RIEMANN SURFACES AND DISCRETE GROUPS TKC Lent 2007

1. Let $f : \Omega \to \mathbb{C}$ be a k-times continuously differentiable function on a domain $\Omega \subset \mathbb{C}$. Show that, for $z_o \in \Omega$, there are complex numbers $(a_{r,s})$ with
$$f(z_o + w) = \sum_{r+s \leqslant k} a_{r,s} w^r \overline{w}^s \ + \ o(|w|^k)$$
as $w \to 0$. Find the corresponding formulae for $\partial f/\partial z$ and $\partial f/\partial \overline{z}$. Show that when $\partial f/\partial \overline{z} = 0$ on Ω then $a_{r,s} = 0$ for $s > 0$ (so "f is a function of w alone").
2. Let $f : \Omega \to \Omega'; z \mapsto f(z) = w$ and $g : \Omega' \to \mathbb{C}$ be smooth functions. Prove the chain rule:
$$\frac{\partial (gf)}{\partial z} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{w}}\overline{\left(\frac{\partial f}{\partial \overline{z}}\right)}.$$
3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Show that the partial sums converge locally uniformly to f on $\{z \in \mathbb{C} : |z| < R\}$ but need not converge uniformly.
4. Let Ω be a domain in $\mathbb{C}$. For a compact set $K \subset \Omega$ and an open set $U \subset \mathbb{C}$ set
$$M(K, U) = \{f \in \mathcal{O}(\Omega) : f(K) \subset U\}.$$
These sets form a sub-basis for a topology on $\mathcal{O}(\Omega)$ called the *compact-open* topology. Show that this co-incides with the topology of locally uniform convergence.
5. Every harmonic function $u : A \to \mathbb{R}$ on the annulus $A = \{z \in \mathbb{C} : r < |z| < R\} \quad 0 \leqslant r < R \leqslant \infty)$ can be expressed as
$$u(z) = b \log |z| \ + \ \mathrm{Re}\, a(z)$$
for some $b \in \mathbb{R}$ and some analytic function $a : A \to \mathbb{C}$.
6. Every harmonic function on a domain Ω is the real part of an analytic function if, and only if, Ω is simply connected.
7. Use Cauchy's representation theorem (Cauchy's Integral Formula) to prove that
$$u(0) = \int_0^{2\pi} u(e^{i\theta}) \, \frac{d\theta}{2\pi}$$
for each function $u \in \mathcal{H}(\mathbb{D})$. Let T be the Möbius transformation $z \mapsto (z + z_o)/(1 + \overline{z_o} z)$. Show that $z \mapsto u(Tz)$ is in $\mathcal{H}(\mathbb{D})$ and hence deduce the Poisson integral formula.
8. Use the residue theorem (or Cauchy's representation formula) to prove that a function f analytic on a domain containing $\overline{\mathbb{D}}$ satisfies
$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \mathrm{Re}\, f(w) \, \frac{w + z}{w(w - z)} \, dw \ + \ i \mathrm{Im}\, f(0)$$
for $z \in \mathbb{D}$. (This is due to Schwarz, 1870.) Deduce the Poisson integral formula.
9. A continuous function $u : \Omega \to \mathbb{R}$ on a domain $\Omega \subset \mathbb{C}$ has the *mean value property* if, for each $z \in \Omega$ there exists $r(z) > 0$ with $\{w : |w - z| < r(z)\} \subset \Omega$ and
$$u(z) = \int_0^{2\pi} u(z + re^{i\theta}) \, \frac{d\theta}{2\pi}$$
for $0 < r < r(z)$. Prove that if such a function has a local maximum at $z \in \Omega$ then it is constant on a neighbourhood of z. Prove that u has the mean value property if, and only if, u is harmonic.
10. For $z \in \mathbb{D}$ find
$$\sup \big(u(z) \ : \ u : \mathbb{D} \to \mathbb{R}^+ \text{ is harmonic, } u(0) = 1\big) .$$
[Try $u \in \mathcal{H}(\mathbb{D})$ first.] Which functions attain the supremum? For $z_1, z_2 \in \mathbb{D}$ find
$$\sup \big(u(z_2) \ : \ u : \mathbb{D} \to \mathbb{R}^+ \text{ is harmonic, } u(z_1) = 1\big)$$
and
$$\inf \big(u(z_2) \ : \ u : \mathbb{D} \to \mathbb{R}^+ \text{ is harmonic, } u(z_1) = 1\big) .$$

11. Show that Harnack's theorem fails if we do not demand that the sequence of harmonic functions is increasing. That is, find a sequence of harmonic functions which converge at each point of a domain to a limit function which is not harmonic.
12. Let p be a polynomial in one complex variable which has no repeated zeros. Show that
$$\{(w,z) : w^2 = p(z)\}$$
is a (connected) Riemann surface. What happens if p does have repeated zeros?
13. Show that
$$R = \{(w,z) \in \mathbb{C}^2 : w^2 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)\}$$
is a Riemann surface provided that the four complex numbers are distinct. Prove that it may be made into a compact Riemann surface by adjoining two points. Prove that this compact surface is homeomorphic to a torus (i.e. $S^1 \times S^1$).
14. Prove that $\pi_1(R, z_0)$ is a group. Show that $\pi_1(R,z)$ is isomorphic to $\pi_1(R,z_o)$ for any $z \in R$. (The isomorphism is not natural.) Calculate $\pi_1(R,z_o)$ for the following Riemann surfaces: (a) $\mathbb{D}$, (b) an annulus, (c) a torus, (d) $\mathbb{C} \setminus \{0,1\}$.
15. Let $\psi : (M, w_o) \to (R, z_o)$ be a regular covering of R and $\pi : (\hat{R}, \hat{z}_o) \to (R, z_o)$ a universal covering. Then there is a covering $f : (\hat{R}, \hat{z}_o) \to (M, w_o)$. Prove that the following two conditions are equivalent.
   (a) If $T \in \operatorname{Aut} \pi$ then there is an unique $S \in \operatorname{Aut} \psi$ with $Sf = fT$.
   (b) $\operatorname{Aut} f = \{T \in \operatorname{Aut} \pi : fT = f\}$ is a normal subgroup of $\operatorname{Aut} \pi$ and the quotient $\operatorname{Aut} \pi / \operatorname{Aut} f$ is isomorphic to $\operatorname{Aut} \psi$.
16. Show that $\mathbb{P}, \mathbb{C}$ and $\mathbb{D}$ are all simply connected and that no two of them are conformally equivalent.
17. Exhibit explicitly a universal covering $\pi : \mathbb{D} \to \{z \in \mathbb{C} : r < |z| < 1\}$ for each $0 \leqslant r < 1$. Identify the group $\operatorname{Aut} \pi$. [Hint: exp.]
18. Exhibit explicitly a universal covering $\pi : \mathbb{C} \to \{z \in \mathbb{C} : 0 < |z| < \infty\}$. Identify the group $\operatorname{Aut} \pi$.
19. Prove that the Study metric is indeed a metric.
20. Show that for $T \in \mathrm{GL}(2, \mathbb{C})$ the map $[\mathbf{z}] \mapsto [T\mathbf{z}]$ is a continuous map from $\mathbb{P}(\mathbb{C}^2)$ to itself. When is it an isometry?
21. If $\mathbf{u}, \mathbf{v}$ is an orthogonal basis for $\mathbb{C}^2$ prove that the map
$$\theta : \mathbb{P}(\mathbb{C}^2) \setminus [\mathbf{u}] \; ; \; [\mathbf{z}] \mapsto \frac{\langle \mathbf{u}, \mathbf{z} \rangle}{\langle \mathbf{v}, \mathbf{z} \rangle}$$
is a chart for the Riemann surface $\mathbb{P}(\mathbb{C}^2)$. What are the transition maps for two such charts?
22. [This assumes a little knowledge of algebraic geometry.] Let $\mathbf{z} \in \mathbb{C}^N$ be a row vector. Then $\mathbf{z}^*\mathbf{z} = \overline{\mathbf{z}}^t\mathbf{z}$ is in the <u>real</u> vector space $\mathrm{Her}(N)$ of Hermitian matrices. What is the dimension of the real projective space $\mathbb{P}(\mathrm{Her}(N))$? Show that
$$J : \mathbb{P}(\mathbb{C}^N) \to \mathbb{P}(\mathrm{Her}(N)) \; ; \; [\mathbf{z}] \mapsto [\mathbf{z}^*\mathbf{z}]$$
is a well defined, injective map and that its image is a projective variety (i.e. the set where a collection of homogeneous polynomials vanish). When $N = 2$, the image is a conic in $\mathbb{P}(\mathbb{R}^4)$ isomorphic to the sphere. [Thus J generalizes the identification of $\mathbb{P}(\mathbb{C}^2)$ with S^2.]
23. A *divisor* on a <u>compact</u> Riemann surface is a function $d : R \to \mathbb{Z}$ which is zero except at a finite set of points. These form a commutative group $\mathcal{D}$. The map
$$\delta : \mathcal{D} \to \mathbb{Z} \qquad ; \qquad d \mapsto \sum (d(z) : z \in R)$$
is a homomorphism. Let $\mathcal{D}_0$ be its kernel.
   (a) Let f be a meromorphic function on R which is not identically zero, so $f \in \mathcal{M}(R)^\times$. Then f has finitely many zeros and poles. Let (f) be the divisor which is $\deg f(z)$ at any zero z, $-\deg f(z)$ at any pole z, and zero elsewhere. Show that this gives a homomorphism of commutative groups
$$\mathcal{M}(R)^\times \to \mathcal{D}_0 \qquad ; \qquad f \mapsto (f).$$
   Find the kernel of this homomorphism. The quotient $\mathcal{D}_0 / \{(f) : f \in \mathcal{M}(R)^\times\}$ is called the *divisor class group* of R.
   (b) Show that the divisor class group of $\mathbb{P}$ is trivial.

Let $T : z \mapsto (az+b)/(cz+d)$ be a Möbius transformation.

24. Consider the chordal metric on $\mathbb{P}$ and show that T multiplies the length of an infinitesimally short curve at z by the factor
$$\frac{|T'(z)|(1+|z|^2)}{1+|T(z)|^2} = \frac{|ad-bc|(1+|z|^2)}{|az+b|^2+|cz+d|^2}.$$
Show that the maximum and minimum values of this quantity are
$$s+\sqrt{s^2-1} \qquad \text{and} \qquad s-\sqrt{s^2-1}$$
where
$$s = \frac{|a|^2+|b|^2+|c|^2+|d|^2}{2|ad-bc|}.$$
[Hint: Think about $\mathbb{P}$ as $\mathbf{P}(\mathbb{C}^2)$.]

25. Let $Z(T) = \{S \in \text{Möb} : ST = TS\}$.
    (a) Show that $Z(T)$ is a subgroup of Möb.
    (b) Find which groups (up to isomorphism) can arise as $Z(T)$ for some Möbius transformation T .

26. Let A be a 2×2 complex matrix with trace equal to 0. Show that the series
$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$
converges and prove the following properties.
    (a) If $AB = BA$ then $\exp(A+B) = \exp A \exp B$.
    (b) $\{\exp tA : t \in \mathbb{R}\}$ is a commutative group under multiplication of matrices.
    (c) The function $f(t) = \det \exp tA$ satisfies $f'(t) = f(t)\,\mathrm{tr}\,A = 0$. Hence $\exp tA \in SL(2,\mathbb{C})$.

    Let $\exp tA$ now denote the Möbius transformation determined by the matrix $\exp tA$. Show that every Möbius transformation is equal to $\exp A$ for some matrix A . Is the choice of A unique? For $z \in \mathbb{P}$ the images of z under the Möbius transformations $\exp tA$ for $t \in \mathbb{R}$ trace out a curve. Which curves can arise in this way? Sketch examples. (The groups $\{\exp tA : t \in \mathbb{R}\}$ for some A are the 1-parameter subgroups of the Lie group Möb.)