COMPLEX DIFFERENTIAL EQUATIONS – Example Sheet 1

TKC Lent 2008

1. Let (K_n) be a compact exhaustion of a domain $D \subset \mathbb{C}$. Show that a sequence of continuous functions $f_n : D \to \mathbb{C}$ converge locally uniformly on D if and only if they converge for the metric

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \min\left(1, \sup\{|f(z) - g(z)| : z \in K_n\}\right) .$$

2. Let $f : H^+ = \{x + iy : y > 0\} \to \mathbb{C}$ be a bounded analytic function on the upper half plane with $f(iy) \to \ell$ as $y \searrow 0$. Prove that $f(z)$ converges uniformly to ℓ in any cone of the form:

$$\{x + iy \in H^+ : |x| \leqslant ky\}$$

[*Hint: Consider* $f_n(z) = f(z/n)$.]

3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Show that the partial sums converge locally uniformly to f on $\{z \in \mathbb{C} : |z| < R\}$ but need not converge uniformly.

   Give an example of a function f for which the partial sums do converge uniformly on the disc of convergence.

4. A power series $f(z) = \sum a_n z^n$ has radius convergence R with $0 < R < \infty$. Show that there is at least one *singular point* w with $|w| = R$: that is a point w for which f can not be continued analytically to any neighbourhood of w.

   If $a_n \geqslant 0$ for each $n \in \mathbb{N}$, prove that R is a singular point. (Pringsheim's theorem.)

   Show that the (lacunary) power series

$$\sum z^{2^n}$$

   has radius of convergence 1 and every point on the unit circle is a singular point.

5. Solve the differential equation:

$$f'(z) = \frac{f(z) - z}{z^2} \qquad ; \qquad f(0) = 0 .$$

   [*Write the answer as an integral.*]

   Explain why this can not be solved as a power series about 0.

6. Let $T_n, T : M \to M$ be contraction mappings on a complete metric space M, with fixed points w_n, w respectively. If $T_n \to T$ uniformly, is it necessarily true that $w_n \to w$?

7. Let $f : [0,1] \to [0,\infty)$ be a continuous function with $f(0) = 0$ and $\lim_{t \searrow 0} \dfrac{f(t)}{t} = 0$. Show that, if f satisfies

$$f(t) \leqslant \int_0^t \frac{f(u)}{u}\, du \qquad \text{for all } t \in [0,1]$$

   then f is identically 0.

8. Let $f, g : [0,1] \to [0,\infty)$ be continuous functions that satisfy

$$f(t) \leqslant g(t) + K \int_0^t (t-u) f(u)\, du \qquad \text{for all } t \in [0,1] .$$

   Show that

$$f(t) \leqslant g(t) + K^{1/2} \int_0^t \sinh\left(K^{1/2}(t-u)\right) g(u)\, du .$$

9. Are there any non-trivial functions $f : [0,1] \to [0,\infty)$ that satisfy

$$f'(t) \leqslant -1 - f(t)^2 \qquad \text{for all } t \in [0,1] \ ?$$

10. Solve $f'(z) = f(z)$; $f(0) = 1$ explicitly by finding successive approximations starting from the constant function 1.

    Solve $f'(z) = 1 + f(z)^2$; $f(0) = 0$ explicitly by finding successive approximations starting from the identity function $z \mapsto z$.

11. Find all of the solutions of $f'(z) = 2f(z)^{1/2}$ when we take a branch of the square root. (Note that there is one exceptional solution with $f(0) = 0$.)

12. Let $f_1, f_2 : D \to \mathbb{C}$ be two analytic functions on a domain $D \subset \mathbb{C}$ that are linearly independent over $\mathbb{C}$. Show that there is a (non-trivial) second order, linear differential equation

$$f''(z) + a_1(z)f'(z) + a_0(z)f(z) = 0$$

which has f_1 and f_2 as solutions. Where are the singular points of this differential equation?

13. *Eisenstein series.* Show that, for $k \geqslant 2$, the series

$$\varepsilon_k(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^k}$$

converges locally uniformly on $\mathbb{C}$ to give a meromorphic function. Prove the following properties of these functions.

    (a) Each ε_k is periodic with period 1.

    (b) Each ε_k has a pole of order k at each integer and nowhere else.

    (c) $\varepsilon_k(x + iy) \to 0$ as $y \to \pm\infty$ uniformly for $x \in \mathbb{R}$.

    (d) $\varepsilon_k'(z) = -k\varepsilon_{k+1}(z)$.

    Prove that a meromorphic function $f : \mathbb{C} \to \mathbb{P}$ with period 1 can be written as a series:

$$f(z) = \sum_{n\in\mathbb{Z}} f_n \exp 2\pi i n z$$

that converges locally uniformly. Deduce that each ε_k is a rational function of $\exp 2\pi i z$.

Prove that

$$\varepsilon_2(z) = \frac{\pi^2}{\sin^2 \pi z} \; .$$

14. *Eisenstein series (continued).* Show that the function

$$\varepsilon_1(z) = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{z-n} + \frac{1}{n}$$

defines a meromorphic function on $\mathbb{C}$ with $\varepsilon_1'(z) = -\varepsilon_2(z)$. Solve this differential equation to find an explicit formula for ε_1.

Solve the equation

$$f'(z) = \varepsilon_1(z)f(z)$$

and hence find an infinite product for $\sin \pi z$.

15. Write $1/(z-n)$ as a Laurent series about 0. Hence find the Laurent series for ε_1 about 0. (Write the coefficients in terms of the Riemann ζ function

$$\zeta(s) = \sum_{n\in\mathbb{N}} n^{-s} \; .)$$

What is its radius of convergence?

Find the Laurent series for each ε_k about 0.

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \; .$$

Please send any comments or corrections to me at: t.k.carne@dpmms.cam.ac.uk .