## 7.1 Subharmonic Functions

Let $u : R \to \mathbb{R}$ be a continuous function on the Riemann surface $R$. A *disc* $\Delta$ in $R$ is $\phi^{-1}\mathbb{D}$ when $\phi : U \to \phi U \supset \overline{\mathbb{D}}$ is a chart for $R$. Then

$$\partial\mathbb{D} \to \mathbb{R} \quad ; \quad z \mapsto u(\phi^{-1}z)$$

is a continuous function so its Poisson integral gives a continuous function $v : \overline{\mathbb{D}} \to \mathbb{R}$ which is harmonic on $\mathbb{D}$. Define $u_\Delta : R \to \mathbb{R}$ by

$$u_\Delta(z) = \begin{cases} u(z) & \text{if } z \notin \Delta \\ v(\phi z) & \text{if } z \in \Delta \ . \end{cases}$$

Then $u_\Delta$ is continuous on $R$ and harmonic on $\Delta$.

The function $u$ is clearly harmonic if, and only if, $u_\Delta = u$ for every disc. It is difficult to construct harmonic functions directly so we introduce a weaker condition which is easier to work with. The **continuous** function $u : R \to \mathbb{R}$ is *subharmonic* if $u_\Delta \geqslant u$ for every disc $\Delta$ in $R$. Similarly, $u$ is *superharmonic* if $u_\Delta \leqslant u$ for every disc $\Delta$. We will usually prove results for subharmonic functions and tacitly assume the corresponding results for superharmonic functions. The following properties are elementary:

**Proposition 7.1.1**

*A continuous function $u : R \to \mathbb{R}$ is subharmonic if, and only if, $-u$ is superharmonic.*
*A continuous function $u : R \to \mathbb{R}$ is harmonic if, and only if, it is both subharmonic and superharmonic.*
*If $f : S \to R$ is conformal and $u : R \to \mathbb{R}$ is continuous and subharmonic then $uf : S \to \mathbb{R}$ is subharmonic.*
*Let $(u_n)$ be continuous subharmonic functions on $R$.*
- *$\lambda_1 u_1 + \lambda_2 u_2$ is subharmonic for $\lambda_1, \lambda_2 > 0$.*
- *$\max(u_1, u_2)$ is subharmonic.*
- *If $u_n \to u$ locally uniformly then $u$ is subharmonic.*

□

To obtain further examples of subharmonic functions we prove:

**Proposition 7.1.2**

*A $C^2$ function $u : R \to \mathbb{R}$ is subharmonic if, and only if, the Laplacian $\triangle u_\alpha$ of $u_\alpha = u\phi_\alpha^{-1} : \phi_\alpha U_\alpha \to \mathbb{R}$ is non-negative for every chart $\phi_\alpha : U_\alpha \to \phi_\alpha U_\alpha \subset \mathbb{C}$.*

Note that $u_\beta = u_\alpha t_{\alpha\beta}$ for the analytic transition map $t_{\alpha\beta}$, so

$$\triangle u_\beta(z) = \triangle u_\alpha(t_{\alpha\beta}(z))|t'_{\alpha\beta}(z)|^2.$$

Hence, $\triangle u_\alpha(\phi_\alpha z) \geqslant 0$ at a point $z \in U_\alpha \cap U_\beta$ if, and only if, $\triangle u_\beta(\phi_\beta z) \geqslant 0$.

*Proof:*

It is sufficient to prove this when $R$ is a domain in $\mathbb{C}$.

Suppose first that $\triangle u \geqslant \varepsilon > 0$ on $R$ and $\Delta$ is a disc in $R$. The continuous function $u - u_\Delta$ attains its supremum $s$ at some point $z_o$ in the compact set $\overline{\Delta}$. If $s > 0$ then $z_o \in \Delta$. Now $u - u_\Delta$ has a Taylor expansion

$$(u - u_\Delta)(z_o + z) = A + Bz + \overline{B}\overline{z} + Cz^2 + \overline{C}\overline{z}^2 + Dz\overline{z} + o(|z|^2).$$

If this has a local maximum at $z_o$ then we must have $B = C = 0$ and $D \leqslant 0$. Since $D = \partial^2 u(z_o)/\partial\overline{z}\partial z = \frac{1}{4}\triangle u(z_o)$ we see that $\triangle u(z_o) \leqslant 0$ which is a contradiction. Therefore, $s = 0$ and so $u \leqslant u_\Delta$.

If $\triangle u \geqslant 0$ then $u_\varepsilon(z) = u(z) + \varepsilon z\overline{z}$ satisfies $\triangle u_\varepsilon \geqslant 4\varepsilon > 0$. So $u_\varepsilon \leqslant u_{\varepsilon\ \Delta}$. As $\varepsilon \to 0$ the functions $u_\varepsilon$ converge locally uniformly to $u$, so $u \leqslant u_\Delta$ and $u$ is subharmonic.

Conversely, suppose that $u$ is subharmonic and $z_o \in R \subset \mathbb{C}$. Let $\Delta$ be the disc $\{z : |z - z_o| < r\}$. Then

$$u_\Delta(z_o) = \int_0^{2\pi} u(z_o + re^{i\theta})\, \frac{d\theta}{2\pi}.$$

If $u$ has a Taylor expansion

$$u(z_o + z) = u(z_o) + Bz + \overline{B}\overline{z} + Cz^2 + \overline{C}\overline{z}^2 + Dz\overline{z} + o(|z|^2)$$

then

$$u_\Delta(z_o) = u(z_o) + Dr^2 + o(r^2).$$

By hypothesis $u_\Delta(z_o) \geqslant u(z_o)$ so $\frac{1}{4}\triangle u(z_o) = D \geqslant 0$. □

**Proposition 7.1.3** The maximum principle for subharmonic functions

*If $u : R \to \mathbb{R}$ is a continuous subharmonic function which attains its maximum value then it is constant.*

*Proof:*

Let $S = \sup(u(z) : z \in R)$ and $A = u^{-1}(S)$. Since $u$ is continuous, $A$ is closed. We will show that $A$ is also open.

Suppose that $z_o \in A$ and let $\phi : U \to V$ be a chart with $\phi(z_o) = 0$. Then $v = u \circ \phi^{-1} : V \to \mathbb{R}$ is continuous and subharmonic with a maximum value $S$ attained at $0$. Hence, for each $r > 0$ with $\{z : |z| \leqslant r\} \subset V$, we have

$$S = v(0) \leqslant \int_0^{2\pi} v(re^{i\theta})\, \frac{d\theta}{2\pi} \leqslant \int_0^{2\pi} S\, \frac{d\theta}{2\pi} = S$$

by Poisson's formula. Thus equality must hold throughout. Since $v$ is continuous, $v(re^{i\theta}) = S$ for all $\theta$. This shows that $v$ is constant on a neighbourhood of $0$, so $u$ is constant on a neighbourhood of $z_o$.

Thus $A$ is both open and closed. Since $R$ is connected we must have $A = \emptyset$ or $A = R$. □

Note that the Proposition actually gives a little more. A function $u : R \to \mathbb{R}$ is *locally subharmonic* if it is subharmonic on a neighbourhood of each point. The proof shows that locally subharmonic functions satisfy the maximum principle. Now suppose that $\Delta$ is any disc in $R$ and consider $u - u_\Delta$. This is locally subharmonic and $0$ outside $\Delta$. So it must attain its maximum value and this must be $0$. Consequently, $u \leqslant u_\Delta$. So we have shown that every locally subharmonic function is subharmonic.

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**Exercises**

1. A continuous function $u : \Omega \to \mathbb{R}$ on a domain $\Omega \subset \mathbb{C}$ has the property S if, for each $z_o \in \Omega$, there is a $\rho > 0$ such that

$$\int_0^{2\pi} u(z_o + re^{i\theta})\, \frac{d\theta}{2\pi} \geqslant u(z_o) \quad \text{for } 0 \leqslant r < \rho.$$

   Prove that

   (a) if $u$ has property S and attains its maximum value then it is constant.

   (b) $u$ has the property S if, and only if, $u$ is subharmonic.

2. Let $u : \Omega \to \mathbb{R}$ be a continuous subharmonic function on a domain $\Omega \subset \mathbb{C}$. Show that, for $r \geqslant 0$, the function $\int_0^{2\pi} u(z_o + re^{i\theta})\, d\theta/2\pi$ is continuous and increasing.

Let $\phi : \mathbb{C} \to [0, \infty)$ be a smooth function with

$\phi(w) = \phi(|w|)$ for all $w \in \mathbb{C}$.

$\phi(w) = 0$ for $|w| > 1$.

$\int_{\mathbb{C}} \phi(w)\, du \wedge dv = \int_0^\infty \phi(r) 2\pi\, rdr = 1$ where $w = u + iv$.

For $\varepsilon > 0$ set $\phi_\varepsilon(w) = \varepsilon^2 \phi(w/\varepsilon)$, and $\Omega_\varepsilon = \{z \in \Omega : \mathbb{D}(z, \varepsilon) \subset \Omega\}$. Define $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by

$$u_\varepsilon(z) = \int_{|w| \leqslant \varepsilon} u(z + w)\phi_\varepsilon(w)\, du \wedge dv.$$

Prove that

(a)
$$\frac{\partial u_\varepsilon}{\partial z}(z) = \frac{\partial}{\partial z} \int_{\mathbb{C}} u(w)\phi_\varepsilon(w - z)\, du \wedge dv = -\int_{\mathbb{C}} u(w) \frac{\partial \phi_\varepsilon}{\partial z}(w - z)\, du \wedge dv.$$

(b) $u_\varepsilon$ is a smooth function on $\Omega_\varepsilon$.

(c) Each $u_\varepsilon$ is subharmonic and $u_\varepsilon \to u$ locally uniformly as $\varepsilon \to 0$.

Thus each continuous subharmonic function is the locally uniform limit of smooth (and hence $C^2$) subharmonic functions.

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* The most useful definition of subharmonic functions is that they are **distributions** $\theta$ on $R$ with $\triangle\theta \geqslant 0$ in the sense that
$$\langle \triangle\theta, f \rangle = \langle \theta, \triangle f \rangle \geqslant 0$$
for every *test* function $f$ which is smooth and has compact support in $\mathbb{R}$. It can then be proved that there is an upper semi-continuous function $u : R \to [-\infty, \infty)$ with $\langle \theta, f \rangle = \int_R uf$. The function $u$ need not be continuous. Hence the most common definition of subharmonic functions only requires that they are upper semi-continuous. Because we will only use them to construct harmonic functions from their limits, we never need this extra generality. *

A *Perron family* on $R$ is a family $\mathcal{F}$ of subharmonic functions on $R$ which satisfy:

(a) if $u \in \mathcal{F}$ and $\Delta$ is a disc in $R$ then $u_\Delta \in \mathcal{F}$.

(b) if $u_1, u_2 \in \mathcal{F}$ then $\max(u_1, u_2) \in \mathcal{F}$.

The purpose of the definition is:

**Theorem 7.1.4** Perron families.

*If $\mathcal{F}$ is a Perron family on $R$ then $\sup \mathcal{F}$ is either $+\infty$ on all of $R$ or else it is a harmonic function on $R$.*

*Proof:*

Let $h(z) = \sup(u(z) : u \in \mathcal{F})$ and choose $z_o \in R$. Let $\Delta$ be a disc containing $z_o$. Then there exist $a_n \in \mathcal{F}$ with $a_n(z_o) \nearrow h(z_o)$. So $u_n = \max(a_1, a_2, \ldots, a_n)$ form an increasing sequence in $\mathcal{F}$ with $u_n(z_o) \nearrow h(z_o)$. Also, $v_n = u_{n\,\Delta}$ will be an increasing sequence in $\mathcal{F}$ with $v_n(z_o) \nearrow h(z_o)$. Each $v_n$ is harmonic on $\Delta$ so Harnack's theorem shows that $v = \sup(v_n : n \in \mathbb{N})$ is either identically $+\infty$ on $\Delta$ or else it is harmonic on $\Delta$. Clearly $v \leqslant h$ with equality at $z_o$.

Suppose that $v(z) < h(z)$ at some $z \in \Delta$. Then we can find $b_n \in \mathcal{F}$ with $b_n(z) \nearrow h(z)$. Set $\tilde{a}_n = \max(a_n, b_n) \in \mathcal{F}$ and define $\tilde{u}_n, \tilde{v}_n$ and $\tilde{v}$ as above using $\tilde{a}_n$ in place of $a_n$. Then $\tilde{v}$ is harmonic on $\Delta$ with $\tilde{v} \geqslant v$ and
$$\tilde{v}(z_o) = h(z_o) = v(z_o) \quad ; \quad \tilde{v}(z) = h(z) > v(z).$$

So the function $\tilde{v} - v$ is harmonic on $\Delta$, not constant and has a minimum at $z_o$. This is impossible, so $v(z) = h(z)$ throughout $\Delta$. In particular, $h$ is harmonic on $\Delta$.

Hence the set $h^{-1}(+\infty)$ is both open and closed in $R$ so it is either $R$ or else $\emptyset$. In the first case $h$ is identically $+\infty$. In the second $h$ is finite and so is harmonic. $\square$

A *harmonic majorant* for a family $\mathcal{F}$ of functions on $R$ is a harmonic function $h : R \to \mathbb{R}$ with $h \geqslant u$ for each $u \in \mathcal{F}$.

**Corollary 7.1.5**

*If $\mathcal{F}$ is a family of continuous, subharmonic functions on $R$ which has a harmonic majorant, then there is an unique least harmonic majorant.*

*Proof*:

Let
$$\mathcal{H} = \{h : R \to \mathbb{R} : h \text{ is harmonic and } h \geqslant u \text{ for all } u \in \mathcal{F}\}$$
and
$$\overline{\mathcal{F}} = \{u : R \to \mathbb{R} : u \text{ is continuous, subharmonic and } h \geqslant u \text{ for all } h \in \mathcal{H}\}.$$
Then $\mathcal{F} \subset \overline{\mathcal{F}}$. It is apparent that $\overline{\mathcal{F}}$ is a Perron family so $k = \sup \overline{\mathcal{F}}$ is either $+\infty$ or else harmonic. It can not be $+\infty$ since $\mathcal{H}$ is not empty. So $k$ is harmonic. Then $k \in \mathcal{H} \cap \overline{\mathcal{F}}$ so $k$ must be the unique least element of $\mathcal{H}$. $\square$

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**Exercises**

3. Let $u : \mathbb{D} \to \mathbb{R}$ be continuous and subharmonic. Show that the least harmonic majorant of $u$ is given by
$$\lim_{r \to 1-} \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \frac{d\theta}{2\pi} = \sup_{r<1} \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \frac{d\theta}{2\pi}.$$

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## 7.2 Dirichlet's Problem

We will now show how to construct harmonic functions with specified properties by taking the suprema of suitable Perron families. As a first example we will try to solve Dirichlet's problem:

Let $\Omega$ be a relatively compact domain in $R$ and $b : \partial\Omega \to \mathbb{R}$ a continuous function. Find a continuous function $B : \overline{\Omega} \to \mathbb{R}$ which is harmonic on $\Omega$ and has $B|\partial\Omega = b$.

There need not be a solution to the Dirichlet problem. For example, if $\Omega = \{z : 0 < |z| < 1\} \subset \mathbb{C}$ and
$$b : \partial\Omega \to \mathbb{R} \quad ; \quad z \mapsto \begin{cases} 1 & \text{if } z = 0 \\ 0 & \text{if } |z| = 1 \end{cases}$$
then any solution $B$ would have a removable singularity at 0. Then $B$ would be harmonic on $\mathbb{D}$ and 0 on $\partial\mathbb{D}$ so it would be identically 0, which was forbidden. If there is a solution it is unique because of the maximum principle. The domain $\Omega$ is *regular for the Dirichlet problem* if there is a solution to the Dirichlet problem for every continuous function $b : \partial\Omega \to \mathbb{R}$.

Let $\mathcal{F} = \{u : \overline{\Omega} \to \mathbb{R} : u$ is continuous, subharmonic on $\Omega$ and $u|\partial\Omega \leqslant b\}$. This is a Perron family on $\Omega$ which contains the constant function $\inf b$ and is bounded above by $\sup b$. So its supremum is a harmonic function $h : \Omega \to \mathbb{R}$. If there is a solution $B$ to the Dirichlet problem, then each $u \in \mathcal{F}$ satisfies $u \leqslant B$ on $\Omega$. So $h \leqslant B$. Also $B \in \mathcal{F}$ so $B \leqslant h$. Therefore, if there is a solution, it is $h$. We will show that $h$ is indeed a solution when the boundary $\partial\Omega$ is sufficiently regular.

A *barrier* at $\zeta \in \partial\Omega$ is a continuous function $\beta : \overline{\Omega} \to \mathbb{R}$ which is subharmonic on $\Omega$ and has $\beta(z) \leqslant 0$ on $\overline{\Omega}$ with equality if, and only if, $z = \zeta$. If $\beta$ is a barrier at $\zeta$ and $U$ is an open neighbourhood of $\zeta$ then $\beta|\overline{\Omega \cap U}$ is obviously a barrier at $\zeta$ for $\Omega \cap U$. Conversely, suppose that $\gamma$ is a barrier at $\zeta$ for $\Omega \cap U$. Then $\gamma$ is bounded on $\overline{\Omega} \cap \partial U$, say $\gamma(z) < -m < 0$ for $z \in \overline{\Omega} \cap \partial U$. Set

$$\beta(z) = \begin{cases} \max(\gamma(z), -m) & \text{for } z \in \overline{\Omega} \cap U \\ -m & \text{for } z \in \overline{\Omega} \setminus U \ . \end{cases}$$

Then $\beta$ is easily seen to be continuous on $\overline{\Omega}$ and subharmonic on $\Omega$, so it is a barrier at $\zeta$ for $\Omega$. Thus the existence of a barrier depends only on the shape of $\Omega$ in the neighbourhood of $\zeta$.

**Theorem 7.2.1**

*A relatively compact domain $\Omega$ in $R$ is regular for the Dirichlet problem if, and only if, there is a barrier at each point of $\partial\Omega$.*

*Proof:*

Suppose that $\Omega$ is regular. For $\zeta \in \partial\Omega$ we can certainly find a continuous function $b : \partial\Omega \to \mathbb{R}$ with $b(z) \leqslant 0$ and equality if, and only if, $z = \zeta$. Let $B : \overline{\Omega} \to \mathbb{R}$ be the corresponding solution to the Dirichlet problem. Then $B$ is continuous on $\overline{\Omega}$ and harmonic on $\Omega$. Since $b$ is not constant, neither is $B$, and so the maximum principle shows that $B(z) < 0$ for every $z \in \overline{\Omega}$ except $\zeta$. Thus $B$ is a barrier at $\zeta$.

For the converse, suppose that $b : \partial\Omega \to \mathbb{R}$ is continuous and let $h = \sup \mathcal{F}$ as above. We will show that $h(z) \to b(\zeta)$ as $z \in \Omega$ tends to any boundary point $\zeta \in \partial\Omega$, so $h$ will be the desired solution of the Dirichlet problem.

Let $\gamma$ be a barrier at $\zeta \in \partial\Omega$. Consider $c_1, c_2$ with $c_1 < b(\zeta) < c_2$. There is an open neighbourhood $U$ of $\zeta$ with $c_1 < b(z) < c_2$ for all $z \in U \cap \partial\Omega$. Since $\gamma \leqslant 0$ we have $c_1 + K\gamma(z) < b(z) < c_2 - K\gamma(z)$ for each $z \in \partial\Omega \cap U$ and any positive $K$. Since $\gamma$ is bounded away from 0 on $\overline{\Omega} \setminus U$, we can choose $K$ so large that

$$c_1 + K\gamma(z) < b(z) < c_2 - K\gamma(z) \quad \text{ for } z \in \partial\Omega.$$

This implies that $z \mapsto c_1 + K\gamma(z)$ is a member of $\mathcal{F}$ so

$$c_1 + K\gamma(z) \leqslant h(z) \quad \text{ for } z \in \Omega.$$

Taking the limit as $z \to \zeta$ gives $c_1 \leqslant \liminf_{z\to\zeta} h(z)$ . Similarly, $s : z \mapsto c_2 - K\gamma(z)$ is superharmonic on $\Omega$. If $u \in \mathcal{F}$ then $u - s$ is subharmonic on $\Omega$, continuous on $\overline{\Omega}$ and negative on $\partial\Omega$. The maximum principle shows that $u - s$ is negative on all of $\overline{\Omega}$ so $u \leqslant s$. Therefore

$$h(z) \leqslant c_2 - K\gamma(z) \quad \text{ for } z \in \Omega$$

and so $\limsup_{z\to\zeta} h(z) \leqslant c_2$.

Together these inequalities show that $h(z)$ tends to $b(\zeta)$ as $z \to \zeta$ from within $\Omega$. So the function which is $b$ on $\partial\Omega$ and $h$ on $\Omega$ is the desired solution for the Dirichlet problem. □

For domains with suitably smooth boundaries it is easy to construct barriers. As a simple but useful example, suppose that there is a chart $\phi : U \to V \subset \mathbb{C}$ at $\zeta \in \partial\Omega$ with $\phi(\zeta) = 0$ and $\phi(\overline{\Omega})$ disjoint from the strictly negative real axis. Then the principle branch $\sigma$ of the square root is continuous on $\overline{\Omega} \cap U$ and analytic on $\Omega \cap U$. Its real part $\beta$ is 0 at $\zeta$ and strictly positive at every other point of $\overline{\Omega} \cap U$. So $\beta$ is a barrier at $\zeta$ for $\Omega \cap U$. If $\partial\Omega$ is piecewise $C^1$ then we can certainly find such charts at each boundary point.

We can readily adapt these results to domains $\Omega \subset R$ which are not relatively compact provided the boundary values are bounded, continuous functions. We say that $\Omega$ is regular for the Dirichlet problem if, for each bounded, continuous function $b : \partial\Omega \to \mathbb{R}$ there is a bounded, continuous function $B : \overline{\Omega} \to \mathbb{R}$ which is harmonic on $\Omega$ and agrees with $b$ on the boundary $\partial\Omega$. Then

**Theorem 7.2.1'**

*A domain $\Omega$ in $R$ is regular for the Dirichlet problem if, and only if, there is a barrier at each point of $\partial\Omega$.*

□

Note that the function $B$ need no longer be uniquely determined by $b$.

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**Exercises**

4. Prove that a relatively compact domain with piecewise smooth boundary is regular for the Dirichlet problem.

-5. Show that there cannot be a barrier at an isolated boundary point.

6. Show that every relatively compact domain $\Omega$ in $R$ is contained within another relatively compact domain $\Omega'$ which is regular for the Dirichlet problem.

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## 7.3 Green's Functions

We will now consider the related problem of constructing the Green's function for a Riemann surface $R$.

Let $u : R \setminus \{z_o\} \to \mathbb{R}$ be a harmonic function and $\phi : U \to \phi(U) = \mathbb{D}$ a chart with $\phi(z_o) = 0$. We want $u(z)$ to behave like $c \log|\phi(z)|$ for $z$ near $z_o$. More formally, we will say that $u$ has a *logarithmic singularity* with coefficient $c$ at $z_o$ if there is a harmonic function $h$ on a neighbourhood of $z_o$ with

$$u(z) = c \log|\phi(z)| + h(z) \ .$$

The crucial part of this is that $h$ is harmonic at $z_o$. The value of $c$ does not depend on which chart we choose. (Indeed,

$$c = \frac{1}{\pi i} \int_\gamma \partial u$$

for a small closed curve $\gamma$ winding once around $z_o$.) We will sometimes write "$u(z) \sim c \log|\phi(z)|$ near $z_o$" as an abbreviation for "$u$ has a logarithmic singularity with coefficient $c$ at $z_o$".

Let $\mathcal{G}(z_o)$ be the set of all positive harmonic functions $g : R \setminus \{z_o\} \to \mathbb{R}^+$ which have a logarithmic singularity at $z_o$ with coefficient $-1$. The least element of $\mathcal{G}(z_o)$ is called the *Green's function* for $R$ with a *pole* at $z_o$. For this definition to make sense we need:

**Proposition 7.3.1**

*If $\mathcal{G}(z_o)$ is non-empty, then it has a least element.*

*Proof:*

Let $\mathcal{F}(z_o)$ be the set of continuous subharmonic functions $u : R \setminus \{z_o\} \to \mathbb{R}$ which have compact support and

$$z \mapsto u(z) + \log|\phi z| \qquad (z \in U)$$

continuous and subharmonic on $U$. This is clearly a Perron family on $R \setminus \{z_o\}$. It contains

$$z \mapsto \begin{cases} -\log|\phi(z)| & \text{if } z \in U \\ 0 & \text{if } z \notin U\ . \end{cases}$$

so it is non-empty.

Let $u \in \mathcal{F}(z_o)$ and $g \in \mathcal{G}(z_o)$. Then $u$ is zero outside some compact set $K$. The difference $u - g$ is continuous and subharmonic on all of $R$, including $z_o$. It is negative outside $K$ and, by the maximum principle (Proposition 7.1.3), it must be negative within $K$. Therefore $u \leqslant g$. The Perron family $\mathcal{F}(z_o)$ is thus bounded above and so its supremum $s : R \setminus \{z_o\} \to \mathbb{R}$ is harmonic. We also have

$$s(z) + \log|\phi z| \quad = \quad \sup(u(z) + \log|\phi z| : u \in \mathcal{F}(z_o))$$

so $s$ has a logarithmic singularity at $z_o$ with coefficient $-1$. Thus $s \in \mathcal{G}(z_o)$ and $s \leqslant g$. □

We will denote the Green's function for $R$ with a pole at $z_o$ by $g_{z_o}$ (or $g$ if there is no possibility of confusion). If $\inf g = m > 0$, then $g - m$ would be a smaller element of $\mathcal{G}(z_o)$. So $\inf g$ must be 0. The following result shows that, if the Green's function with a pole at $z_o$ exists, then so does the Green's function with pole at any other point of $R$.

**Theorem 7.3.2**

*The following conditions on the Riemann surface $R$ are equivalent.*

(a) *There is a function $s : R \to [0, \infty)$ which is non-constant, positive, continuous and superharmonic .*

(b) *For $z_o \in R$ there is a Green's function $g$ on $R$ with pole at $z_o$.*

*Proof:*

If (b) holds, then set $s(z) = \min(g(z), 1)$. Since $g(z)$ takes values close to 0 and $\infty$, the function $s$ is not constant. It is certainly continuous, superharmonic and positive, so (a) holds.

Now suppose that (a) holds. Let $\Delta = \phi^{-1}\mathbb{D}$ and $\Delta' = \phi^{-1}\{z : |z| < \frac{1}{2}\}$. We will first construct the *harmonic measure* of $\Delta'$, that is a continuous function $\omega : R \to \mathbb{R}$ which is 1 on $\Delta'$, harmonic on $R \setminus \Delta'$ and not constant. The collection

$$\mathcal{A} = \{u : R \to \mathbb{R} : u \text{ continuous with compact support, subharmonic on } R \setminus \overline{\Delta'} \text{ and } u \leqslant 1\}$$

is a Perron family on $R \setminus \overline{\Delta'}$. The function

$$u_o(z) = \begin{cases} 1 & \text{if } z \in \Delta' \\ -\log|\phi(z)|/\log 2 & \text{if } z \in \Delta \setminus \Delta' \\ 0 & \text{if } z \in R \setminus \Delta \end{cases}$$

is in $\mathcal{A}$. Also, every $u \in \mathcal{A}$ satisfies $u \leqslant 1$. So the supremum $\omega = \sup \mathcal{A}$ is harmonic on $R \setminus \Delta'$ with $u_o \leqslant \omega \leqslant 1$. It follows that $\omega(z) \to 1$ as $z \to \partial\Delta'$ so we can extend $\omega$ continuously to all of $R$ by setting $\omega(z) = 1$ for $z \in \overline{\Delta'}$. We may assume that $\inf s = 0$ for otherwise we could replace $s$ by $s - \inf s$. Since $s$ is superharmonic it cannot attain this infimum, so $\inf(s(z) : z \in \overline{\Delta'}) = m > 0$. The maximum principle for subharmonic functions shows that each $u \in \mathcal{A}$ satisfies $u \leqslant s/m$ so $\omega \leqslant s/m$. In particular, $\inf(\omega(z) : z \in R) = 0$. Hence $\omega$ is not constant. By the maximum principle for harmonic functions, $0 < \omega(z) < 1$ for $z \in R \setminus \overline{\Delta'}$.

Now that we have the harmonic measure we can construct a positive superharmonic function $v$ on $R \setminus \{z_o\}$ with $v(z) \sim -\log|\phi(z)|$ as $z \to z_o$. We can choose $c$ with $\sup(\omega(z) : z \in \partial\Delta) < c < 1$. Define $v$ by

$$v(z) = \begin{cases} Ac - \log|\phi(z)| & \text{if } z \in \Delta' \\ \min(Ac - \log|\phi(z)|, A\omega(z)) & \text{if } z \in \Delta \setminus \Delta' \\ A\omega(z) & \text{if } z \in R \setminus \Delta \end{cases}$$

for some $A > \log 2/(1-c)$. It is apparent that $v$ is continuous and superharmonic on the interiors of the three regions $\Delta', \Delta \setminus \Delta'$, and $R \setminus \Delta$. The only difficulty is at the boundaries. If $z \in \partial\Delta'$ then

$$Ac - \log|\phi(z)| = Ac + \log 2 < A = A\omega(z)$$

so $v$ is continuous and superharmonic at $z$ because $Ac - \log|\phi(z)|$ is. If $z \in \partial\Delta$ then

$$Ac - \log|\phi(z)| = Ac > A\sup(\omega(w) : w \in \partial\Delta) \geqslant A\omega(z)$$

so $v$ is continuous and superharmonic at $z$ because $A\omega(z)$ is.

If $u \in \mathcal{F}(z_o)$ then $u - v$ is subharmonic on all of $R$ (even near $z_o$ where it is $h(z) - Ac$). Outside of the compact support of $u$ it is negative. The maximum principle then shows that $u - v \leqslant 0$ on all of $R$. So $\sup \mathcal{F}(z_o) \leqslant v$. Hence $\sup \mathcal{F}(z_o)$ is finite and so it is the desired Green's function. □

We will call $R$ *hyperbolic* if it has a Green's function. For example, the disc $\mathbb{D}$ has a Green's function

$$z \mapsto -\log|z|$$

with pole at 0, so it is hyperbolic.

We will call $R$ *elliptic* if it is compact. An elliptic surface can not be hyperbolic, for if $g$ were a Green's function it would attain its infimum and that would contradict the minimum principle for harmonic functions. Hence the classes of elliptic and hyperbolic surfaces are disjoint. The Riemann sphere $\mathbb{C}_\infty$ is an example of an elliptic Riemann surface.

We will call $R$ *parabolic* if it is neither elliptic nor hyperbolic. The complex plane $\mathbb{C}$ is parabolic. For it is certainly not compact and, if $g$ were a Green's function, it would have a removable singularity at $\infty$ and so give a Green's function for $\mathbb{C}_\infty$.

**Corollary 7.3.3**

*If $\Omega$ is a domain in $R$ and there is a barrier at one point $\zeta \in \partial\Omega$ then $\Omega$ is hyperbolic and the Green's function $g$ satisfies $g(z) \to 0$ as $z \to \zeta$ with $z \in \Omega$.*

*Proof:*

Let $\beta$ be the barrier at $\zeta$. Then $-\beta$ is a non-constant, positive, superharmonic function on $\Omega$ so $\Omega$ is certainly hyperbolic and has a Green's function $g$ with pole at $z_o$. Recall from Proposition 7.3.1 that $g$ is the supremum of $\mathcal{F}(z_o)$ which consists of continuous subharmonic functions $u$ which are zero outside some compact subset of $\Omega$. We will extend any such function $u$ continuously to $\partial\Omega$ by setting it equal to 0 there. Let $V$ be a compact neighbourhood of $z_o$ contained in $\Omega$. Then $\partial V$ is compact so we can find $K$ with $g(z) \leqslant K$ for $z \in \partial V$. This implies that $u(z) \leqslant K$ for $z \in \partial V$. However, $u$ is 0 on $\partial\Omega$ so the maximum principle for subharmonic functions gives

$$u(z) \leqslant K \qquad \text{for} \quad z \in \Omega \setminus V.$$

The barrier $\beta$ has $\beta(z) \leqslant 0$ for $z \in \overline{\Omega}$ and $\beta(z) = 0$ if, and only if, $z = \zeta$. Hence, we can find $m > 0$ with $\beta(z) \leqslant -m$ for $z \in \partial V$. It follows that the subharmonic function $z \mapsto u(z) + (K/m)\beta(z)$ is at most 0 on $\partial\Omega \cup \partial V$. So the maximum principle gives

$$u(z) + (K/m)\beta(z) \leqslant 0 \qquad \text{for} \quad z \in \Omega \setminus V.$$

Taking the supremum over all $u$ gives

$$g(z) \leqslant -(K/m)\beta(z) \qquad \text{for} \quad z \in \Omega \setminus V.$$

Hence $g(z) \to 0$ as $z \to \zeta$ with $z \in \Omega$. □

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**Exercises**

-7. If $g$ is the Green's function for $R$ with pole at $z_o$ and $f : S \to R$ is a conformal equivalence, then $gf$ is the Green's function on $S$ with pole at $f^{-1}(z_o)$.

8. Find the Green's function on the unit disc with a pole at any specified point (and prove that it is the Green's function).

9. Let $g$ be the Green's function on a domain $\Omega \subset \mathbb{C}$ with a pole at $z_o$ and let $f$ be a smooth function on $\Omega$ with compact support within $\Omega$. Prove that

$$\int_\Omega \triangle f(z) g(z)\, d\overline{z} \wedge dz = -4\pi i g(z_o).$$

Hint: Stokes' Theorem.

(This means that $g$ defines a distribution with $\triangle g = -4\pi i \delta_{z_o}$. In partial differential equations and applied mathematics this is often taken as the definition of the Green's function.)

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