8 8 THE RIEMANN MAPPING THEOREM

8.1 Simply Connected Surfaces

Our aim is to prove the Riemann Mapping Theorem which states that every simply connected Riemann surface R is conformally equivalent to $\mathbb{D}, \mathbb{C}$ or $\mathbb{P}$. If R is hyperbolic it must be conformally equivalent to $\mathbb{D}$. Similarly, if R is parabolic or elliptic it must be conformally equivalent to $\mathbb{C}$ or $\mathbb{P}$ respectively.

Theorem 8.1.1

Let R be a simply connected Riemann surface and $z_o \in R$. If $g : R \setminus \{z_o\} \to \mathbb{R}$ is harmonic and has a logarithmic singularity at z_o with coefficient -1, then there is a holomorphic function $G : R \to \mathbb{C}$ with $g(z) = -\log |G(z)|$ for $z \in R$.

Proof:

Let $\phi_\alpha : U_\alpha \to \phi_\alpha U_\alpha$ be an atlas of charts for R with each U_α simply connected. If $z_o \notin U_\alpha$ then there is a holomorphic function $F_\alpha : U_\alpha \to \mathbb{C}$ with $g = \Re F_\alpha$. So we can set $G_\alpha = \exp -F_\alpha : U_\alpha \to \mathbb{C}$. If $z_o \in U_\alpha$ then $z \mapsto g(z) + \log |\phi_\alpha(z) - \phi_\alpha(z_o)|$ is harmonic on U_α so it is the real part of a holomorphic function F_α. Set $G_\alpha(z) = (\phi_\alpha(z) - \phi_\alpha(z_o)) \exp F_\alpha(z)$. The functions $G_\alpha : U_\alpha \to \mathbb{C}$ satisfy $g(z) = -\log |G_\alpha(z)|$ for $z \in U_\alpha$.

Note that the functions G_α are only determined up to a scalar multiple of modulus 1. Suppose that $U_\alpha \cap U_\beta \neq \emptyset$. Then $|G_\alpha(z)| = |G_\beta(z)|$ on the intersection. Hence, G_α/G_β is equal to a constant ω of modulus 1 on a component of $U_\alpha \cap U_\beta$ and therefore ωG_β is a holomorphic continuation of G_α. It follows that we can continue G_α analytically along any path in R. Since R is simply connected, the Monodromy theorem shows that there is a single holomorphic function $G : R \to \mathbb{C}$ with $g(z) = -\log |G(z)|$. □

8.2 Hyperbolic Surfaces

Theorem 8.2.1 The Riemann Mapping Theorem for Hyperbolic Surfaces

A simply connected, hyperbolic Riemann surface R is conformally equivalent to the unit disc.

Proof:

Let g_{z_o} be the Green's function for R with pole at z_o. Then Proposition 8.1.1 shows that there is a holomorphic function $G_{z_o} : R \to \mathbb{D}$ with $g_{z_o}(w) = -\log|G_{z_o}(w)|$. Choose one particular value for z_o and let $G = G_{z_o}$. We will prove that G is conformal by proving a series of intermediate results **A** $-$ **F**.

First observe that the minimality property of the Green's function implies a similar property for G.

A If $F : R \to \mathbb{D}$ is any holomorphic function with a zero at z_o, then $|F(z)| \leqslant |G(z)|$ for each $z \in R$.

For, if we set $f(z) = -\log|F(z)|$, then f is positive and harmonic on $R \setminus F^{-1}(0)$ and has a logarithmic singularity at each zero z of F with coefficient $\deg F(z)$. If $u \in \mathcal{F}(z_o)$ then $v = u - f$ is subharmonic on $R \setminus F^{-1}(0)$. Also v is negative outside the support of u and tends to $-\infty$ at each point of $F^{-1}(0)$ except z_o. At z_o we also have v tending to $-\infty$ unless $\deg F(z_o) = 1$ in which case v is subharmonic on a neighbourhood of z_o. The maximum principle for subharmonic functions (Proposition 8.1.3) implies that $v \leqslant 0$ and so $u \leqslant f$. Consequently $g_{z_o} \leqslant f$ and $|G| \geqslant |F|$, as required.

We can also say exactly when we get equality:

B $|G(z)| = |F(z)|$ at some $z \in R \setminus \{z_o\}$ if, and only if, $F = \omega G$ for some complex number ω of unit modulus.

For the function $a : R \to \mathbb{C},\ w \mapsto F(w)/G(w)$ is holomorphic but achieves its maximum modulus at the point z. Hence a is a constant with modulus 1.

Now we can adapt this to holomorphic functions which do not have zeros at z_o:

C For any holomorphic function $F : R \to \mathbb{D}$ we have

$$\left| \frac{F(z) - F(z_o)}{1 - \overline{F(z_o)}F(z)} \right| \leqslant |G(z)| \ .$$

If equality holds at one point $z \in R\setminus\{z_o\}$, then $F = TG$ for some Möbius transformation $T \in \mathrm{Aut}\,\mathbb{D}$ and equality holds everywhere.

For we can apply the previous results to $z \mapsto (F(z) - F(z_o))/(1 - \overline{F(z_o)}F(z))$.

D For each $w \in R$ we have $G_w = TG$ for some $T \in \mathrm{Aut}\,\mathbb{D}$.

We apply **C** to the holomorphic function G_w to obtain:

$$\left| \frac{G_w(z) - G_w(z_o)}{1 - \overline{G_w(z_o)}G_w(z)} \right| \leqslant |G(z)|.$$

For $z = w$ we obtain, $|G_w(z_o)| \leqslant |G_{z_o}(w)|$. Interchanging the rôles of w and z_o we see that $|G_w(z_o)| = |G_{z_o}(w)|$ so equality holds for this particular value of z. Hence $G_w = TG_{z_o}$ for some $T \in \mathrm{Aut}\,\mathbb{D}$.

E G is injective.

Suppose that $G(w) = G(w')$. Then, by **D**, we also have $0 = G_w(w) = G_w(w')$. The only zero of G_w is at w so $w = w'$.

F G is surjective.

Suppose that $a \in \mathbb{D} \setminus \operatorname{Im} G$. Then $z \mapsto (G(z) - a)/(1 - \overline{a}G(z))$ maps R into $\mathbb{D} \setminus \{0\}$. Since R is simply connected this map has a holomorphic square root $\sigma : R \to \mathbb{D} \setminus \{0\}$. Thus

$$G(z) = \frac{\sigma(z)^2 - \sigma(z_o)^2}{1 - \overline{\sigma(z_o)}^2 \sigma(z)^2} = \left(\frac{\sigma(z) - \sigma(z_o)}{1 - \overline{\sigma(z_o)}\sigma(z)} \right) \left(\frac{\sigma(z) + \sigma(z_o)}{1 + \overline{\sigma(z_o)}\sigma(z)} \right) .$$

We know from **C** that

$$\left| \frac{\sigma(z) - \sigma(z_o)}{1 - \overline{\sigma(z_o)}\sigma(z)} \right| \leqslant |G(z)|.$$

So, we see that

$$1 \leqslant \left| \frac{\sigma(z) + \sigma(z_o)}{1 + \overline{\sigma(z_o)}\sigma(z)} \right| .$$

However, the map

$$w \mapsto \frac{w + \sigma(z_o)}{1 + \overline{\sigma(z_o)}w}$$

maps the unit disc into itself, so the last inequality is impossible.

We have now shown that $G : R \to \mathbb{D}$ is injective and surjective, so it is bijective and gives a conformal map of R onto the unit disc. □

Corollary 8.2.2

Every Riemann surface has a countable dense subset.

Proof:

Let Δ be a compact disc in the Riemann surface R. Then Corollary 8.3.3 shows that $R \setminus \Delta$ is hyperbolic. Let $\pi : S \to R \setminus \Delta$ be its universal cover. If f is a positive superharmonic function on $R \setminus \Delta$ then $f\pi$ is a positive superharmonic function on S. Thus S is a simply connected hyperbolic Riemann surface. By the theorem, S is conformally equivalent to $\mathbb{D}$, so S has a countable dense set. Consequently $\pi(S) = R \setminus \Delta$ has a countable dense set and so does R. □

It follows immediately from this that every Riemann surface has a *compact exhaustion*, that is an increasing sequence of compact subsets whose union is the entire surface. Indeed, with a little care we can ensure that each of the compact sets has a real analytic boundary.

Exercises

1. Let K be a compact subset of the non-compact Riemann surface R. Prove that we can cover K by finitely many discs so that the union of the discs is a domain Ω which is regular for the Dirichlet problem. Let g be a Green's function for Ω. Show that, for suitable small $\varepsilon > 0$, the set $\{z \in \Omega : g(z) > \varepsilon\}$ contains K and has a real analytic boundary.

   Hence every Riemann surface has a compact exhaustion by sets with real analytic boundaries.

Recall that, if $\pi : \mathbb{D} \to R$ is a universal covering of a Riemann surface R, then R is conformally equivalent to the quotient of $\mathbb{D}$ by the subgroup $\mathrm{Aut}(\pi)$ of $\mathrm{Aut}(\mathbb{D})$. This subgroup is a discrete subgroup of $\mathrm{Aut}(\mathbb{D})$ or, equivalently, it acts discontinuously on $\mathbb{D}$. We call such a subgroup a *Fuchsian group*. When G is a Fuchsian group, the quotient $\mathbb{D}/G$ is a Riemann surface. The following theorem shows when the quotient $\mathbb{D}/G$ is hyperbolic. Note that it can also be elliptic or parabolic.

Let R be a hyperbolic Riemann surface. Then there is a non-constant, positive, continuous, superharmonic map $s : R \to \mathbb{R}^+$. There is a universal covering $\pi : \widehat{R} \to R$ and $s\pi$ is a non-constant, positive, continuous, superharmonic map on $\widehat{R}$. So $\widehat{R}$ is hyperbolic and hence conformally equivalent to $\mathbb{D}$. Thus every hyperbolic Riemann surface is conformally equivalent to a quotient of $\mathbb{D}$ by a Fuchsian group.

Theorem 8.2.3

The quotient $\mathbb{D}/G$ by a Fuchsian group G is hyperbolic if, and only if, $\sum(\exp -\rho(0, T(0)) : T \in G)$ converges. If $q : \mathbb{D} \to \mathbb{D}/G$ is the quotient map then the Green's function g_{qw} for $\mathbb{D}/G$ with pole at qw satisfies

$$g_{qw}(qz) = -\sum_{T\in G} \log \left| \frac{z - Tw}{1 - \overline{Tw}z} \right| .$$

Proof:

First note that, as $|z| \to 1$ we have:

$$|z| = \tanh \tfrac{1}{2}\rho(0, z) = \frac{1 - e^{-\rho(0,z)}}{1 + e^{-\rho(0,z)}} \sim 1 - 2e^{-\rho(0,z)} .$$

Therefore,

$$1 - |z| \sim -\log |z| \sim 2e^{-\rho(0,z)} .$$

This shows that the three series $\sum 1 - |T(0)|$, $\sum -\log |T(0)|$ and $\sum \exp -\rho(0, T(0))$ all converge or diverge together. The triangle inequality shows that

$$e^{-\rho(0,w)} \left(\sum e^{-\rho(0,z)}\right) \leqslant \sum e^{-\rho(w,z)} \leqslant e^{\rho(0,w)} \left(\sum e^{-\rho(0,z)}\right)$$

so we deduce that if the series $\sum e^{-\rho(w,T(0))}$ converges for one value of $w \in \mathbb{D}$, then it converges for all $w \in \mathbb{D}$ locally uniformly.

Suppose that $\sum(\exp -\rho(0, T(0)) : T \in G)$ converges. Then $\sum \exp -\rho(w, T(0))$ converges and hence

$$h(w) = \sum -\log \left| \frac{w - T(0)}{1 - \overline{T(0)}w} \right|$$

converges locally uniformly on $\mathbb{D}$ by comparison with it. This gives a harmonic function with logarithmic singularities, coefficient -1, at each point of $G(0)$. Furthermore, $h(T(w))$ is simply a re-arrangement of the series for $h(w)$, so $h(Tw) = h(w)$ for $w \in \mathbb{D}$. Hence, there is a function $\widetilde{h} : \mathbb{D}/G \to \mathbb{R}$ which is positive harmonic with a logarithmic singularity at $q(0)$. Consequently, $\mathbb{D}/G$ is hyperbolic and its Green's function g with a pole at $q(0)$ satisfies $g \leqslant \widetilde{h}$.

For the converse, we will adapt the Poisson – Jensen formula (Theorem 5.2.1) to harmonic functions.

Lemma 8.2.4 Poisson-Jensen formula

Let S be a finite subset of $\mathbb{D}$ and $u : \overline{\mathbb{D}} \setminus S \to \mathbb{R}$ a continuous function which is harmonic on $\mathbb{D} \setminus S$ with logarithmic singularities at each $s \in S$ with coefficient $c(s)$. Then

$$u(z) = \sum_{s\in S} c(s) \log \left| \frac{z - s}{1 - \overline{s}z} \right| + \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} u(e^{i\theta})\, d\theta$$

for each $z \in \mathbb{D} \setminus S$.

Proof:

For the function
$$v(z) = u(z) - \sum c(s) \log \left| \frac{z-s}{1-\overline{s}z} \right|$$
is continuous on $\overline{\mathbb{D}}$ and harmonic on all of $\mathbb{D}$. Its boundary values are $v(e^{i\theta}) = u(e^{i\theta})$. We apply Poisson's formula to v to obtain:
$$v(z) = \int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2} u(e^{i\theta})\, d\theta$$
which is the required formula. □

Suppose that g is the Green's function with pole at $q(0)$ for $\mathbb{D}/G$. Then $u : \mathbb{D} \to \mathbb{R},\ w \mapsto g(q(w))$ is positive harmonic on $\mathbb{D}$ with logarithmic singularities, coefficient -1, at each point of $G(0)$. For each $r < 1$ (with $r \neq |T(0)|$) we can apply the lemma to the function $z \mapsto u(rz)$ to obtain:
$$u(z) = \sum_{|T(0)|<r} -\log \left| \frac{z - r^{-1}T(0)}{1 - r^{-1}\overline{T(0)}z} \right| + \int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2} u(re^{i\theta})\, d\theta \ .$$
Since $u \geqslant 0$ this implies that
$$u(z) \geqslant \sum_{|T(0)|<r} -\log \left| \frac{z - r^{-1}T(0)}{1 - r^{-1}\overline{T(0)}z} \right|$$
and, letting r increase to 1, we obtain
$$u(z) \geqslant \sum_{T\in G} -\log \left| \frac{z - T(0)}{1 - \overline{T(0)}z} \right| = h(z) \ .$$
Since u is finite on $\mathbb{D} \setminus G(0)$, the series on the right must converge. In particular,
$$\sum_{T\in G\setminus\{I\}} \log|T(0)|$$
converges. Therefore, $\sum \exp -\rho(0, T(0))$ converges.

Furthermore, we have shown in the first part of the proof that $g(q(z)) \geqslant h(z)$ and in the second part that $g(q(z)) \leqslant h(z)$. Therefore $g(q(z)) = h(z)$.

For any $w \in \mathbb{D}$ the map $S : z \mapsto (z+w)/(1+\overline{w}z)$ is in $\mathrm{Aut}\,\mathbb{D}$. So the map
$$\widetilde{q} : \mathbb{D} \to \mathbb{D}/G \quad ; \quad z \mapsto q(Sz)$$
is another universal covering with $\widetilde{q}0 = qw$ and $\mathrm{Aut}\,\widetilde{q} = \{S^{-1}TS : T \in G\}$. Since $g_{qw} = g_{\widetilde{q}0}$, we obtain the desired result. □

Note that we can write the formula for the Green's function in the invariant form:
$$g_{qw}(qz) = -\sum_{T\in G} \log\tanh \tfrac{1}{2}\rho(z, Tw).$$

Corollary 8.2.5

If R is a hyperbolic Riemann surface then the Green's function is symmetric:
$$g_w(z) = g_z(w) \qquad \text{for } w, z \in R.$$

This is usually proved by using Stokes' theorem. That approach has the advantage of applying to the Green's function on an arbitrary manifold. The symmetry of the Green's function reflects the fact that the Laplacian is self-adjoint. Because of this symmetry we often write $g_w(z)$ as $g(w,z)$. Then g is a smooth function on the set where $w \neq z$.

Proof:

Let $q : \mathbb{D} \to \mathbb{D}/G \cong R$ be the quotient map. Then each $T \in G$ is an isometry for the hyperbolic metric so

$$g_{qw}(qz) = -\sum_{T\in G} \log\tanh \tfrac{1}{2}\rho(z, Tw) = -\sum_{T\in G} \log\tanh \tfrac{1}{2}\rho(T^{-1}z, w) = g_{qz}(qw).$$

$\square$

8.3 Parabolic Surfaces

Let R be a non-compact Riemann surface and Δ a disc in R. Corollary 8.2.2 implies that there is an increasing sequence (Ω_n) of relatively compact domains in R with $\overline{\Delta} \subset \Omega_0$, $\bigcup \Omega_n = R$ and each Ω_n regular for the Dirichlet problem. By Corollary 7.3.3, each Ω_n has a Green's function with a pole at $w \in \Delta$ which extends continuously to $\partial\Omega_n$ and is 0 there. Let $g_n(w, \cdot)$ denote this Green's function. The maximum principle shows that $g_n(w, z)$ increases with n. If the supremum is finite, it is the Green's function for R and so R is hyperbolic. Hence, when R is parabolic, the sequence must tend to infinity. However, we will show that we can choose the domains Ω_n and constants M_n so that $g_n(w, z) - M_n$ converges to a finite harmonic function of z. This will be an analogue of the Green's function for the parabolic surface.

Proposition 8.3.1

Let $M_n = \sup\{g_n(w, z) : z \in \partial\Delta\}$ then the sequence of functions $(g_n(w, \cdot) - M_n)$ has a subsequence which converges locally uniformly on $R \setminus \{w\}$ to a function $q(w, \cdot) : R \setminus \{w\} \to \mathbb{R}$ which is harmonic with a logarithmic singularity at w having coefficient -1, and satisfies $q(w, z) \leqslant 0$ for $w \in R \setminus \Delta$.

(If R is parabolic, then q can not be bounded below.)

Proof:

Set $u_n(z) = M_n - g_n(w, z)$. Then each u_n is a positive harmonic function on $\Omega_n \setminus \overline{\Delta}$. Let K be a compact subset of $R \setminus \overline{\Delta}$ which, for convenience, we will assume contains $\partial\Omega_0$. Then $K \subset \Omega_n$ for n sufficiently large, say $n \geqslant N(K)$. Harnack's inequality gives a constant $c(K)$ with

$$u_n(z) \leqslant c(K) u_n(z') \qquad \text{for } n \geqslant N(K) \text{ and } z, z' \in K \tag{1}$$

We will show that there is a $z' \in \partial\Omega_0$ with $u_n(z') \leqslant M_0$.

Set $h_n(z) = g_n(w, z) - g_0(w, z)$. Then h_n has a removable singularity at w and gives a continuous function on $\overline{\Omega_0}$ which is harmonic on Ω_0. At some point $\zeta \in \partial\Delta$ we have $g_n(w, \zeta) = M_n$, so $h_n(\zeta) = M_n - g_0(w, \zeta) \geqslant M_n - M_0$. The maximum principle implies that there is a $z' \in \partial\Omega_0$ with $h_n(z') \geqslant M_n - M_0$. However, $h_n(z') = g_n(w, z') - g_0(w, z') = g_n(w, z')$, so

$$u_n(z') = M_n - g_n(w, z') \leqslant M_0 \ .$$

Putting this into (1) above gives:

$$u_n(z) \leqslant c(K) M_0 \qquad \text{for } n \geqslant N(K) \text{ and } z \in K \ .$$

This shows that the functions (u_n) are a normal family on $R \setminus \overline{\Delta}$. Therefore, there is a subsequence which converges locally uniformly on $R \setminus \overline{\Delta}$. By discarding some of the terms of the sequence we can suppose that

$$g_n(w, z) - M_n \to q(w, z) \quad \text{locally uniformly on } R \setminus \overline{\Delta} \ .$$

In particular, $g_n(w, z) \to q(w, z)$ uniformly on $\partial\Omega_0$. The functions $h_n(z) - M_n = g_n(w, z) - M_n - g_0(w, z)$ are continuous on $\overline{\Omega_0}$ and harmonic on Ω_0. On the boundary $\partial\Omega_0$ they equal $g_n(w, z) - M_n$ so they converge uniformly there. By the maximum principle, they converge uniformly on all of $\overline{\Omega_0}$. The limit is a harmonic function which extends $q(w, \cdot) - g_0(w, \cdot)$. Therefore,

$$g_n(w, z) - M_n \to q(w, z) \quad \text{locally uniformly on } R \setminus \{w\}$$

and $q(w, \cdot)$ has the same singularity at w as does $g_0(w, \cdot)$. We chose M_n so that $g_n(w, z) - M_n \leqslant 0$ for $z \in R \setminus \Delta$. Therefore, $q(w, z) \leqslant 0$ for $z \in R \setminus \Delta$. $\square$

We need to apply the previous result at two different point w and w' in Δ. We will write M_n' for the supremum $\sup\{g_n(w',z) : z \in \partial\Delta\}$. We can find a subsequence giving convergence for w and then a subsequence of that which gives convergence for w'. Thus, we may assume that both

$$\begin{aligned} g_n(w,z) - M_n \to q(w,z) &\quad \text{locally uniformly on } R \setminus \{w\}, \text{ and} \\ g_n(w',z) - M_n' \to q(w',z) &\quad \text{locally uniformly on } R \setminus \{w'\}\ . \end{aligned}$$

However, Corollary 8.2.4 shows that $g_n(w,w') = g_n(w',w)$. So we see that the sequence $(M_n' - M_n)$ converges to $q(w,w') - q(w',w)$. It is therefore bounded.

The functions

$$k_n(z) = (g_n(w,z) - M_n) - (g_n(w',z) - M_n')$$

converge uniformly to $q(w,z) - q(w',z)$ on $\partial\Delta$, so they are uniformly bounded there. On $\partial\Omega_n$ we have $k_n(z) = M_n' - M_n$, so they are also uniformly bounded there. Since they are harmonic we find that there is a constant S with

$$|k_n(z)| \leqslant S \qquad \text{for all } z \in \Omega_n \setminus \Delta\ .$$

Taking the limit as $n \to \infty$ we obtain:

Proposition 8.3.2

We may choose the functions $q(w,\cdot)$ in Proposition 8.3.1 so that

$$a(z) = q(w,z) - q(w',z)$$

is bounded on $R \setminus \Delta$.

□

We can now prove the Riemann Mapping Theorem for parabolic Riemann surfaces. First note that

Proposition 8.3.3

Let R be a parabolic Riemann surface. A bounded holomorphic function $f : R \to \mathbb{C}$ must be constant.

Proof:

The real part (or imaginary part) of f is a bounded harmonic function, say u. So $u - \inf u$ is a positive, bounded, (super)harmonic function on R. Since R is parabolic, this must be constant. □

Theorem 8.3.4 The Riemann Mapping Theorem for parabolic surfaces

A simply connected, parabolic Riemann surface R is conformally equivalent to the complex plane.

Proof:

Let w be a fixed point in a disc $\Delta \subset R$. Construct the function $q(w,\cdot)$ as above. Theorem 8.1.1 shows that there is a holomorphic function $Q_w : R \to \mathbb{C}$ with $q(w,z) = -\log|Q_w(z)|$. The properties of $q(w,\cdot)$ imply that:

(a) Q_w has exactly one zero, at w;

(b) $|Q_w(z)| \geqslant 1$ for $z \in R \setminus \Delta$.

For any other point $w' \in \Delta$ we can construct a similar holomorphic function $Q_{w'} : R \to \mathbb{C}$ with a single zero at w' and there is a constant S with:

(c) $e^{-S}|Q_w(z)| \leqslant |Q_{w'}(z)| \leqslant e^S|Q_w(z)|$ for $z \in R \setminus \Delta$.

We will prove that $Q_w : R \to \mathbb{C}$ is conformal.

The function $z \mapsto Q_{w'}(z) - Q_{w'}(w)$ has a zero at w so $F : z \mapsto (Q_{w'}(z) - Q_{w'}(w))/Q_w(z)$ is an holomorphic function. Properties (b) and (c) above show that

$$|F(z)| \leqslant e^S + |Q_{w'}(w)| \quad \text{for } z \in R \setminus \Delta \ .$$

Therefore, F is bounded on all of R. By Theorem 8.3.3, it is constant, c. This proves that, $Q_{w'} = cQ_w + Q_{w'}(w)$. Thus $Q_{w'}$ is $T \circ W_w$ for some Euclidean similarity T. This is true for all choices of w' within the disc Δ. By covering a path from w to w' by a finite number of such discs, we see that it holds for any $w' \in R$.

If $Q_w(z_1) = Q_w(z_2)$ then $Q_{z_1} = T \circ Q_w$ for some Euclidean similarity T. Therefore $0 = Q_{z_1}(z_1) = Q_{z_1}(z_2)$. But Q_{z_1} has only one zero, at z_1, so $z_1 = z_2$. Thus Q_w is injective.

Suppose that $a \in \mathbb{C} \setminus \mathrm{Im}\, Q_w$. Then $Q_w - a$ would have a holomorphic square root $\sigma : R \to \mathbb{C}$. Since Q_w is injective, $\sigma(z_1) \neq -\sigma(z_2)$ for $z_1, z_2 \in R$. Hence, σ cannot take values in the open set $-\sigma(\Delta)$ and so $z \mapsto 1/(\sigma(z) + \sigma(w))$ would be a bounded holomorphic function. Because R is parabolic, this must be constant and so Q_w must be constant. This is impossible, so Q_w must be surjective.

Thus $Q_w : R \to \mathbb{C}$ is holomorphic and bijective, so it is conformal. $\square$

8.4 Elliptic Surfaces

Let R be an elliptic Riemann surface and $\zeta \in R$. Then $R \setminus \{\zeta\}$ can not be hyperbolic or its Green's function would have a removable singularity at ζ. So $R \setminus \{\zeta\}$ is parabolic. As in the previous section we can choose a disc Δ in $R \setminus \{\zeta\}$ and, for each $w \in \Delta$, a harmonic function

$$q(w, \cdot) : R \setminus \{w, \zeta\} \to \mathbb{R}$$

which has a logarithmic singularity at w with coefficient -1. For $w, w' \in \Delta$ we can choose these functions so that $a(z) = q(w, z) - q(w', z)$ is bounded outside Δ. Then a has a removable singularity at ζ. Removing it we obtain a harmonic function $a : R \setminus \{w, w'\} \to \mathbb{R}$ which has logarithmic singularities at w and w' with coefficients -1 and $+1$ respectively.

Theorem 8.4.1 The Riemann Mapping Theorem for Elliptic surfaces

A simply connected, elliptic Riemann surface R is conformally equivalent to the Riemann sphere.

Proof:

By mimicing the proof of Theorem 8.1.1 we can construct a meromorphic function $F : R \to \mathbb{P}$ with $f = -\log |F|$. Then F has precisely one zero, at w, so it is of degree 1. Hence F is bijective and therefore conformal. $\square$

Exercises

2. Show that, for any distinct points z_o, w in any Riemann surface R there is a harmonic function $f : R \setminus \{z_o, w\} \to \mathbb{R}$ which has logarithmic singularities at z_o and w with coefficients $+1$ and -1 respectively.
3. Let R be an elliptic Riemann surface and z_o, ζ distinct points of R. Show that the function $q(z_o, \cdot) : R \setminus \{z_o, \zeta\} \to \mathbb{R}$ on the parabolic surface $R \setminus \{\zeta\}$ has a logarithmic singularity at ζ with coefficient -1 (and at z_o with coefficient $+1$).

8.5 Meromorphic Functions

Let R be a hyperbolic or parabolic Riemann surface. For each $z_o \in R$ there is a harmonic function

$$q_{z_o} : R \setminus \{z_o\} \to \mathbb{R}$$

which has a logarithmic singularity at z_o with coefficient -1. The derivative ∂q_{z_o} is then a meromorphic 1-form on R which has a single pole at z_o. If $z_o \neq z_1$ then the 1-forms ∂q_{z_o} and ∂q_{z_1} have their poles at different places, so $\partial q_{z_o}/\partial q_{z_1}$ is a non-constant meromorphic function on R by Proposition 2.4.2.

Similarly, if R is an elliptic Riemann surface, then we have constructed harmonic functions

$$f_{z_o z_1} : R \setminus \{z_o, z_1\} \to \mathbb{R}$$

which have logarithmic singularities at z_o and z_1 with coefficients $+1$ and -1 respectively. So $\partial f_{z_o z_1}$ is a meromorphic 1-form with exactly two poles, at z_o and z_1. The ratio $\partial f_{z_o z_1}/\partial f_{z_2 z_3}$ is a non-constant meromorphic function on R.

Theorem 8.5.1

Every Riemann surface R has a non-constant meromorphic function. □

Exercises

4. *Show that every Riemann surface is triangulable.*

Let R be any Riemann surface. Then we know that it has a universal covering $\pi : \widehat{R} \to R$ and $\widehat{R}$ is conformally equivalent to $\mathbb{P}$, $\mathbb{C}$ or $\mathbb{D}$. Thus every Riemann surface is conformally equivalent to the quotient of $\mathbb{P}$, $\mathbb{C}$ or $\mathbb{D}$ by a subgroup of their automorphism group. We know which groups can arise for $\mathbb{P}$ or $\mathbb{C}$ (Theorem 3.3.3 and 4.3.2). So:

Theorem 8.5.2 The uniformization theorem

A Riemann surface is conformally equivalent to one of the following:

$\mathbb{P}$;

$\mathbb{C}$, $\mathbb{C}/\mathbb{Z} \cong \mathbb{C} \setminus \{0\}$, $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ *with* $\Im\tau > 0$;

$\mathbb{D}/G$ *for* G *a Fuchsian group.*

□

We have shown that there are natural metrics on each of the surfaces $\mathbb{P}$, $\mathbb{C}$ and $\mathbb{D}$ and the groups which we quotient out by in the above theorem are all isometries for these metrics. Hence every Riemann surface inherits a natural metric from its universal cover.

For example, the map
$$\mathbb{R}^2_+ \to \mathbb{D} \setminus \{0\} \; ; \quad z \mapsto \exp iz$$
is the universal cover for the punctured disc $\mathbb{D} \setminus \{0\}$. So we see that the hyperbolic metric on the punctured disc is
$$ds = \frac{1}{|z| \log \frac{1}{|z|}} |dz| \; .$$

Theorem 8.5.3 Little Picard theorem

A non-constant meromorphic function $f : \mathbb{C} \to \mathbb{P}$ takes every value in $\mathbb{P}$ with at most two exceptions.

The exponential function shows that two exceptional values can occur.

Proof:

Suppose that f does not take three values w_1, w_2, w_3. Then $\mathbb{P} \setminus \{w_1, w_2, w_3\}$ is a Riemann surface. By Theorem 8.5.2 its universal cover is $\mathbb{D}$. Let $\pi : \mathbb{D} \to \mathbb{P} \setminus \{w_1, w_2, w_3\}$ be a universal covering. The monodromy theorem enables us to construct a *lift* of f, that is a holomorphic map $F : \mathbb{C} \to \mathbb{D}$ with $\pi F = f$. Liouville's theorem shows that F, and hence f, is constant. □

Theorem 8.5.4 Picard's Great Theorem

Let $f : \{z \in \mathbb{C} : |z| > R\} \to \mathbb{P}$ be an analytic function that never takes the three distinct values w_1, w_2, w_3. Then f has either a removable singularity or a pole at ∞.

Proof:

The map $\phi : \mathbb{D} \setminus \{0\} \to \{z \in \mathbb{C} : |z| > R\}$ which sends z to R/z is conformal. So we may consider $g = f \circ \phi : \mathbb{D} \setminus \{0\} \to \mathbb{P}$ instead of f and show that g has a removable singularity or a pole at 0.

Both $\mathbb{D} \setminus \{0\}$ and $\mathbb{P} \setminus \{w_1, w_2, w_3\}$ have the disc as their universal cover and so they have hyperbolic metrics. The Schwarz – Pick lemma shows that the analytic map $g : \mathbb{D} \setminus \{0\} \to \mathbb{P} \setminus \{w_1, w_2, w_3\}$ is a contraction for these hyperbolic metrics.

We found the hyperbolic metric on $\mathbb{D} \setminus \{0\}$ above. For this metric, the circle $C(r) = \{z : |z| = r\}$ has hyperbolic length $L(C(r)) = 2\pi / \log(1/r)$, which decreases to 0 as r decreases to 0.

Let Δ_j be disjoint closed neighbourhoods of the points w_j in $\mathbb{P}$. The there is a $\delta > 0$ such that any pair of points chosen from different sets Δ_j must be distance at least δ apart. For r sufficiently small, we have $L(C(r)) < \delta$, so the hyperbolic length of $g(C(r))$ is also less than δ. This means the $g(C(r))$ can not meet more than one of the sets $\Delta_1, \Delta_2, \Delta_3$.

At least one of the sets Δ_k must have $g(C_r)$ disjoint from Δ_k for arbitrarily small r. So we can choose a sequence of radii r_n tending to 0 with each $g(C(r_n))$ not meeting Δ_k. By composing g with a Möbius transformation we may make $w_k = \infty$. Then the complement of Δ_k is bounded within some disc $D(0, K)$. For each r_n, the curve $g(C(n))$ does not meet Δ_k so it lies in $D(0, K)$.

By the maximum modulus principle, g is bounded by K on the entire annulus $\{z : r_m < |z| < r_n\}$. This implies that g is bounded on all of $\{z : 0 < |z| < r_1\}$. Consequently, g has a removable singularity at 0.

It follows that g has a removable singularity or a pole at 0, and f has a removable singularity or a pole at ∞. □