=0}^{N} a_r t^r$ on E then f is differentiable at x and

$$f'(x) = \sum_{r=1}^{N} r a_r x^{r-1}$$

.

The next result is slightly harder to prove than it looks. (we split the proof into two halves depending on whether $f'(x) \neq 0$ or $f'(x) = 0$.

Lemma 6.7. **[Chain rule]** *Let U and V be subsets of $\mathbb{R}$. Suppose $f : U \to \mathbb{R}$ is such that $f(t) \in V$ for all $t \in U$. If f is differentiable at $x \in U$ and $g : V \to \mathbb{R}$ is differentiable at $f(x)$, then the composition $g \circ f$ is differentiable at x with*

$$(g \circ f)'(x) = f'(x)g'(f(x)).$$

7 The mean value theorem

We have almost finished our project of showing that the horrid situation revealed by Example 1.1 can not occur for the reals.

Our first step is to prove Rolle's theorem.

Theorem 7.1. **[Rolle's theorem]** *If $g : [a, b] \to \mathbb{R}$ is a continuous function with g differentiable on (a, b) and $g(a) = g(b)$, then we can find a $c \in (a, b)$ such that $g'(c) = 0$.*

A simple tilt gives the famous mean value theorem.

Theorem 7.2. **(The mean value theorem).** *If $f : [a, b] \to \mathbb{R}$ is a continuous function with f differentiable on (a, b), then we can find a $c \in (a, b)$ such that*

$$f(b) - f(a) = (b - a)f'(c).$$

We now have the results so long desired.

Lemma 7.3. *If $f : [a, b] \to \mathbb{R}$ is a continuous function with f differentiable on (a, b), then the following results hold.*

(i) If $f'(t) > 0$ for all $t \in (a, b)$ then f is strictly increasing on $[a, b]$. (That is, $f(y) > f(x)$ whenever $b \geq y > x \geq a$.)

(ii) If $f'(t) \geq 0$ for all $t \in (a, b)$ then f is increasing on $[a, b]$. (That is, $f(y) \geq f(x)$ whenever $b \geq y > x \geq a$.)

(iii) **[The constant value theorem]** *If $f'(t) = 0$ for all $t \in (a, b)$ then f is constant on $[a, b]$. (That is, $f(y) = f(x)$ whenever $b \geq y > x \geq a$.)*

Notice that since we deduce Lemma 7.3 from the mean value theorem we can not use it in the proof of Rolle's theorem.

The mean value theorem has many important consequences, some of which we look at in the remainder of the section.

We start by looking at inverse functions.

Lemma 7.4. *(i) Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then f is injective if and only if it is strictly increasing (that is $f(t) > f(s)$ whenever $a \leq s < t \leq b$) or strictly decreasing.*

(ii) Suppose $f : [a, b] \to \mathbb{R}$ is continuous and strictly increasing. Let $f(a) = c$ and $f(b) = d$. Then the map $f : [a, b] \to [c, d]$ is bijective and f^{-1} is continuous on $[c, d]$.

Lemma 7.5. **[Inverse rule]** *Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and $f'(x) > 0$ for all $x \in [a, b]$. Let $f(a) = c$ and $f(b) = d$. Then the map $f : [a, b] \to [c, d]$ is bijective and f^{-1} is differentiable on $[c, d]$ with*

$$f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))}.$$

In the opinion of the author the true meaning of the inverse rule and the chain rule only becomes clear when we consider higher dimensions in the next analysis course.

We now prove a form of Taylor's theorem.

Theorem 7.6. **[nth mean value theorem]** *Suppose that $b > a$ and $f : [a, b] \to \mathbb{R}$ is $n + 1$ times differentiable. Then*

$$f(b) - \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(b - a)^j = \frac{f^{(n+1)}(c')}{(n + 1)!}(b - a)^{n+1}$$

for some c' with $a < c' < b$.

This gives us a global and a local Taylor's theorem.

Theorem 7.7. **[Global Taylor theorem]** *Suppose that $b > a$ and $f : [a, b] \to \mathbb{R}$ is $n + 1$ times differentiable. If x, $t \in [a, b]$*

$$f(t) = \sum_{j=0}^{n} \frac{f^{(j)}(x)}{j!}(t - x)^j + \frac{f^{(n+1)}(x + \theta(t - x))}{(n + 1)!}(t - x)^{n+1}$$

for some $\theta \in (0, 1)$.

Theorem 7.8. **[Local Taylor theorem]** *Suppose that $\delta > 0$ and $f : (x - \delta, x + \delta) \to \mathbb{R}$ is n times differentiable on $(x - \delta, x + \delta)$ and $f^{(n)}$ is continuous at x. Then*

$$f(t) = \sum_{j=0}^{n} \frac{f^{(j)}(x)}{j!}(t - x)^j + \epsilon(t)(t - x)^n$$

where $\epsilon(t) \to 0$ as $t \to x$.

Notice that the local Taylor theorem always gives us some information but that the global one is useless unless we can find a useful bound on the $n+1$th derivative.

To reinforce this warning we consider a famous example of Cauchy.

Exercise 7.9. *Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by*

$$\begin{aligned} &F(0) = 0 \\ &F(x) = \exp(-1/x^2) \qquad \textit{otherwise.} \end{aligned}$$

(i) Prove by induction, using the standard rules of differentiation, that F is infinitely differentiable at all points $x \neq 0$ and that, at these points,

$$F^{(n)}(x) = P_n(1/x)\exp(-1/x^2)$$

where P_n is a polynomial which need not be found explicitly.

(ii) Explain why $x^{-1}P_n(1/x)\exp(-1/x^2) \to 0$ as $x \to 0$.

(iii) Show by induction, using the definition of differentiation, that F is infinitely differentiable at 0 with $F^{(n)}(0) = 0$ for all n. [Be careful to get this part of the argument right.]

(iv) Show that

$$F(x) = \sum_{j=0}^{\infty} \frac{F^{(j)}(0)}{j!} x^j$$

if and only if $x = 0$. (The reader may prefer to say that 'The Taylor expansion of F is only valid at 0'.)

(v) Why does part (iv) not contradict the local Taylor theorem (Theorem 7.8)?

Since examiners are fonder of the global Taylor theorem than it deserves I shall go through the following example.

Example 7.10. *Assuming the standard properties of the exponential function show that*

$$\exp x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

for all x.

Please note that in a pure mathematics question many (or even most) of the marks in a question of this type will depend on estimating the remainder term. [In methods questions you may simply be asked to 'find the Taylor's series' without being asked to prove convergence.]

8 Complex variable

The field $\mathbb{C}$ of complex numbers resembles the field $\mathbb{R}$ of real numbers in many ways but not in all.

Lemma 8.1. *We can not define an order on $\mathbb{C}$ which will behave in the same way as $>$ for $\mathbb{R}$.*

However there is sufficient similarity for us to define limits, continuity and differentiability. (We have already seen some of this in Definition 3.1 and Exercise 3.2.)

Definition 8.2. *Let E be a subset of $\mathbb{C}$, f be some function from E to $\mathbb{C}$, and z some point of E. If $l \in \mathbb{C}$ we say that $f(w) \to l$ as $w \to z$ [or, if we wish to emphasise the restriction to E that $f(w) \to l$ as $w \to z$ through values $w \in E$] if, given $\epsilon > 0$, we can find a $\delta(\epsilon) > 0$ [read 'a delta depending on epsilon'] such that*

$$|f(w) - l| < \epsilon$$

for all $w \in E$ with $0 < |w - z| < \delta(\epsilon)$.

As usual there is no real loss if the reader initially takes $E = \mathbb{C}$.

Definition 8.3. *Let E be a subset of $\mathbb{C}$. We say that a function $f : E \to \mathbb{C}$ is continuous at $z \in E$ if and only if $f(w) \to f(z)$ as $w \to z$.*

Exercise 8.4. *Let E be a subset of $\mathbb{C}$, f, g be some functions from E to $\mathbb{C}$, and z some point of E.*

(i) The limit is unique. That is, if $f(w) \to l$ and $f(w) \to k$ as $w \to z$ then $l = k$.

(ii) If $z \in E' \subseteq E$ and $f(w) \to l$ as $w \to z$ through values $w \in E$, then $f(w) \to l$ as $w \to x$ through values $w \in E'$.

(iii) If $f(u) = c$ for all $u \in E$ then $f(w) \to c$ as $w \to z$.

(iv) If $f(w) \to l$ and $g(w) \to k$ as $w \to z$ then $f(w) + g(w) \to l + k$.

(v) If $f(w) \to l$ and $g(w) \to k$ as $w \to z$ then $f(w)g(w) \to lk$.

(vi) If $f(w) \to l$ as $w \to z$, $f(u) \neq 0$ for each $u \in E$ and $l \neq 0$ then $f(w)^{-1} \to l^{-1}$.

Exercise 8.5. *Suppose that E is a subset of $\mathbb{C}$, that $z \in E$, and that f and g are functions from E to $\mathbb{C}$.*

(i) If $f(u) = c$ for all $u \in E$, then f is continuous on E.

(ii) If f and g are continuous at z, then so is $f + g$.

(iii) Let us define $f \times g : E \to \mathbb{C}$ by $f \times g(u) = f(u)g(u)$ for all $u \in E$. Then if f and g are continuous at z, so is $f \times g$.

(iv) Suppose that $f(u) \neq 0$ for all $u \in E$. If f is continuous at z so is $1/f$.

(v) If $z \in E' \subset E$ and f is continuous at z then the restriction $f|_{E'}$ of f to E' is also continuous at z.

(vi) If $J : \mathbb{C} \to \mathbb{C}$ is defined by $J(z) = z$ for all $z \in \mathbb{C}$, then J is continuous on $\mathbb{C}$.

(vii) Every polynomial P is continuous on $\mathbb{C}$.

(viii) Suppose that P and Q are polynomials and that Q is never zero on some subset E of $\mathbb{C}$. Then the rational function P/Q is continuous on E (or, more precisely, the restriction of P/Q to E is continuous.)

Exercise 8.6. *Let U and V be subsets of $\mathbb{C}$. Suppose $f : U \to \mathbb{C}$ is such that $f(z) \in V$ for all $z \in U$. If f is continuous at $w \in U$ and $g : V \to \mathbb{C}$ is continuous at $f(w)$, then the composition $g \circ f$ is continuous at w.*

Definition 8.7. *Let E be a subset of $\mathbb{C}$. A function $f : E \to \mathbb{C}$ is differentiable at $z \in E$ with derivative $f'(z)$ if*

$$\frac{f(w) - f(z)}{w - z} \to f'(z)$$

as $w \to z$.

If f is differentiable at each point $z \in E$ we say that f is a differentiable function on E.

Exercise 8.8. *Let E be a subset of $\mathbb{C}$, f some function from E to $\mathbb{C}$, and z some point of E. Show that if f is differentiable at z then f is continuous at z.*

Exercise 8.9. *Let E be a subset of $\mathbb{C}$, f, g be some functions from E to $\mathbb{C}$, and z some point of E. Prove the following results.*

(i) If $f(u) = c$ for all $u \in E$ then f is differentiable at z with $f'(z) = 0$.

(ii) If f and g are differentiable at z then so is their sum $f + g$ and

$$(f + g)'(z) = f'(z) + g'(z).$$

(iii) If f and g are differentiable at z then so is their product $f \times g$ and

$$(f \times g)'(z) = f'(z)g(z) + f(z)g'(z).$$

(iv) If f is differentiable at z and $f(u) \neq 0$ for all $u \in E$ then $1/f$ is differentiable at z and

$$(1/f)'(z) = -f'(z)/f(z)^2.$$

(v) If $f(u) = \sum_{r=0}^{N} a_r u^r$ on E then f is differentiable at z and

$$f'(z) = \sum_{r=1}^{N} r a_r z^{r-1}$$

.

Exercise 8.10. [Chain rule] *Let U and V be subsets of $\mathbb{C}$. Suppose $f : U \to \mathbb{C}$ is such that $f(z) \in V$ for all $t \in U$. If f is differentiable at $w \in U$ and $g : V \to \mathbb{R}$ is differentiable at $f(w)$, then the composition $g \circ f$ is differentiable at w with*

$$(g \circ f)'(w) = f'(w)g'(f(w)).$$

In spite of these similarities the subject of complex differentiable functions is very different from that of real differentiable functions. It turns out that 'well behaved' complex functions need not be differentiable.

Example 8.11. *Consider the map $\Gamma : \mathbb{C} \to \mathbb{C}$ given by $\Gamma(z) = z^*$. The function Γ is nowhere differentiable.*

Because complex differentiability is so much more restrictive than real differentiability we can prove stronger theorems about complex differentiable functions. For example it can be shown that such functions can be written locally as power series[2] (contrast the situation in the real case revealed by Example 7.9). To learn more go to the course P3 on complex methods.

9 Power series

In this section we work in $\mathbb{C}$ unless otherwise stated. We start with a very useful observation.

Lemma 9.1. *If $\sum_{n=0}^{\infty} a_n z_0^n$ converges and $|z| < |z_0|$ then $\sum_{n=0}^{\infty} a_n z^n$ converges.*

This gives us the following basic theorem on power series.

Theorem 9.2. *Suppose that $a_n \in \mathbb{C}$. Then either $\sum_{n=0}^{\infty} a_n z^n$ converges for all $z \in \mathbb{C}$, or there exists a real number R with $R \geq 0$ such that*
(i) $\sum_{n=0}^{\infty} a_n z^n$ converges if $|z| < R$,
(ii) $\sum_{n=0}^{\infty} a_n z^n$ diverges if $|z| > R$.

[2] The syllabus says that this fact is part of this course. In this one instance I advise you to ignore the syllabus.

We call R the radius of convergence of $\sum_{n=0}^\infty a_n z^n$. If $\sum_{n=0}^\infty a_n z^n$ converges for all z we write $R = \infty$. The following useful strengthening is left to the reader as an exercise.

Exercise 9.3. *Suppose that $\sum_{n=0}^\infty a_n z^n$ has radius of convergence R. Then the sequence $|a_n z^n|$ is unbounded if $|z| > R$ and $\sum_{n=0}^\infty a_n z^n$ converges absolutely if $|z| < R$.*

Note that we say nothing about what happens on the circle of convergence.

Example 9.4. *(i) $\sum_{n=1}^\infty n^{-2} z^n$ has radius of convergence 1 and converges for all z with $|z| = 1$.*

(ii) $\sum_{n=1}^\infty z^n$ has radius of convergence 1 and diverges for all z with $|z| = 1$.

A more complicated example is given in Exercise 18.16.

It is a remarkable fact that we can operate with power series in the same way as polynomials (within the radius of convergence). In particular we shall show that we can differentiate term by term.

Theorem 9.5. *If $\sum_{n=0}^\infty a_n z^n$ has radius of convergence R and we write $f(z) = \sum_{n=0}^\infty a_n z^n$ then f is differentiable at all points z with $|z| < R$ and*

$$f'(z) = \sum_{n=1}^\infty n a_n z^{n-1}.$$

The proof is starred in the syllabus. We use three simple observations.

Lemma 9.6. *If $\sum_{n=0}^\infty a_n z^n$ has radius of convergence R then given any $\epsilon > 0$ we can find a $K(\epsilon)$ such that $\sum_{n=0}^\infty |a_n z^n| < K(\epsilon)$ for all $|z| \leq R - \epsilon$.*

Lemma 9.7. *If $\sum_{n=0}^\infty a_n z^n$ has radius of convergence R then so do $\sum_{n=1}^\infty n a_n z^{n-1}$ and $\sum_{n=2}^\infty n(n-1) a_n z^{n-2}$.*

Lemma 9.8. *(i) $\binom{n}{r} \leq n(n-1)\binom{n-2}{r-2}$ for all $2 \leq r \leq n$.*

(ii) $|(z+h)^n - z^n - nhz^{n-1}| \leq n(n-1)(|z| + |h|)^{n-2}|h|^2$ for all $z,\ h \in \mathbb{C}$.

In this course we shall mainly work on the real line. Restricting to the real line we obtain the following result.

Theorem 9.9. *(i) If $a_j \in \mathbb{R}$ there exists a unique R (the radius of convergence) with $0 \leq R \leq \infty$ such that $\sum_{n=0}^{\infty} a_n x^n$ converges for all real x with $|x| < R$ and diverges for all real x with $x > R$.*

(ii) If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence R and we write $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then f is differentiable at all points x with $|x| < R$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

10 The standard functions

In school you learned all about the functions exp, log, sin and cos and about the behaviour of x^{α}. Nothing that you learned was wrong (we hope) but you might be hard pressed to prove all the facts you know in a coherent manner,

To get round this problem, we start from scratch making new definitions and assuming nothing about these various functions. One of your tasks is to make sure that the lecturer does not slip in some unproved fact. On the other hand you must allow your lecturer to choose definitions which allow an easy development of the subject rather than those that follow some 'historic', 'intuitive' or 'pedagogically appropriate' path[3].

Let us start with the exponential function. Throughout this section we shall restrict ourselves to the real line.

Lemma 10.1. *The sum $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has infinite radius of convergence.*

We can thus define a function $e : \mathbb{R} \to \mathbb{R}$ by

$$e(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(We use $e(x)$ rather than $\exp(x)$ to help us avoid making unjustified assumptions.)

Theorem 10.2. *(i) The function $e : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable with $e'(x) = e(x)$.*

(ii) $e(x + y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

(iii) $e(x) > 0$ for all $x \in \mathbb{R}$.

(iv) e is a strictly increasing function.

(v) $e(x) \to \infty$ as $x \to \infty$, $e(x) \to 0$ as $x \to -\infty$.

(vi) $e : \mathbb{R} \to (0, \infty)$ is a bijection.

[3] If you want to see a treatment along these lines see the excellent text of Burn[3].

It is worth stating some of our results in the language of group theory.

Lemma 10.3. *The mapping e is an isomorphism of the group $(\mathbb{R}, +)$ and the group $((0, \infty), \times)$.*

Since $e : \mathbb{R} \to (0, \infty)$ is a bijection we can consider the inverse function $l : (0, \infty) \to \mathbb{R}$.

Theorem 10.4. *(i) $l : (0, \infty) \to \mathbb{R}$ is a bijection. We have $l(e(x)) = x$ for all $x \in \mathbb{R}$ and $e(l(t)) = t$ for all $t \in (0, \infty)$.*

(ii) The function $l : (0, \infty) \to \mathbb{R}$ is everywhere differentiable with $l'(t) = 1/t$.

(iii) $l(uv) = l(u) + l(v)$ for all u, $v \in (0, \infty)$.

We now write $e(x) = \exp(x)$, $l(t) = \log t$ (the use of ln is unnecessary). If $\alpha \in \mathbb{R}$ and $x > 0$. we write $r_\alpha(x) = \exp(\alpha \log x)$.

Theorem 10.5. *Suppose x, $y > 0$ and α, $\beta \in \mathbb{R}$. Then*

(i) $r_\alpha(xy) = r_\alpha(x) r_\alpha(y)$,

(ii) $r_{\alpha+\beta}(x) = r_\alpha(x) r_\beta(x)$,

(iii) $r_\alpha(r_\beta(x)) = r_{\alpha\beta}(x)$

(iv) $r_1(x) = x$.

Exercise 10.6. *Use the results of Theorem 10.5 to show that if n is a strictly positive integer and $x > 0$ then*

$$r_n(x) = \underbrace{xx \ldots x}_{n}.$$

Thus, if we write $x^\alpha = r_\alpha(x)$, our new notation is *consistent* with our old school notation when α is rational but gives, in addition, a valid definition when α is irrational.

Lemma 10.7. *(i) If α is real, $r_\alpha : (0, \infty) \to (0, \infty)$ is everywhere differentiable and $r'_\alpha(x) = \alpha r_{\alpha-1}(x)$.*

(ii) If $x > 0$ and we define $f_x(\alpha) = x^\alpha$ then $f_x : \mathbb{R} \to (0, \infty)$ is everywhere differentiable and $f'_x(\alpha) = \log x f_x(\alpha)$.

Finally we look at the trigonometric functions.

Lemma 10.8. *The sums $\sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}$ have infinite radius of convergence.*

We can thus define functions s, $c : \mathbb{R} \to \mathbb{R}$ by

$$s(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!} \text{ and } c(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}$$

Lemma 10.9. *(i) The functions s, $c : \mathbb{R} \to \mathbb{R}$ are everywhere differentiable with $s'(x) = c(x)$ and $c'(x) = -s(x)$.*

(ii) $s(x + y) = s(x)c(y) + c(x)s(y)$, $c(x + y) = c(x)c(y) - s(x)s(y)$ and $s(x)^2 + c(x)^2 = 1$ for all x, $y \in \mathbb{R}$.

(iii) $s(-x) = -s(x)$, $c(-x) = c(x)$ for all $x \in \mathbb{R}$.

Exercise 10.10. *Write down what you consider to be the chief properties of* sinh *and* cosh*. (They should convey enough information to draw a reasonable graph of the two functions.)*

(i) Obtain those properties by starting with the definitions

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \text{ and } \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

and proceeding along the lines of Lemma 10.9.

(ii) Obtain those properties by starting with the definitions

$$\sinh x = \frac{\exp x - \exp(-x)}{2} \text{ and } \cosh x = \frac{\exp x + \exp(-x)}{2}$$

We have not yet proved one of the most remarkable properties of the sine and cosine functions, their periodicity.

Theorem 10.11. *Let s and c be as in Lemma 10.9.*

(i) If $c(a) = 0$ and $c(b) = 0$ then $s(b - a) = 0$.

(ii) We have $s(x) > 0$ for all $0 < x \leq 1$.

(iii) There exists a unique $\omega \in [0, 2]$ such that $c(\omega) = 0$.

(iv) $s(\omega) = 1$.

(v) $s(x + 4\omega) = s(x)$, $c(x + 4\omega) = c(\omega)$.

(vi) The function c is strictly decreasing from 1 to -1 as x runs from 0 to 2ω,

We now **define** $\pi = 2\omega$. If $\mathbf{x}$ and $\mathbf{y}$ are non-zero vectors in $\mathbb{R}^m$ we know by the Cauchy-Schwarz inequality that $|\mathbf{x}.\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ and we may **define** the angle between the two vectors to be that θ with $0 \leq \theta \leq \pi$ which satisfies

$$\cos\theta = \frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

We can also justify the standard use of polar coordinates.

Lemma 10.12. *If $(x, y) \in \mathbb{R}^2$ and $(x, y) \neq (0, 0)$ then there exist a unique $r > 0$ and θ with $0 \leq \theta < 2\pi$ such that*

$$x = r\cos\theta, \ \ y = r\sin\theta.$$

You may be unhappy with a procedure which reduces geometry to analysis. It is possible to produce treatments which soften the blow[4] but what we can not do is to justify analytic results by appealing to geometry and then appeal to analysis to justify the geometry.

11 Onwards to the complex plane

This section contains useful background material which does not really form part of the course. If time is short I shall omit it entirely.

The mean value theorem **fails** for differentiable functions $f : \mathbb{C} \to \mathbb{C}$. (See Example 11.5.) However, the constant value theorem holds.

Theorem 11.1. *If $f : \mathbb{C} \to \mathbb{C}$ is differentiable and $f'(z) = 0$ for all $z \in \mathbb{C}$ then f is constant.*

The proof of Theorem 11.1 that I shall give is very ad hoc and you will meet better ones later.

Since the constant value theorem holds we can extend the proof of Theorem 10.2 (ii) to this wider context and obtain a version of the exponential function for complex numbers. In this section we work in $\mathbb{C}$ unless otherwise stated.

Lemma 11.2. *The sum $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has infinite radius of convergence.*

We can thus define a function $\exp : \mathbb{C} \to \mathbb{C}$ by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Theorem 11.3. *(i) The function $e : \mathbb{C} \to \mathbb{C}$ is everywhere differentiable with $e'(z) = e(z)$.*

(ii) $e(z + w) = e(z)e(w)$ for all $z, w \in \mathbb{C}$.

Notice that the remaining parts of Theorem 10.2 are either meaningless or false (compare Theorem 10.2 (vi) with Lemma 11.4 (iii) which shows that $\exp : \mathbb{C} \to \mathbb{C}$ is not injective). We must be very careful in making the transition from real to complex.

We obtain a series of famous formulae.

[4]The matter is more subtle than it looks. Classical Euclidean geometry is 'weaker' than the geometry required to justify analysis and if you wished to obtain analysis from geometry you would need to add extra axioms.

Lemma 11.4. *(i) If θ is real*

$$\exp i\theta = \cos\theta + i\sin\theta.$$

(ii) If x and y are real

$$\exp(x+iy) = (\exp x)(\cos y + i\sin y).$$

(iii) exp *is periodic with period $2\pi i$, that is to say*

$$\exp(z+2\pi i) = \exp z$$

for all $z \in \mathbb{C}$.

(iv) $\exp(\mathbb{C}) = \mathbb{C}\setminus\{0\}$.

Example 11.5. *Observe that* $\exp 0 = \exp 2\pi i = 1$ *but* $\exp'(z) = \exp(z) \neq 0$ *for all $z \in \mathbb{C}$. Thus the mean value theorem does not hold for differentiable functions $f : \mathbb{C} \to \mathbb{C}$.*

It is also possible to extend the definition of sine and cosine to the complex plane but the reader is warned that the behaviour of the new functions may be somewhat unexpected. Since these extended functions most certainly do not form part of this course (though you will be expected to know them after the Complex Methods course) their study is left as an exercise.

Exercise 11.6. *(i) Explain why the infinite sums*

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \ \text{and}\ \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

converge everywhere and are differentiable everywhere with $\sin' z = \cos z$, $\cos' z = -\sin z$.

(ii) Show that

$$\sin z = \frac{\exp iz - \exp(-iz)}{2i},\ \cos z = \frac{\exp iz + \exp(-iz))}{2}$$

and

$$\exp iz = \cos z + i\sin z$$

for all $z \in \mathbb{C}$.

(iii) Show that

$$\sin(z+w) = \sin z\cos w + \cos z\sin w,\ \cos(z+w) = \cos z\cos w - \sin z\sin w,$$

and $(\sin z)^2 + (\cos z)^2 = 1$ *for all* $z,\ w \in \mathbb{C}$.

(iv) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$ *for all* $z \in \mathbb{C}$.

(v) $\sin$ *and* $\cos$ *are* 2π *periodic in the sense that*

$$\sin(z + 2\pi) = \sin z \text{ and } \cos(z + 2\pi) = \cos z$$

for all $z \in \mathbb{C}$.

(vi) If x *is real then* $\sin ix = i \sinh x$ *and* $\cos ix = \cosh x$.

(vii) Recover the addition formulae for $\sinh$ *and* $\cosh$ *by setting* $z = ix$ *and* $w = iy$ *in part (iii).*

(ix) Show that $|\sin z|$ *and* $|\cos z|$ *are bounded if* $|\Im z| \leq K$ *for some* K *but that* $|\sin z|$ *and* $|\cos z|$ *are unbounded on* $\mathbb{C}$.

However, as you were already shown in the Algebra and Geometry course and will be shown again in the Complex Methods course

There is no logarithm defined on all of $\mathbb{C} \setminus \{0\}$.

Exercise 11.7. *Suppose, if possible, that there exists a continuous* $L : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ *with* $\exp(L(z)) = z$ *for all* $z \in \mathbb{C} \setminus \{0\}$.

(i) If θ *is real, show that* $L(\exp(i\theta)) = i(\theta + 2\pi n(\theta))$ *for some* $n(\theta) \in \mathbb{Z}$.

(ii) Define $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(\theta) = \frac{1}{2\pi}\left(\frac{L(\exp i\theta) - L(1)}{i} - \theta\right).$$

Show that f *is a well defined continuous function, that* $f(\theta) \in \mathbb{Z}$ *for all* $\theta \in \mathbb{R}$, *that* $f(0) = 0$ *and that* $f(2\pi) = -1$.

(iii) Show that the statements made in the last sentence of (ii) are incompatible with the intermediate value theorem and deduce that no function can exist with the supposed properties of L.

(iv) Discuss informally what connection, if any, the discussion above has with the existence of the international date line.

A similar argument shows that it is not possible to produce a continuous square root on the complex plane.

Exercise 11.8. *Show by modifyiing the argument of Exercise 11.7, that there does not exist a continuous* $S : \mathbb{C} \to \mathbb{C}$ *with* $S(z)^2 = z$ *for all* $z \in \mathbb{C}$.

More generally, it is not possible to define continuous non-integer powers z^α on the complex plane. (Of course, $z \mapsto z^n$ is well behaved if n is an integer.) However, in the special case when x is real and strictly positive we can define

$$x^z = \exp(z \log x)$$

without problems and this enables us to write $\exp z = e^z$ where $e = \exp 1$. Surprisingly, Exercise 11.7 is not an end but a beginning of much important mathematics – but that is another story.

12 The Riemann integral

At school we are taught that an integral is the area under a curve. If pressed the framer of this definition might reply that everybody knows what area is, but then everybody knows what honey tastes like. But does honey taste the same to you as it does to me? Perhaps the question is unanswerable but, for many practical purposes, it is sufficient that we agree on what we call honey.

In order to agree on what an integral is, we need a definition which does not depend on intuition. It is important that, as far as possible, the properties of our formally defined integral shall agree with our intuitive ideas on area **but we have to prove this agreement** and not simply assume it.

In this section we introduce a notion of integral due to Riemann. For the moment we only attempt to define our integral for bounded functions on bounded intervals.

Let $f : [a, b] \to \mathbb{R}$ be a function such that there exists a K with $|f(x)| \leq K$ for all $x \in [a, b]$. [To see the connection with 'the area under the curve' it is helpful to suppose initially that $0 \leq f(x) \leq K$. However, all the definitions and proofs work more generally for $-K \leq f(x) \leq K$.]

A dissection $\mathcal{D}$ of $[a, b]$ is a finite subset of $[a, b]$ containing the end points a and b. By convention, we write

$$\mathcal{D} = \{x_0, x_1, \ldots, x_n\} \text{ with } a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b.$$

We define the *upper sum* and *lower sum* associated with $\mathcal{D}$ by

$$S(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x),$$
$$s(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x)$$

[Observe that, *if the integral* $\int_a^b f(t)\,dt$ *exists,* then the upper sum ought to provide an upper bound and the lower sum a lower bound for that integral.]

The next lemma is obvious but useful.

Lemma 12.1. *If $\mathcal{D}$ and $\mathcal{D}'$ are dissections with $\mathcal{D}' \supseteq \mathcal{D}$ then*

$$S(f, \mathcal{D}) \geq S(f, \mathcal{D}') \geq s(f, \mathcal{D}') \geq s(f, \mathcal{D}).$$

The next lemma is again hardly more than an observation but it is the key to the proper treatment of the integral.

Lemma 12.2. [Key integration property] *If $f : [a, b] \to \mathbb{R}$ is bounded and $\mathcal{D}_1$ and $\mathcal{D}_2$ are two dissections, then*

$$S(f, \mathcal{D}_1) \geq S(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq s(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq s(f, \mathcal{D}_2). \qquad \bigstar$$

The inequalities $\bigstar$ tell us that, whatever dissection you pick and whatever dissection I pick, your lower sum cannot exceed my upper sum. There is no way we can put a quart in a pint pot[5].

Since $S(f, \mathcal{D}) \geq -(b-a)K$ for all dissections $\mathcal{D}$ we can define the *upper integral* as $I^*(f) = \inf_{\mathcal{D}} S(f, \mathcal{D})$. We define the *lower integral* similarly as $I_*(f) = \sup_{\mathcal{D}} s(f, \mathcal{D})$. The inequalities $\bigstar$ tell us that these concepts behave well.

Lemma 12.3. *If $f : [a, b] \to \mathbb{R}$ is bounded, then $I^*(f) \geq I_*(f)$.*

[Observe that, *if the integral $\int_a^b f(t)\,dt$ exists,* then the upper integral ought to provide an upper bound and the lower integral a lower bound for that integral.]

If $I^*(f) = I_*(f)$, we say that f is Riemann integrable and we write

$$\int_a^b f(x)\,dx = I^*(f).$$

The following lemma provides a convenient criterion for Riemann integrability.

Lemma 12.4. *(i) A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if, given any $\epsilon > 0$, we can find a dissection $\mathcal{D}$ with*

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \epsilon.$$

(ii) A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable with integral I if and only if, given any $\epsilon > 0$, we can find a dissection $\mathcal{D}$ with

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \epsilon \text{ and } |S(f, \mathcal{D}) - I| \leq \epsilon.$$

Many students are tempted to use Lemma 12.4 (ii) as the *definition* of the Riemann integral. The reader should reflect that, without the inequality $\bigstar$, it is not even clear that such a definition gives a unique value for I. (This is only the first of a series of nasty problems that arise if we attempt to develop the theory without first proving $\bigstar$, so I strongly advise the reader not to take this path.)

We now prove a series of standard results on the integral.

[5] Or put a litre in a half litre bottle.

Lemma 12.5. *(i) The function $J : [a,b] \to \mathbb{R}$ given by $J(t) = 1$ is integrable and*

$$\int_a^b 1\,dx = b - a.$$

(ii) If $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable, then so is $f + g$ and

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(iii) If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then $-f$ is and

$$\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx.$$

(iv) If $\lambda \in \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then λf is Riemann integrable and

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx.$$

(v) If $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable functions with $f(t) \geq g(t)$ for all $t \in [a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

Lemma 12.6. *(i) If $f : [a,b] \to \mathbb{R}$ is Riemann integrable then so is f^2.*

(ii) If $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable, then so is the product fg.

Lemma 12.7. *(i) If $f : [a,b] \to \mathbb{R}$ is Riemann integrable then so is $f_+(t) = \max(f(t), 0)$.*

(ii) If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then $|f|$ is Riemann integrable and

$$\int_a^b |f(x)|\,dx \geq \left| \int_a^b f(x)\,dx \right|.$$

Notice that we often only need the much weaker inequality

$$\left| \int_a^b f(x)\,dx \right| \leq \sup_{t \in [a,b]} |f(t)|(b-a)$$

usually stated as

$$|\text{integral}| \leq \text{length} \times \text{sup}.$$

The next lemma is also routine in its proof but continues our programme of showing that the integral has all the properties we expect.

Lemma 12.8. *Suppose that $a \leq c \leq b$ and that $f : [a,b] \to \mathbb{R}$ is a bounded function. Then, f is Riemann integrable on $[a,b]$ if and only if $f|_{[a,c]}$ is Riemann integrable on $[a,c]$ and $f|_{[c,b]}$ is Riemann integrable on $[c,b]$. Further, if f is Riemann integrable on $[a,b]$, then*

$$\int_a^b f(x)\,dx = \int_a^c f|_{[a,c]}(x)\,dx + \int_c^b f|_{[c,b]}(x)\,dx.$$

In a very slightly less precise and very much more usual notation we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

There is a standard convention which says that, if $b \geq a$ and f is Riemann integrable on $[a,b]$, we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

It is, however, a convention that requires care in use.

Exercise 12.9. *Suppose that $b \geq a$, $\lambda, \mu \in \mathbb{R}$, and f and g are Riemann integrable. Which of the following statements are always true and which are not? Give a proof or counter-example. If the statement is not always true, find an appropriate correction and prove it.*

(i) $\displaystyle\int_b^a \lambda f(x) + \mu g(x)\,dx = \lambda \int_b^a f(x)\,dx + \mu \int_b^a g(x)\,dx.$

(ii) If $f(x) \geq g(x)$ for all $x \in [a,b]$, then $\displaystyle\int_b^a f(x)\,dx \geq \int_b^a g(x)\,dx.$

13 Some properties of the integral

Not all bounded functions are Riemann integrable

Lemma 13.1. *If $f : [0,1] \to \mathbb{R}$ is given by*

$$\begin{aligned} f(x) &= 1 \qquad \text{when } x \text{ is rational,} \\ f(x) &= 0 \qquad \text{when } x \text{ is irrational,} \end{aligned}$$

then f is not Riemann integrable

This does not worry us unduly, but makes it more important to show that the functions we wish to be integrable actually are.

Our first result goes back to Riemann (indeed, essentially, to Newton and Leibniz).

Lemma 13.2. *(i) If $f : [a,b] \to \mathbb{R}$ is increasing. then f is Riemann integrable.*

(ii)If $f : [a,b] \to \mathbb{R}$ can be written as $f = f_1 - f_2$ with $f_1, f_2 : [a,b] \to \mathbb{R}$ increasing, then f is Riemann integrable.

(iii) If $f : [a,b] \to \mathbb{R}$ is piecewise monotonic, then f is Riemann integrable.

It should be noted that the results of Lemma 13.2 do not require f to be continuous. (For example, the Heaviside function, given by $H(t) = 0$ for $t < 0$, $H(t) = 1$ for $t \geq 0$ is increasing but not continuous.) It is quite hard to find a continuous function which is not the difference of two increasing functions but an example is given in Exercise 19.18.

The proof of the next result is starred. Next year you will see a simpler proof based on a different idea (that of uniform continuity).

Theorem 13.3. *If $f : [a,b] \to \mathbb{R}$ is continuous then f is Riemann integrable.*

We complete the discussion of integration and this course with a series of results which apply only to *continuous* functions.

Our first result is an isolated, but useful, one.

Lemma 13.4. *If $f : [a,b] \to \mathbb{R}$ is continuous, $f(t) \geq 0$ for all $t \in [a,b]$ and*

$$\int_a^b f(t)\,dt = 0,$$

it follows that $f(t) = 0$ for all $t \in [a,b]$.

Exercise 13.5. *Let $a \leq c \leq b$. Give an example of a Riemann integrable function $f : [a,b] \to \mathbb{R}$ such that $f(t) \geq 0$ for all $t \in [a,b]$ and*

$$\int_a^b f(t)\,dt = 0,$$

but $f(c) \neq 0$.

We now come to the justly named fundamental theorem of the calculus.

Theorem 13.6. [The fundamental theorem of the calculus] *Suppose that $f : (a,b) \to \mathbb{R}$ is a continuous function and that $u \in (a,b)$. If we set*

$$F(t) = \int_u^t f(x)\,dx,$$

then F is differentiable on (a,b) and $F'(t) = f(t)$ for all $t \in (a,b)$.

Exercise 13.7. *(i) Let H be the Heaviside function $H : \mathbb{R} \to \mathbb{R}$ given by $H(x) = 0$ for $x < 0$, $H(x) = 1$ for $x \geq 0$. Calculate $F(t) = \int_0^t H(x)\,dx$ and show that F is not differentiable at 0. Where does our proof of Theorem 13.6 break down?*

(ii) Let $f(0) = 1$, $f(t) = 0$ otherwise. Calculate $F(t) = \int_0^t f(x)\,dx$ and show that F is differentiable at 0 but $F'(0) \neq f(0)$. Where does our proof of Theorem 13.6 break down?

Sometimes we think of the fundamental theorem in a slightly different way.

Theorem 13.8. *Suppose that $f : (a, b) \to \mathbb{R}$ is continuous, that $u \in (a, b)$ and $c \in \mathbb{R}$. Then there is a unique solution to the differential equation $g'(t) = f(t)$ $[t \in (a, b)]$ such that $g(u) = c$.*

We call the solutions of $g'(t) = f(t)$ *indefinite integrals* (or, simply, *integrals*) of f.

Yet another version of the fundamental theorem is given by the next theorem.

Theorem 13.9. *Suppose that $g : [a, b] \to \mathbb{R}$ has continuous derivative. Then*

$$\int_a^b g'(t)\,dt = g(b) - g(a).$$

Theorems 13.6 and 13.9 show that (under appropriate circumstances) integration and differentiation are inverse operations and the the theories of differentiation and integration are subsumed in the greater theory of the calculus.

We use the fundamental theorem of the calculus to prove the formulae for integration by substitution and integration by parts.

Theorem 13.10. [Change of variables for integrals] *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $g : [\gamma, \delta] \to \mathbb{R}$ is differentiable with continuous derivative. Suppose further that $g([\gamma, \delta]) \subseteq [a, b]$. Then, if $c, d \in [\gamma, \delta]$, we have*

$$\int_{g(c)}^{g(d)} f(s)\,ds = \int_c^d f(g(x))g'(x)\,dx.$$

Exercise 13.11. *The following exercise is traditional.*

(i) Show that integration by substitution, using $x = 1/t$, gives

$$\int_a^b \frac{dx}{1 + x^2} = \int_{1/b}^{1/a} \frac{dt}{1 + t^2}$$

when $b > a > 0$.

(ii) If we set $a = -1$, $b = 1$ in the formula of (i), we obtain

$$\int_{-1}^{1} \frac{dx}{1+x^2} \stackrel{?}{=} -\int_{-1}^{1} \frac{dt}{1+t^2}$$

Explain this apparent failure of the method of integration by substitution.

(iii) Write the result of (i) in terms of $\tan^{-1}$ and prove it using standard trigonometric identities.

Lemma 13.12. *Suppose that $f : [a,b] \to \mathbb{R}$ has continuous derivative and $g : [a,b] \to \mathbb{R}$ is continuous. Let $G : [a,b] \to \mathbb{R}$ be an indefinite integral of g. Then, we have*

$$\int_a^b f(x)g(x)\,dx = [f(x)G(x)]_a^b - \int_a^b f'(x)G(x)\,dx.$$

We obtain the following version of Taylor's theorem by repeated integration by parts.

Theorem 13.13. [A global Taylor's theorem with integral remainder] *If $n \geq 1$ and $f : (-a,a) \to \mathbb{R}$ is n times differentiable with continuous nth derivative, then*

$$f(t) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} t^j + R_n(f,t)$$

for all $|t| < a$, where

$$R_n(f,t) = \frac{1}{(n-1)!} \int_0^t (t-x)^{n-1} f^{(n)}(x)\,dx.$$

In the opinion of the lecturer this form is powerful enough for most purposes and is a form that is easily remembered and proved for examination.

14 Infinite integrals

The reader may already be familiar with definitions of the following type.

Definition 14.1. *If $f : [a,b] \to \mathbb{R}$ and $M, P \geq 0$ let us write*

$$f_{M,P}(x) = \begin{cases} f(x) & \text{if } -P \leq f(x) \leq M \\ M & \text{if } f(x) > M \\ -P & \text{if } f(x) < -P. \end{cases}$$

If $f_{M,P}$ is Riemann integrable for each M, $P \geq 0$ and

$$\int_a^b f_{M,P}(x)\,dx \to L$$

as $M, P \to \infty$ then we say that f is Riemann integrable and

$$\int_a^b f(x)\,dx = L$$

Definition 14.2. *If $f : [a, \infty) \to \mathbb{R}$ is such that $f|_{[a,X]}$ is Riemann integrable for each $X > a$ and $\int_a^X f(x)\,dx \to L$ as $X \to \infty$, then we say that $\int_a^\infty f(x)\,dx$ exists with value L.*

It must be said that neither Definition 14.1 nor Definition 14.2 are more than ad hoc.

In the rest of this section we look at Definition 14.2.

Lemma 14.3. *Suppose $f : [a, \infty) \to \mathbb{R}$ is such that $f|_{[a,X]}$ is Riemann integrable on $[a, X]$ for each $X > a$. If $f(x) \geq 0$ for all x, then $\int_a^\infty f(x)\,dx$ exists if and only if there exists a K such that $\int_a^X f(x)\,dx \leq K$ for all X.*

We use Lemma 14.3 to prove the integral comparison test.

Lemma 14.4. *If $f : [1, \infty) \to \mathbb{R}$ is a decreasing function with $f(x) \to 0$ as $x \to \infty$ then*

$$\sum_{n=1}^{\infty} f(n) \text{ and } \int_1^\infty f(x)\,dx$$

either both diverge or both converge.

Example 14.5. *If $\alpha > 1$ then $\sum_{n=1}^\infty n^{-\alpha}$ converges. If $\alpha \leq 1$ then $\sum_{n=1}^\infty n^{-\alpha}$ diverges.*

This is really as far as we need to go, but I will just add one further remark.

Lemma 14.6. *Suppose $f : [a, \infty) \to \mathbb{R}$ is such that $f|_{[a,X]}$ is Riemann integrable on $[a, X]$ for each $X > a$. If $\int_a^\infty |f(x)|\,dx$ exists, then $\int_a^\infty f(x)\,dx$ exists.*

It is natural to state Lemma 14.6 in the form 'absolute convergence of the integral implies convergence'.

Speaking broadly, infinite integrals $\int_a^\infty f(x)\,dx$ work well when they are absolutely convergent, that is to say, $\int_a^\infty |f(x)|\,dx < \infty$, but are full of traps for the unwary otherwise. This is not a weakness of the Riemann integral but inherent in any mathematical situation where an object only exists 'by virtue of the cancellation of two infinite objects'. (Question 19.17 gives an example of an integral which is convergent but not absolutely convergent.)

15 Further reading

The two excellent books Spivak's *Calculus* [5] and J. C. Burkill's *A First Course in Mathematical Analysis* [2] both cover the course completely and should be in your college library[6]. Burkill's book is more condensed and Spivak's more leisurely. A completely unsuitable but interesting version of the standard analysis course is given by Berlinski's *A Tour of the Calculus* [1] — Spivak rewritten by Sterne with additional purple passages by the Ankh-Morpork tourist board. I have written *A Companion to Analysis* [4] which covers this course at a higher level together with the next analysis course. It is available off the web but is unlikely to be as suitable for beginners as Spivak and Burkill. If you do download it, remember that you are under a moral obligation to send me an e-mail about any mistakes you find.

References

[1] D. Berlinski *A Tour of the Calculus* Mandarin Paperbacks 1997.

[2] J. C. Burkill *A First Course in Mathematical Analysis* CUP, 1962.

[3] R. P. Burn *Numbers and Functions* CUP, 1992.

[4] T. W. Körner *A Companion to Analysis* for the moment available via my home page http://www.dpmms.cam.ac.uk/~twk/ .

[5] M. Spivak, *Calculus* Addison-Wesley/Benjamin-Cummings, 1967.

16 First example sheet

Students vary in how much work they are prepared to do. On the whole, exercises from the main text are reasonably easy and provide good practice in the ideas. Questions and parts of questions marked with * are not intended to be hard but cover topics less central to the present course.

Q 16.1. Let $a_n \in \mathbb{R}$. We say that $a_n \to \infty$ as $n \to \infty$ if, given $K > 0$, we can find an $n_0(K)$ such that $a_n \geq K$ for all $n \geq n_0(K)$.

(i) Write down a similar definition for $a_n \to -\infty$.

(ii) Show that $a_n \to -\infty$ if and only if $-a_n \to \infty$.

(iii) If $a_n \neq 0$ for all n, show that, if $a_n \to \infty$, it follows that $1/a_n \to 0$ as $n \to \infty$.

[6] A quieter version of the JCR with shelves of books replacing the bar.

(iv) Is it true that, if $a_n \neq 0$ for all n and $1/a_n \to 0$, then $a_n \to \infty$ as $n \to \infty$? Give a proof or a counter example.

Q 16.2. Prove that, if

$$a_1 > b_1 > 0 \text{ and } a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{2a_n b_n}{a_n + b_n},$$

then $a_n > a_{n+1} > b_{n+1} > b_n$. Prove that, as $n \to \infty$, a_n and b_n both tend to the limit $\sqrt{(a_1 b_1)}$.

Use this result to give an example of an increasing sequence of rational numbers tending to a limit l which is not rational.

Q 16.3. (Exercise 2.2.) Show that a decreasing sequence of real numbers bounded below tends to a limit.
[Hint. If $a \leq b$ then $-b \leq -a$.]

Q 16.4. (i) By using the binomial theorem, or otherwise, show that, if $\eta > 0$ and n is a positive integer, then

$$(1 + \eta)^n \geq \eta n.$$

Deduce that $(1 + \eta)^n \to \infty$ as $n \to \infty$.

(ii) By using the binomial theorem, or otherwise, show that, if $\eta > 0$, then

$$(1 + \eta)^n \geq \eta^2 \frac{n(n-1)}{2}.$$

Deduce that $n^{-1}(1 + \eta)^n \to \infty$ as $n \to \infty$.

(iii) Show that, if k is any positive integer and $a > 1$, then $n^{-k}a^n \to \infty$ as $n \to \infty$. [Thus 'powers beat polynomial growth'.]

(iv) Show that if k is any positive integer and $1 > a \geq 0$, then $n^k a^n \to 0$ as $n \to \infty$.

Q 16.5. If $a \in \mathbb{R}$ and $a \neq -1$ describe the behaviour of

$$\frac{a^n - 1}{a^n + 1}$$

as $n \to \infty$. (That is, for each value of a say whether the sequence converges or not, and, if it converges, say what it converges to. For certain values of a you may find it useful to divide top and bottom by a^n.)

Q 16.6. (Exercises 3.2 and 3.3.)

We work in $\mathbb{C}$.

(i) The limit is unique. That is, if $a_n \to a$ and $a_n \to b$ as $n \to \infty$, then $a = b$.

(ii) If $a_n \to a$ as $n \to \infty$ and $n(1) < n(2) < n(3) \ldots$, then $a_{n(j)} \to a$ as $j \to \infty$.

(iii) If $a_n = c$ for all n, then $a_n \to c$ as $n \to \infty$.

(iv) If $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then $a_n + b_n \to a + b$.

(v) If $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then $a_n b_n \to ab$.

(vi) If $a_n \to a$ as $n \to \infty$ and $a_n \neq 0$ for each n and $a \neq 0$, then $a_n^{-1} \to a^{-1}$.

(vii) Explain why there is no result in Exercise 3.2 corresponding to part (vii) of Lemma 2.1.

Q 16.7. ***[Cesáro summation]** If $a_j \in \mathbb{R}$ we write

$$\sigma_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

(i) Show that, if $a_n \to 0$, then $\sigma_n \to 0$. [Hint: Given $\epsilon > 0$ we can find an $j_0(\epsilon)$ such that $|a_j| < \epsilon/2$ for all $j \geq j_0(\epsilon)$.]

(ii) Show that, if $a_n \to a$, then $\sigma_n \to a$.

(iii) If $a_n = (-1)^n$ show that a_n does not converge but σ_n does.

(iv) If $a_{2^m + r} = (-1)^m$ for $0 \leq r \leq 2^m - 1$, $m \geq 0$ show that σ_n does not converge.

(v) If σ_n converges show that $n^{-1} a_n \to 0$ as $n \to \infty$.

Q 16.8. In this question you may assume the standard properties of the exponential function including the relation $\exp x \geq 1 + x$ for $x \geq 0$.

(i) Suppose that $a_n \geq 0$. Show that

$$\sum_{j=1}^{n} a_j \leq \prod_{j=1}^{n} (1 + a_j) \leq \exp\left(\sum_{j=1}^{n} a_j\right).$$

Deduce, carefully, that $\prod_{j=1}^{n}(1 + a_j)$ tends to a limit as $n \to \infty$ if and only if $\sum_{j=1}^{n} a_j$ does.

(ii)* Euler made use of these ideas as follows. Let p_k be the kth prime. By observing that

$$\frac{1}{1 - p_k^{-1}} \geq \sum_{r=0}^{m} p_k^{-r}$$

show that

$$\prod_{k=1}^{n} \frac{1}{1 - p_k^{-1}} \geq \sum_{u \in S(N,n)} \frac{1}{u}$$

where $S(N,n)$ is the set of all integers of the form

$$u = \prod_{k=1}^{n} p_k^{m_k}$$

with $0 \leq m_k \leq N$.

By letting $N \to \infty$ show that

$$\prod_{k=1}^{n} \frac{1}{1-p_k^{-1}} \geq \sum_{u \in S(n)} \frac{1}{u}$$

where $S(n)$ is the set of all integers whose only prime factors are p_1, p_2, ..., p_n. Deduce that there are infinitely many primes (what can could you say about $S(n)$ if there were only n primes?) and show that

$$\prod_{k=1}^{n} \frac{1}{1-p_k^{-1}} \to \infty$$

as $n \to \infty$.

Conclude that

$$\sum_{k=1}^{\infty} \left(\frac{1}{1-p_k^{-1}} - 1 \right) \text{ diverges.}$$

Show that $|1-(1-p_k^{-1})^{-1}| \leq 2p_k^{-1}$ and deduce that

$$\sum_{k=1}^{\infty} \frac{1}{p_k} \text{ diverges.}$$

Q 16.9. (i) If a_j is an integer with $0 \leq a_j \leq 9$ show *from the fundamental axiom* that

$$\sum_{j=1}^{\infty} a_j 10^{-j}$$

exists. Show that $0 \leq \sum_{j=1}^{\infty} a_j 10^{-j} \leq 1$, carefully quoting any theorems that you use.

(ii)* If $0 \leq x \leq 1$, show that we can find integers x_j with $0 \leq x_j \leq 9$ such that

$$x = \sum_{j=1}^{\infty} x_j 10^{-j}.$$

(iii)* If a_j and b_j are integers with $0 \leq a_j, b_j \leq 9$ and $a_j = b_j$ for $j < N$, $a_N > b_N$ show that

$$\sum_{j=1}^{\infty} a_j 10^{-j} \geq \sum_{j=1}^{\infty} b_j 10^{-j}.$$

Give the precise necessary and sufficient condition for equality and prove it.

Q 16.10.* (We work with the same ideas as in Example 1.1.)

(i) Find a differentiable function $f : \mathbb{Q} \to \mathbb{Q}$ such that $f'(x) = 1$ for all $x \in \mathbb{Q}$ but $f(0) > f(1)$.

(ii) If we define $g : \mathbb{Q} \to \mathbb{Q}$ by $g(x) = (x^2 - 2)^{-2}$ show that g is continuous but unbounded on the set of x with $|x| \leq 4$.

Q 16.11. In each of the following cases determine an integer N (not necessarily the least such integer) with the property that if $m \geq N$ the mth partial sum of the series $\sum_{n=1}^{\infty} a_n$ differs from the sum of the series by less than 0.01:

(i) $a_n = 1/n(n+1)$;
(ii) $a_n = 2^{-n}$;
(iii)* $a_n = n2^{-n}$;
(iv)* $a_n = n^{-n}$.

Q 16.12. For what values of real β is

$$\sum_{n=1}^{\infty} \frac{n^\beta}{n^{2\beta} - n^\beta + 1}$$

convergent and for which divergent. Prove your answer. (You may assume that $\sum_{n=1}^{\infty} n^{-\beta}$ converges if $\beta > 1$ and diverges otherwise.)

Q 16.13.* (i) Let us write

$$S_n = \sum_{r=0}^{n} \frac{1}{r!}.$$

Show by induction, or otherwise, that $1/r! \leq 2^{-r+1}$ for $r \geq 1$ and deduce that $S_n \leq 3$. Show, from first principles, that S_n converges to a limit (which, with the benefit of extra knowledge, we call e).

(ii) Show that, if $n \geq 2$ and $r \geq 0$ then

$$\frac{n!}{(n+r)!} \leq \frac{1}{3^r}.$$

Deduce carefully that, if $m \geq n \geq 2$,

$$0 \leq n!(S_m - S_n) \leq \frac{1}{2}.$$

and that

$$0 < n!(e - S_n) \leq \frac{1}{2}.$$

Deduce that $n!e$ is not an integer for any n and conclude that e is irrational.

(iii) Show similarly that $\displaystyle\sum_{r=0}^{\infty} \frac{1}{(2r)!}$ is irrational.

17 Second example sheet

Students vary in how much work they are prepared to do. On the whole, exercises from the main text are reasonably easy and provide good practice in the ideas. Questions and parts of questions marked with * are not intended to be hard but cover topics less central to the present course.

Q 17.1. (Exercises 4.7 and 4.9.)

(i)Define the greatest lower bound in the manner of Definition 4.2, prove its uniqueness in the manner of Lemma 4.3 and state and prove a result corresponding to Lemma 4.4.

(ii) Use Lemma 4.8 and Theorem 4.5 to show that any non-empty set in $\mathbb{R}$ with a lower bound has a greatest lower bound.

Q 17.2. Suppose that A and B are non-empty bounded subsets of $\mathbb{R}$. Show that

$$\sup\{a+b : a\in A,\ b\in B\} = \sup A + \sup B.$$

The last formula is more frequently written

$$\sup_{a\in A,\ b\in B} a+b = \sup_{a\in A} a + \sup_{b\in B} b.$$

Suppose, further that a_n and b_n are bounded sequences of real numbers. For each of the following statements either give a proof that it is always true or an example to show that it is sometimes false.

(i) $\sup_n(a_n+b_n) = \sup_n a_n + \sup_n b_n$.

(ii) $\sup_n(a_n+b_n) \leq \sup_n a_n + \sup_n b_n$.

(iii) $\sup_n(a_n+b_n) \geq \sup_n a_n + \inf_n b_n$.

(iv) $\sup_{a\in A,\ b\in B} ab = (\sup_{a\in A} a)(\sup_{b\in B} b)$.

(v) $\inf_{a\in A,\ b\in B} a+b = \inf_{a\in A} a + \inf_{b\in B} b$.

Q 17.3. (Exercise 5.5) Prove the following results.

(i) Suppose that E is a subset of $\mathbb{R}$ and that $f : E \to \mathbb{R}$ is continuous at $x\in E$. If $x\in E' \subset E$ then the restriction $f|_{E'}$ of f to E' is also continuous at x.

(ii) If $J : \mathbb{R} \to \mathbb{R}$ is defined by $J(x) = x$ for all $x\in\mathbb{R}$, then J is continuous on $\mathbb{R}$.

(iii) Every polynomial P is continuous on $\mathbb{R}$.

(iv) Suppose that P and Q are polynomials and that Q is never zero on some subset E of $\mathbb{R}$. Then the rational function P/Q is continuous on E (or, more precisely, the restriction of P/Q to E is continuous.)

Q 17.4. (Exercises 5.8 to 5.10.)

(i) Show that any real polynomial of odd degree has at least one root. Is the result true for polynomials of even degree? Give a proof or counterexample.

(ii)* Suppose that $g : [0,1] \to [0,1]$ is a continuous function. By considering $f(x) = g(x) - x$, or otherwise, show that there exists a $c \in [0,1]$ with $g(c) = c$. (Thus every continuous map of $[0,1]$ into itself has a fixed point.) Give an example of a bijective (but, necessarily, non-continuous) function $h : [0,1] \to [0,1]$ such that $h(x) \neq x$ for all $x \in [0,1]$.
[Hint: First find a function $H : [0,1] \setminus \{0,\ 1,\ 1/2\} \to [0,1] \setminus \{0,\ 1,\ 1/2\}$ such that $H(x) \neq x$.]

(iii)* Every mid-summer day at six o'clock in the morning, the youngest monk from the monastery of Damt starts to climb the narrow path up Mount Dipmes. At six in the evening he reaches the small temple at the peak where he spends the night in meditation. At six o'clock in the morning on the following day he starts downwards, arriving back at the monastery at six in the evening. Of course, he does not always walk at the same speed. Show that, none the less, there will be some time of day when he will be at the same place on the path on both his upward and downward journeys.

Q 17.5. (Exercise 6.2.)

Let E be a subset of $\mathbb{R}$. Show that function $f : E \to \mathbb{R}$ is continuous at $x \in E$ if and only if $f(y) \to f(x)$ as $y \to x$.

Q 17.6. (Exercise 6.3.)

Let E be a subset of $\mathbb{R}$, f, g be some functions from E to $\mathbb{R}$, and x some point of E. (i) The limit is unique. That is, if $f(y) \to l$ and $f(y) \to k$ as $y \to x$ then $l = k$.

(ii) If $x \in E' \subseteq E$ and $f(y) \to l$ as $y \to x$ through values $y \in E$, then $f(y) \to l$ as $y \to x$ through values $y \in E'$.

(iii) If $f(t) = c$ for all $t \in E$ then $f(y) \to c$ as $y \to x$.

(iv) If $f(y) \to l$ and $g(y) \to k$ as $y \to x$ then $f(y) + g(y) \to l + k$.

(v) If $f(y) \to l$ and $g(y) \to k$ as $y \to x$ then $f(y)g(y) \to lk$.

(vi) If $f(y) \to l$ as $y \to x$, $f(t) \neq 0$ for each $t \in E$ and $l \neq 0$ then $f(t)^{-1} \to l^{-1}$.

(vii) If $f(t) \leq L$ for each $t \in E$ and $f(y) \to l$ as $y \to x$ then $l \leq L$.

Q 17.7. (Exercise 6.6.)

Let E be a subset of $\mathbb{R}$, f, g be some functions from E to $\mathbb{R}$, and x some point of E. Prove the following results.

(i) If $f(t) = c$ for all $t \in E$ then f is differentiable at x with $f'(x) = 0$.

(ii) If f and g are differentiable at x then so is their sum $f+g$ and

$$(f+g)'(x) = f'(x) + g'(x).$$

(iii) If f and g are differentiable at x then so is their product $f \times g$ and

$$(f \times g)'(x) = f'(x)g(x) + f(x)g'(x).$$

(iv) If f is differentiable at x and $f(t) \neq 0$ for all $t \in E$ then $1/f$ is differentiable at x and

$$(1/f)'(x) = -f'(x)/f(x)^2.$$

(v) If $f(t) = \sum_{r=0}^N a_r t^r$ on E then then f is differentiable at x and

$$f'(x) = \sum_{r=1}^N r a_r x^{r-1}$$

.

Q 17.8. We work in the real numbers. Are the following true or false? Give a proof or counterexample as appropriate.

(i) If $\sum_{n=1}^\infty a_n^4$ converges then $\sum_{n=1}^\infty a_n^5$ converges.

(ii) If $\sum_{n=1}^\infty a_n^5$ converges then $\sum_{n=1}^\infty a_n^4$ does.

(iii) If $a_n \geq 0$ for all n and $\sum_{n=1}^\infty a_n$ converges then $na_n \to 0$ as $n \to \infty$.

(iv)* If $a_n \geq 0$ for all n and $\sum_{n=1}^\infty a_n$ converges then $n(a_n - a_{n-1}) \to 0$ as $n \to \infty$.

(v)* If a_n is a decreasing sequence of positive numbers and $\sum_{n=1}^\infty a_n$ converges then $na_n \to 0$ as $n \to \infty$.

(vi)* If a_n is a decreasing sequence of positive numbers and $na_n \to 0$ as $n \to \infty$ then $\sum_{n=1}^\infty a_n$ converges.

[Hint. If necessary, look at Lemmas 14.4 and 14.5.]

(vii)* If $\sum_{n=1}^\infty a_n^2$ converges then $\sum_{n=1}^\infty n^{-1} a_n$ converges.

[Hint: Cauchy-Schwarz]

(viii)* If $\sum_{n=1}^\infty a_n$ converges then $\sum_{n=1}^\infty n^{-1}|a_n|$ converges.

Q 17.9.* **[General principle of convergence]** We say that a sequence x_n of points in $\mathbb{R}$ form a Cauchy sequence if given any $\epsilon > 0$ we can find an $n_0(\epsilon)$ such that

$$|x_n - x_m| < \epsilon \text{ for all } n,\, m \geq n_0(\epsilon).$$

(i) Show that any convergent sequence is a Cauchy sequence.

(ii) Suppose that the points x_n form a Cauchy sequence. Show that, if we can find $n(j) \to \infty$ such that $x_{n(j)} \to x$, it follows that $x_n \to x$. (Thus, if any subsequence of a Cauchy sequence converges, so does the sequence.)

(iii) Show that if the points x_n form a Cauchy sequence the set $\{x_n : n \geq 1\}$ is bounded.

(iv) Use the theorem of Bolzano-Weierstrass to show that any Cauchy sequence converges.

[We have thus shown that a sequence converges if and only if it is a Cauchy sequence. This is the famous Cauchy general principle of convergence. Its generalisation will play an important rôle in next year's analysis course C9.]

Q 17.10.* **[A method of Abel]**

(i) Suppose that a_j and b_j are sequences of complex numbers and that $S_n = \sum_{j=1}^n a_j$ for $n \geq 1$ and $S_0 = 0$. Show that, if $1 \leq u \leq v$ then

$$\sum_{j=u}^{v} a_j b_j = \sum_{j=u}^{v} S_j(b_j - b_{j+1}) - S_{u-1}b_u + S_v b_{v+1}.$$

(This is known as partial summation, for obvious reasons.)

(ii) Suppose now that, in addition, the b_j form a decreasing sequence of positive terms and that $|S_n| \leq K$ for all n. Show that

$$\left| \sum_{j=u}^{v} a_j b_j \right| \leq 3Kb_u.$$

(You can replace $3Kb_u$ by $2Kb_u$ if you are careful but there is no advantage in this.) Deduce that if $b_j \to 0$ as $j \to \infty$ then $\sum_{j=1}^{\infty} a_j b_j$ converges.

Deduce the alternating series test.

(iii) If b_j is a decreasing sequence of positive terms with $b_j \to 0$ as $j \to \infty$ show that $\sum_{j=1}^{\infty} b_j z^j$ converges in the region given by $|z| \leq 1$ and $z \neq 1$. Show by example that we must have the condition $z \neq 1$. Show by example that we must have the condition $|z| \leq 1$.

Q 17.11.* Enter any number on your calculator. Press the sin button repeatedly. What appears to happen? Prove your conjecture using any properties of sin that you need.

18 Third example sheet

Students vary in how much work they are prepared to do. On the whole, exercises from the main text are reasonably easy and provide good practice in the ideas. Questions and parts of questions marked with * are not intended to be hard but cover topics less central to the present course. I know that there is a general belief amongst students, directors of studies and the Faculty

Board that there is a magic set of questions which is suitable for everybody. If there is one, I will be happy to circulate it. For the moment I remark that the unstarred questions on this example sheet represent a good week's work for someone who finds the course hard and the whole sheet represents a good week's work for someone who finds the unstarred questions boring.

Q 18.1. [**Very traditional**] (i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = t^2 \sin 1/t$ for $t \neq 0$, $f(0) = 0$. Show that f is everywhere differentiable and find its derivative. Show that f' is not continuous.
[Deal quickly with the easy part and then go back to the definition to deal with $t = 0$. There are wide selection of counter-examples obtained by looking at $t^\beta \sin t^\alpha$ for various values of α and β.]

(ii) Find an infinitely differentiable function $g : (1, \infty) \to \mathbb{R}$ such that $g(t) \to 0$ but $g'(t) \nrightarrow 0$ as $t \to \infty$.

Q 18.2.* Question 18.1 shows that the derivative of a differentiable function need not be continuous. In spite of this the derivative still obeys Darboux's theorem:- If $f : [a, b] \to \mathbb{R}$ is differentiable and k lies between $f'(a)$ and $f'(b)$ then there is a $c \in [a, b]$ such that $f'(c) = k$. In this question we prove the result.

Explain why there is no loss of generality in supposing that $f'(a) > k > f'(b)$. Set $g(x) = f(x) - kx$. By looking at $g'(a)$ and $g'(b)$ show that g cannot have a maximum at a or b. Use the method of proof of Rolle's theorem to show that there exists a $c \in (a, b)$ with $g'(c) = 0$ and deduce Darboux's theorem.

Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that there does not exist a differentiable function $F : \mathbb{R} \to \mathbb{R}$ with $F' = f$.

Q 18.3. (i) [**Cauchy's mean value theorem**] Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous and that f and g are differentiable on (a, b). Suppose further that $g'(x) \neq 0$ for all $x \in (a, b)$. Explain why $g(a) \neq g(b)$. By applying Rolle's theorem to F where

$$F(x) = (g(b) - g(a))(f(x) - f(a)) - (g(x) - g(a))(f(b) - f(a)),$$

show that there is a $\zeta \in (a, b)$ such that

$$\frac{f'(\zeta)}{g'(\zeta)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(ii) [**L'Hôpital's rule**] Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous and that f and g are differentiable on (a, b). Suppose that $f(a) = g(a) = 0$, that $g'(t)$ does not vanish near a and $f'(t)/g'(t) \to l$ as $t \to a$ through values of $t > a$. Show that $f(t)/g(t) \to l$ as $t \to a$ through values of $t > a$.

Q 18.4. (Exercise 7.9.)

Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} F(0) &= 0 \\ F(x) &= \exp(-1/x^2) \qquad \text{otherwise.} \end{aligned}$$

(i) Prove by induction, using the standard rules of differentiation, that F is infinitely differentiable at all points $x \neq 0$ and that, at these points,

$$F^{(n)}(x) = P_n(1/x) \exp(-1/x^2)$$

where P_n is a polynomial which need not be found explicitly.

(ii) Explain why $x^{-1}P_n(1/x) \exp(-1/x^2) \to 0$ as $x \to 0$.

(iii) Show by induction, using the definition of differentiation, that F is infinitely differentiable at 0 with $F^{(n)}(0) = 0$ for all n. [Be careful to get this part of the argument right.]

(iv) Show that

$$F(x) = \sum_{j=0}^{\infty} \frac{F^{(j)}(0)}{j!} x^j$$

if and only if $x = 0$. (The reader may prefer to say that 'The Taylor expansion of F is only valid at 0'.)

(v) Why does part (iv) not contradict the local Taylor theorem (Theorem 7.8)?

Q 18.5. In this question you may assume the standard properties of sin and cos but not their power series expansion.

(i) By considering the sign of $f_1'(x)$, when $f_1(t) = t - \sin t$, show that

$$t \geq \sin t$$

for all $t \geq 0$.

(ii) By considering the sign of $f_2'(x)$, when $f_2(t) = \cos t - 1 + t^2/2!$, show that

$$\cos t \geq 1 - \frac{t^2}{2!}$$

for all $t \geq 0$.

(iii) By considering the sign of $f_3'(x)$, when $f_3(t) = \sin t - t + t^3/3!$, show that

$$\sin t \geq t - \frac{t^3}{3!}$$

for all $t \geq 0$.

(iv) State general results suggested by parts (i) to (iii) and prove them by induction. State and prove corresponding results for $t < 0$.

(v) Using (iv), show that

$$\sum_{n=0}^{N} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \to \sin t$$

as $N \to \infty$ for all $t \in \mathbb{R}$. State and prove a corresponding result for cos. [This question could be usefully illustrated by computer graphics.]

Q 18.6. (i) Suppose $f : [0,1] \to \mathbb{R}$ is twice differentiable with $f''(t) \geq 0$ for all $t \in [0,1]$. If $f'(0) > 0$ and $f(0) = 0$ explain why $f(t) > 0$ for $t > 0$. If $f'(0) \geq 0$ and $f(0) = f(1) = 0$ what can you say about f and why? If $f'(1) \leq 0$ and $f(0) = f(1) = 0$ what can you say about f and why?

(ii) Suppose $f : [0,1] \to \mathbb{R}$ is twice differentiable with $f''(t) \geq 0$ for all $t \in [0,1]$ and $f(0) = f(1) = 0$. Show that $f(t) \leq 0$ for all $t \in [0,1]$. [Hint: Consider the sign of f'.]

(iii) Suppose $g : [a,b] \to \mathbb{R}$ is twice differentiable with $g''(t) \geq 0$ for all $t \in [a,b]$. By considering the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(t) = g\big((1-t)a + tb\big) - (1-t)g(a) - tg(b)$$

show that

$$g\big((1-t)a + tb\big) \leq (1-t)g(a) + tg(b)$$

for all t with $1 \geq t \geq 0$.

[In other words a twice differentiable function with everywhere positive second derivative is *convex*. Convex functions are considered in the last part of the probability course where you prove the very elegant Jensen's inequality. You should note, however, that not all convex functions are twice differentiable (look at $x \mapsto |x|$).]

Q 18.7.* The results of this question are also useful in the probability course when you study extinction probabilities. Notice that the point of this question is to obtain rigorous proofs of the results stated.

Suppose $a_0 > 0$, $a_1, a_2, \ldots, a_n \geq 0$ and $\sum_{j=0}^n a_j = 1$. We set $P(t) = \sum_{j=0}^n a_j t^j$.

(i) Find $P(0)$, $P(1)$ and $P'(1)$. Show that $P'(t) \geq 0$ and $P''(t) \geq 0$ for all $t \geq 0$.

(ii) Show that the equation $P(t) = t$ has no solution with $0 \leq t < 1$ if $\sum_{j=1}^n ja_j \leq 1$ and exactly one such solution if $\sum_{j=1}^n ja_j > 1$. We write α for the smallest solution of $P(t) = t$ with $0 \leq t \leq 1$.

(iii) If we set $x_0 = 0$ and $x_n = P(x_{n-1})$ for $n \geq 1$ show, by induction, that $x_{n-1} \leq x_n \leq \alpha$.

(iv) Deduce, giving your reasons explicitly, that x_n must converge to a limit β. Show that $0 \leq \beta \leq \alpha$ and $P(\beta) = \beta$. Deduce that $\beta = \alpha$ and so

$$x_n \to \alpha \text{ as } n \to \infty.$$

Q 18.8.* The first proof that transcendental numbers existed is due to Liouville. This question gives a version of his proof.

Let $P(t) = \sum_{j=0}^{N} a_j t^j$ with the a_j integers and $a_N \neq 0$. Let α be a root of P.

(i) Why can we choose a $\delta > 0$ such that $P(t) \neq 0$ for all t with $t \in [\alpha - \delta, \alpha + \delta]$ and $t \neq \alpha$.

(ii) Let δ be as in (i). Explain why there exists a K such that $|P(t) - P(s)| \leq K|t - s|$ for all t, $s \in [\alpha - \delta, \alpha + \delta]$.

(iii) If p and q are integers such that $q \geq 1$ and $P(p/q) \neq 0$ show that $|P(p/q)| \geq q^{-N}$. [Hint: Remember that the coefficients of P are integers.]

(iv) If p and q are integers such that $q \geq 1$, $P(p/q) \neq 0$ and $p/q \in [\alpha - \delta, \alpha + \delta]$ use (ii) and (iii) to show that

$$\left|\alpha - \frac{p}{q}\right| \geq K^{-1} q^{-N}.$$

(v) Show that, if p and q are integers such that $q \geq 1$ and $\alpha \neq p/q$, then

$$\left|\alpha - \frac{p}{q}\right| \geq \min(\delta, K^{-1} q^{-N}).$$

(vi) Show that if α is a root of a polynomial with integer coefficients then there exists a K and an N (depending on α) such that

$$\left|\alpha - \frac{p}{q}\right| \geq K^{-1} q^{-N}$$

whenever p and q are integers with $q \geq 1$ and $\alpha \neq p/q$.

(vii) Explain why $\sum_{n=0}^{\infty} 10^{-n!}$ converges. Call the limit L. Show that

$$\left|L - \sum_{n=0}^{m} 10^{-n!}\right| \leq 2((10)^{m!})^{-m-1}.$$

By taking $q = (10)^{m!}$ and looking at (vi), show that L can not be the root of a polynomial with integer coefficients.

(viii) By looking at

$$\sum_{n=0}^{\infty} \zeta_n 10^{-n!}$$

with $\zeta_n = \pm 1$ show that there are uncountably many transcendentals.

Q 18.9. In this question you may assume standard results on the power function $t \mapsto t^\alpha$.

Use the form of Taylor's theorem given in Theorem 7.7 to show that

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + \dots$$

for all $0 \leq x < 1$. Note that the main part of your task is to estimate the remainder term.

What happens if α is a positive integer?

Why does your argument fail for $-1 < x < 0$? (The result is true but our form of the remainder does not give it. This is one of the reasons why there are so many forms of Taylor's with different remainders.)

Q 18.10.* (i) Suppose that $x_0, x_1, \dots x_n$ are distinct. Set

$$e_j(x) = \prod_{k \neq j} \frac{x - x_k}{x_j - x_k}.$$

What is the value of $e_j(x_k)$ if $j \neq k$ and if $j = k$? Show that given real numbers α_j there is a polynomial of degree n with $f(x_j) = \alpha_j$.

(ii) Suppose $f : [a,b] \to \mathbb{R}$ is $n+1$ times differentiable, $a < x_0 < x_1 < \dots < x_n < b$ and P is a polynomial of degree n with $P(x_j) = f(x_j)$ for $0 \leq j \leq n$. We are interested in the error

$$e(x) = f(x) - P(x)$$

at a point $x \in [a,b]$ with $x \neq x_j$ for all j.

Set

$$F(t) = f(t) - P(t) - e(x)\prod_{k=0}^{n} \frac{t - x_k}{x - x_k}.$$

Explain why F vanishes at least $n+2$ times on $[a,b]$. Explain why F' vanishes at least $n+1$ times on $[a,b]$. By repeating the argument show that $F^{(n+1)}$ must vanish at least once (at ζ, say) on $[a,b]$. Show that

$$e(x) = \frac{1}{(n+1)!} f^{(n+1)}(\zeta) \prod_{k=0}^{n} (x - x_k).$$

Deduce that if $|f^{(n+1)}(t)| \leq M$ for all $t \in [a,b]$ then

$$|f(t) - P(t)| \leq \frac{M}{(n+1)!}|Q(t)|$$

where $Q(t) = \prod_{k=0}^{n}(t - x_k)$.

[The argument just given should remind you of the proof of Theorem 7.6.]

Q 18.11.* Recall from last term (or earlier) that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all real θ and all integers $n \geq 0$.

By taking real parts, show that there is a real polynomial T_n of degree n such that

$$T_n(\cos\theta) = \cos n\theta$$

for all real θ.

Write down $T_0(t)$, $T_1(t)$, $T_2(t)$ and $T_3(t)$ explicitly.

Show that, if $n \geq 1$ then

(a) $|T_n(t)| \leq 1$ for all t with $|t| \leq 1$,

(b) $T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t)$, (why does this result hold for all t?)

(c) the coefficient of t^n in $T_n(t)$ is 2^{n-1}.

Explain why T_{n+1} has $n+1$ zeros x_0, x_1, …, x_n lying in $(-1,1)$. Use (a), (c) and the final result of question 18.10 to show that if $[a,b] = [-1,1]$ and f and P obey the hypotheses of question 18.10 (so that, in particular, $|f^{(n+1)}(t)| \leq M$ for all $t \in [-1,1]$) then

$$|f(t) - P(t)| \leq \frac{M}{(n+1)!}|2^{-n}T_{n+1}(t)| \leq \frac{M}{2^n(n+1)!}$$

The rest of the question just asks for another of useful property of the Tchebychev polynomials T_n. (The modern view is that Tchebychev should have called himself Chebychev. He seems to have preferred Tchebycheff.)

(d) If $n,\ m \geq 0$ then

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{(1-x^2)^{1/2}}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{\pi}{2} & \text{if } n = m \neq 0 \end{cases}$$

What happens if $n = m = 0$?

Q 18.12. (Exercise 8.4.)

Let E be a subset of $\mathbb{C}$, f, g be some functions from E to $\mathbb{C}$, and z some point of E.

(i) The limit is unique. That is, if $f(w) \to l$ and $f(w) \to k$ as $w \to z$ then $l = k$.

(ii) If $z \in E' \subseteq E$ and $f(w) \to l$ as $w \to z$ through values $w \in E$, then $f(w) \to l$ as $w \to z$ through values $w \in E'$.

(iii) If $f(u) = c$ for all $u \in E$ then $f(w) \to c$ as $w \to z$.

(iv) If $f(w) \to l$ and $g(w) \to k$ as $w \to z$ then $f(w) + g(w) \to l + k$.

(v) If $f(w) \to l$ and $g(w) \to k$ as $w \to z$ then $f(w)g(w) \to lk$.

(vi) If $f(w) \to l$ as $w \to z$, $f(u) \neq 0$ for each $u \in E$ and $l \neq 0$ then $f(w)^{-1} \to l^{-1}$.

Q 18.13. (Exercise 8.5 and 8.6.) Suppose that E is a subset of $\mathbb{C}$, that $z \in E$, and that f and g are functions from E to $\mathbb{C}$.

(i) If $f(u) = c$ for all $u \in E$, then f is continuous on E.

(ii) If f and g are continuous at z, then so is $f + g$.

(iii) Let us define $f \times g : E \to \mathbb{C}$ by $f \times g(u) = f(u)g(u)$ for all $u \in E$. Then if f and g are continuous at z, so is $f \times g$.

(iv) Suppose that $f(u) \neq 0$ for all $u \in E$. If f is continuous at z so is $1/f$.

(v) If $z \in E' \subset E$ and f is continuous at z then the restriction $f|_{E'}$ of f to E' is also continuous at z.

(vi) If $J : \mathbb{C} \to \mathbb{C}$ is defined by $J(z) = z$ for all $z \in \mathbb{C}$, then J is continuous on $\mathbb{C}$.

(vii) Every polynomial P is continuous on $\mathbb{C}$.

(viii) Suppose that P and Q are polynomials and that Q is never zero on some subset E of $\mathbb{C}$. Then the rational function P/Q is continuous on E (or, more precisely, the restriction of P/Q to E is continuous.)

(ix) Let U and V be subsets of $\mathbb{C}$. Suppose $f : U \to \mathbb{C}$ is such that $f(z) \in V$ for all $z \in U$. If f is continuous at $w \in U$ and $g : V \to \mathbb{C}$ is continuous at $f(w)$, then the composition $g \circ f$ is continuous at w.

Q 18.14. (Exercises 8.8 8.9 and 8.10.)

Let E be a subset of $\mathbb{C}$, f, g be some functions from E to $\mathbb{C}$, and z some point of E.

Show that if f is differentiable at z then f is continuous at z.

Prove the following results.

(i) If $f(u) = c$ for all $u \in E$ then f is differentiable at z with $f'(z) = 0$.

(ii) If f and g are differentiable at z then so is their sum $f + g$ and

$$(f + g)'(z) = f'(z) + g'(z).$$

(iii) If f and g are differentiable at z then so is their product $f \times g$ and

$$(f \times g)'(z) = f'(z)g(z) + f(z)g'(z).$$

(iv) If f is differentiable at z and $f(u) \neq 0$ for all $u \in E$ then $1/f$ is differentiable at z and

$$(1/f)'(z) = -f'(z)/f(z)^2.$$

(v) If $f(u) = \sum_{r=0}^{N} a_r u^r$ on E then then f is differentiable at z and

$$f'(z) = \sum_{r=1}^{N} r a_r z^{r-1}$$

.

(vi) Let U and V be subsets of $\mathbb{C}$. Suppose $f : U \to \mathbb{C}$ is such that $f(z) \in V$ for all $t \in U$. If f is differentiable at $w \in U$ and $g : V \to \mathbb{R}$ is differentiable at $f(w)$, then the composition $g \circ f$ is differentiable at w with

$$(g \circ f)'(w) = f'(w)g'(f(w)).$$

Q 18.15. (Exercise 9.3.) Suppose that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R. Then the sequence $|a_n z^n|$ is unbounded if $|z| > R$ and $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $|z| < R$.

Q 18.16.* This question requires Abel's test from Exercise 17.10 which is also starred.

(i) Show that $\sum_{n=1}^{\infty} z^n/n$ has radius of convergence 1, converges for all z with $|z| = 1$ and $z \neq 1$ but diverges if $z = 1$.

(ii) Let $|z_1| = |z_2| = \cdots = |z_m| = 1$. Find a power series $\sum_{n=0}^{\infty} a_n z^n$ which has radius of convergence 1 and converges for all z with $|z| = 1$ and $z \neq z_1,\ z_2,\ \ldots,\ z_m$ but diverges if $z = z_j$ for some $1 \leq j \leq m$.

Q 18.17.* (i) Show that there exist power series of all radii of convergence (including 0 and ∞).

(ii) Suppose the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R and the power series $\sum_{n=0}^{\infty} b_n z^n$ has radius of convergence S. Show that, if $R \neq S$ then $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ has radius of convergence $\min(R, S)$.

(iii) Suppose that the conditions of (ii) hold except that $R = S$. Show that the radius of convergence T of $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ satisfies the condition $T \geq R$. Show by means of examples that T can take any value with $T \geq R$. [Hint: Start by thinking of a simple relation between a_n and b_n which will give $T = \infty$.]

Q 18.18.* (i) Suppose that x_n is a bounded sequence of real numbers. Show carefully that $y_n = \sup_{m \geq n} x_m$ is a bounded decreasing sequence and deduce that y_n tends to a limit y say. We write

$$\limsup_{n \to \infty} x_n = y.$$

(ii) Suppose that x_n is a bounded sequence of real numbers. Prove the following two results.

(A) Given any $\epsilon > 0$, we can find an $M(\epsilon)$ such that

$$x_n \leq \limsup_{m\to\infty} x_m + \epsilon$$

for all $n \geq M(\epsilon)$.

(B) Given any $\epsilon > 0$ and any N we can find an integer $P(N,\epsilon) \geq N$ such that

$$x_{P(N,\epsilon)} \geq \limsup_{m\to\infty} x_m - \epsilon.$$

(iii) Using (ii) or otherwise show that if $a_n \in \mathbb{C}$ and the sequence $|a_n|^{1/n}$ is bounded and $\limsup_{n\to\infty} |a_n|^{1/n} \neq 0$ then $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $(\limsup_{n\to\infty} |a_n|^{1/n})^{-1}$.

Show also that, if the sequence $|a_n|^{1/n}$ is unbounded, $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 0, and, if $\limsup_{n\to\infty} |a_n|^{1/n} = 0$, $\sum_{n=0}^{\infty} a_n z^n$ has infinite radius of convergence.

[This result has considerable theoretical significance but I strongly advise using the definition directly rather than relying on the formula.]

(iv) Use the formula of this question to obtain the results of Question 18.17.

Q 18.19. (i) Starting from the observation that

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

show that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for all $|x| < 1$. Show that the result is false or meaningless if $|x| \geq 1$.

(ii) Use term by term differentiation to obtain power series expansions for $(1-x)^{-n}$ for all integer n with $n \geq 1$.

(iii) Find the radius of convergence of $\sum_{n=1}^{\infty} x^n/n$. Use term by term differentiation and the constant value theorem to show that

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

(iv) Use similar ideas to obtain a power series expansion for $\tan^{-1} x$ in a range to be stated.

Q 18.20. (Exercise 10.10.)

Write down what you consider to be the chief properties of sinh and cosh. (They should convey enough information to draw a reasonable graph of the two functions.)

(i) Obtain those properties by starting with the definitions

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \text{ and } \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

and proceeding along the lines of Lemma 10.9.

(ii) Obtain those properties by starting with the definitions

$$\sinh x = \frac{\exp x - \exp(-x)}{2} \text{ and } \cosh x = \frac{\exp x + \exp(-x)}{2}$$

Q 18.21.*(Exercise 11.6.)

(i) Explain why the infinite sums

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \text{ and } \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

converge everywhere and are differentiable everywhere with $\sin' z = \cos z$, $\cos' z = -\sin z$.

(ii) Show that

$$\sin z = \frac{\exp iz - \exp(-iz)}{2i},\ \cos z = \frac{\exp iz + \exp(-iz)}{2}$$

and

$$\exp iz = \cos z + i \sin z$$

for all $z \in \mathbb{C}$.

(iii) Show that

$$\sin(z+w) = \sin z \cos w + \cos z \sin w,\ \cos(z+w) = \cos z \cos w - \sin z \sin w,$$

and $(\sin z)^2 + (\cos z)^2 = 1$ for all z, $w \in \mathbb{C}$.

(iv) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$ for all $z \in \mathbb{C}$.

(v) sin and cos are 2π periodic in the sense that

$$\sin(z + 2\pi) = \sin z \text{ and } \cos(z + 2\pi) = \cos z$$

for all $z \in \mathbb{C}$.

(vi) If x is real then $\sin ix = i \sinh x$ and $\cos ix = \cosh x$.

(vii) Recover the addition formulae for sinh and cosh by setting $z = ix$ and $w = iy$ in part (iii).

(viii) Show that $|\sin z|$ and $|\cos z|$ are bounded if $|\Im z| \leq K$ for some K but that $|\sin z|$ and $|\cos z|$ are unbounded on $\mathbb{C}$.

Q 18.22. *(Exercise 11.7.)

Suppose, if possible, that there exists a continuous $L : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with $\exp(L(z)) = z$ for all $z \in \mathbb{C} \setminus \{0\}$.

(i) If θ is real, show that $L(\exp(i\theta)) = i(\theta + 2\pi n(\theta))$ for some $n(\theta) \in \mathbb{Z}$.

(ii) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(\theta) = \frac{1}{2\pi}\left(\frac{L(\exp i\theta) - L(1)}{i} - \theta\right).$$

Show that f is a well defined continuous function, that $f(\theta) \in \mathbb{Z}$ for all $\theta \in \mathbb{R}$, that $f(0) = 0$ and that $f(2\pi) = -1$.

(iii) Show that the statements made in the last sentence of (ii) are incompatible with the intermediate value theorem and deduce that no function can exist with the supposed properties of L.

(iv) Discuss informally what connection, if any, the discussion above has with the existence of the international date line.

Q 18.23. *(Exercise 11.8.) Show by modifying the argument of Exercise 11.7, that there does not exist a continuous $S : \mathbb{C} \to \mathbb{C}$ with $S(z)^2 = z$ for all $z \in \mathbb{C}$.

19 Fourth example sheet

Students vary in how much work they are prepared to do. On the whole, exercises from the main text are reasonably easy and provide good practice in the ideas. Questions and parts of questions marked with * are not intended to be hard but cover topics less central to the present course.

Q 19.1. (Exercise 12.9.) Suppose that $b \geq a$, $\lambda, \mu \in \mathbb{R}$, and f and g are Riemann integrable. Which of the following statements are always true and which are not? Give a proof or counter-example. If the statement is not always true, find an appropriate correction and prove it.

(i) $\displaystyle\int_b^a \lambda f(x) + \mu g(x)\, dx = \lambda \int_b^a f(x)\, dx + \mu \int_b^a g(x)\, dx$.

(ii) If $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\displaystyle\int_b^a f(x)\, dx \geq \int_b^a g(x)\, dx$.

Q 19.2. (Exercise 13.5.) Let $a \leq c \leq b$. Give an example of a Riemann integrable function $f : [a, b] \to \mathbb{R}$ such that $f(t) \geq 0$ for all $t \in [a, b]$ and

$$\int_a^b f(t)\, dt = 0,$$

but $f(c) \neq 0$.

Q 19.3.* Define $f : [0,1] \to \mathbb{R}$ by $f(p/q) = 1/q$ when p and q are coprime integers with $1 \leq p < q$ and $f(x) = 0$ otherwise.

(i) Show that f is Riemann integrable and find $\int_0^1 f(x)\,dx$.

(ii) At which points is f continuous? Prove your answer.

Q 19.4. (Exercise 13.7).

(i) Let H be the Heaviside function $H : \mathbb{R} \to \mathbb{R}$ given by $H(x) = 0$ for $x < 0$, $H(x) = 1$ for $x \geq 0$. Calculate $F(t) = \int_0^t H(x)\,dx$ and show that F is not differentiable at 0. Where does our proof of Theorem 13.6 break down?

(ii) Let $f(0) = 1$, $f(t) = 0$ otherwise. Calculate $F(t) = \int_0^t f(x)\,dx$ and show that F is differentiable at 0 but $F'(0) \neq f(0)$. Where does our proof of Theorem 13.6 break down?

Q 19.5. (Exercise 13.11.)

The following exercise is traditional.

(i) Show that integration by substitution, using $x = 1/t$, gives

$$\int_a^b \frac{dx}{1+x^2} = \int_{1/b}^{1/a} \frac{dt}{1+t^2}$$

when $b > a > 0$.

(ii) If we set $a = -1$, $b = 1$ in the formula of (i), we obtain

$$\int_{-1}^{1} \frac{dx}{1+x^2} \stackrel{?}{=} -\int_{-1}^{1} \frac{dt}{1+t^2}$$

Explain this apparent failure of the method of integration by substitution.

(iii) Write the result of (i) in terms of $\tan^{-1}$ and prove it using standard trigonometric identities.

Q 19.6. In this question we give an alternative treatment of the logarithm so no properties of the exponential or logarithmic function should be used. You should quote all the theorems that you use, paying particular attention to those on integration.

We set

$$l(x) = \int_1^x \frac{1}{t}\,dt.$$

(i) Explain why $l : (0,\infty) \to \mathbb{R}$ is a well defined function.

(ii) Use the change of variable theorem for integrals to show that

$$\int_x^{xy} \frac{1}{t}\,dt = l(y)$$

for all x, $y > 0$. Deduce that $l(xy) = l(x) + l(y)$.

(iii) Show that l is everywhere differentiable with $l'(x) = 1/x$.

(iv) Show that l is a strictly increasing function.

(v) Show that $l(x) \to \infty$ as $x \to \infty$.

Q 19.7. In the lectures we deduced the properties of the logarithm from those of the exponential. Reverse this by making a list of the properties of the exponential, define exp as the inverse function of log (explaining carefully why you can do this) and using the properties of log found in the previous question (Question 19.6).

Q 19.8.* **[The first mean value theorem for integrals]** Suppose $g : [a, b] \to \mathbb{R}$ is a continuous function such that $g(x) \geq 0$ for all $x \in [a, b]$. If $f : [a, b] \to \mathbb{R}$ is continuous explain why we can find k_1 and k_2 in $[a, b]$ such that

$$f(k_1) \leq f(x) \leq f(k_2)$$

for all $x \in [a, b]$. Deduce carefully that

$$f(k_1)\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq f(k_2)\int_a^b g(x)\,dx$$

and show, stating carefully any theorem that you need, that there exists a $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Does this result remain true if $g(x) \leq 0$ for all $x \in [a, b]$? Does this result remain true if we place no restrictions on g (apart from continuity). In each case give a proof or a counter example. (The first mean value theorem for integrals is used in the numerical analysis course.)

Q 19.9. Use the integral form of the remainder in Taylor's theorem (i.e. Theorem 13.13) to obtain the power series expansion for sin.

Q 19.10. **[The binomial theorem]** If $1 > x \geq t \geq 0$ show that

$$\frac{x - t}{1 + t} \leq x.$$

If $-1 < x \leq t \leq 0$ show that

$$\frac{t - x}{1 + t} \leq -x.$$

Use the integral form of the Taylor theorem to show that, if $|x| < 1$, then

$$(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha - 1)\ldots(\alpha - n + 1)}{n!}ix^n + \ldots.$$

Q 19.11.* **[Cauchy-Schwarz for integrals]** Write $C([a,b])$ for the set of continuous functions $f:[a,b]\to\mathbb{R}$.

(i) If you are doing course P1 verify that $C([a,b])$ is a vector space over $\mathbb{R}$. Is it finite dimensional? Prove your answer.

(ii) If $f,\ g\in C([a,b])$ we write

$$\langle f,g\rangle = \int_a^b f(t)g(t)\,dt.$$

Show that if $f,\ g,\ h\in C([a,b])$ and $\lambda,\ \mu\in\mathbb{R}$ then

(a) $\langle f,f\rangle\geq 0$,
(b) if $\langle f,f\rangle = 0$ then $f=0$.
(c) $\langle f,g\rangle=\langle g,f\rangle$,
(d) $\langle \lambda f+\mu g,h\rangle=\lambda\langle f,h\rangle+\mu\langle g,h\rangle$.

(iii) By imitating the proof of the Cauchy-Schwarz inequality for $\mathbb{R}^n$ in the course C1/C2 show that

$$\langle f,g\rangle^2\leq\langle f,f\rangle\langle g,g\rangle.$$

In other words

$$\left(\int_a^b f(t)g(t)\,dt\right)^2\leq\int_a^b f(t)^2\,dt\int_a^b g(t)^2\,dt.$$

(iv) When do we have equality in the inequality proved in (iii).

Q 19.12.* Explain why

$$\frac{1}{n+1}\leq\int_n^{n+1}\frac{1}{x}\,dx\leq\frac{1}{n}.$$

Hence or otherwise show that if we write

$$T_n=\sum_{r=1}^{n}\frac{1}{r}-\log n$$

we have $T_{n+1}\leq T_n$ for all $n\geq 1$. Show also that $1\geq T_n\geq 0$. Deduce that T_n tends to a limit γ (Euler's constant) with $1\geq\gamma\geq 0$. [It is an indication of how little we know about specific real numbers that, after three centuries, we still do not know whether γ is irrational. G. H. Hardy is said to have offered his chair to anyone who could prove that γ was transcendental.]

(ii) By considering $T_{2n}-T_n$, show that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\cdots=\log 2$$

(iii) By considering $T_{4n} - \frac{1}{2}T_{2n} - \frac{1}{2}T_n$, show that

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \cdots = \frac{3}{2}\log 2$$

The famous example is due to Dirichlet. It gives a specific example where rearranging a non-absolutely convergent sum changes its value.

Q 19.13. [Simple versions of Stirling's formula] (The first part of this question is in the probability course.)

(i) Prove that

$$\int_1^n \log x\,dx \leq \sum_{r=1}^{n} \log r \leq \int_1^n \log x\,dx + \log n.$$

Compute $\int_1^n \log x\,dx$ and hence show that

$$\frac{1}{n}(n!)^{1/n} \to \frac{1}{e}$$

as $n \to \infty$.

(ii)* Show that

$$\int_{n-1/2}^{n+1/2} \log x\,dx - \log n = \int_0^{1/2} \log\left(1 + \frac{t}{n}\right) + \log\left(1 - \frac{t}{n}\right)\,dt.$$

By using the mean value theorem, or otherwise, deduce that

$$\left|\int_{n-1/2}^{n+1/2} \log x\,dx - \log n\right| \leq \frac{4}{3n^2}.$$

(You may replace $4/(3n^2)$ by An^{-2} with A another constant, if you wish.) Deduce that $\int_{1/2}^{N+1/2} \log x\,dx - \log N!$ converges to a limit. Conclude that

$$\frac{n!}{(n+1/2)^{(n+1/2)}}e^n \to C$$

as $n \to \infty$ for some constant C.

(iii)* Find a function $f(n)$ not involving factorials such that

$$f(n)\binom{2n}{n}$$

tends to a limit as $n \to \infty$. You should try for an f in as simple a form as possible. (But, of course, different people may have different views on what simple means.)

(iv)* Show that the result of (ii) implies the result of (i). Give an example of a sequence a_n such that $a_n^{1/n}$ converges but a_n does not.

Q 19.14. (i) Show that $\sum_{n=27}^{\infty} \frac{1}{n \log n \log\log n}$ diverges. Find a value of N such that $\sum_{n=27}^{N} \frac{1}{n \log n \log\log n} \geq 3$. Try and find its numerical value on your calculator.

(ii)* Given $a_n > 0$ such that $\sum_{n=1}^{\infty} a_n$ diverges show that we can find $b_n > 0$ such that $b_n/a_n \to 0$ but $\sum_{n=1}^{\infty} b_n$ diverges.

Given $a_n > 0$ such that $\sum_{n=1}^{\infty} a_n$ converges show that we can find $b_n > 0$ such that $b_n/a_n \to \infty$ but $\sum_{n=1}^{\infty} b_n$ converges.

[These two results show that it is futile to look for some sort of 'super-charged ratio test' to decide the convergence of all possible series.]

Q 19.15. (i) By writing

$$r(r+1)\dots(r+m-1) = A_m\big(r(r+1)\dots(r+m) - (r-1)r\dots(r+m-1)\big),$$

where A_m is to found explicitly, compute $\sum_{r=1}^{N} r(r+1)\dots(r+m-1)$. Deduce that

$$N^{-m-1}\sum_{r=1}^{N} r(r+1)\dots(r+m-1) \to \frac{1}{m+1}.$$

(ii) Show that

$$r(r+1)\dots(r+m-1) - r^m = P(r)$$

where P is a polynomial of degree less than m. Show using (i) and induction, or otherwise, that

$$N^{-m-1}\sum_{r=1}^{N} r^m \to \frac{1}{m+1}.$$

(iii) Use dissections of the form

$$\mathcal{D} = \{0,\ a/n,\ 2a/n,\ \dots,\ a\}$$

to compute

$$\int_0^a x^m\,dx$$

directly from the definition.

(iv) Use dissections of the form

$$\mathcal{D} = \{br^n,\ br^{n-1},\ br^{n-2},\ \dots,\ b\}$$

with $0 < r$ and $r^n = a/b$ to compute

$$\int_a^b x^m\,dx$$

directly from the definition.

Q 19.16.* If A_n and G_n are the arithmetic and geometric means of the n positive integers

$$n+1,\ n+2, \dots,\ n+n$$

show that, as $n \to \infty$,

$$\frac{A_n}{n} \to \frac{3}{2} \ \text{ and } \ \frac{G_n}{n} \to \frac{4}{e}.$$

Deduce that $e \geq 8/3$.

Q 19.17.* (i) Let

$$v_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\, dx.$$

By writing both integrals as integrals from 0 to π, or otherwise, show that $|v_n| \geq |v_{n+1}|$. By using a theorem on the convergence of sums (to be stated) show that the sequence

$$\int_0^{n\pi} \frac{\sin x}{x}\, dx \to L$$

as $n \to \infty$ *through integer values of* n where L is a strictly positive real number.

(ii) Deduce carefully that

$$\int_0^{X} \frac{\sin x}{x}\, dx \to L$$

as $X \to \infty$ *through real values of* X. Thus $\int_0^\infty \frac{\sin x}{x}\, dx$ exists with value L.

(iii)* Let

$$G(\lambda) = \int_0^\infty \frac{\sin \lambda x}{x}\, dx.$$

Show carefully (we have not actually proved a change of variables theorem for infinite integrals) that $G(\lambda)$ exists for all real λ and

$$G(\lambda) = \begin{cases} L & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \\ -L & \text{if } \lambda < 0 \end{cases}$$

[Note that G is not continuous at 0. This is an indication of the unintuitive behaviour which infinite integrals can exhibit.]

Q 19.18.* (i) Suppose f_1, $f_2 : [a, b] \to \mathbb{R}$ are increasing and $g = f_1 - f_2$. Show that there exists a K such that, whenever

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$$

we have

$$\sum_{j=1}^{n} |g(x_j) - g(x_{j-1})| \leq K.$$

(ii) Let $g : [-1, 1] \to \mathbb{R}$ be given by $g(x) = x^2 \sin x^{-4}$ for $x \neq 0$, $g(0) = 0$. Show that g is once differentiable everywhere but that g is not the difference of two increasing functions.

Final Note To Supervisors

Let me reiterate my request for corrections and improvements *particularly to the exercises.* The easiest path for me is e-mail. My e-mail address is `twk@dpmms`.