UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Before Thursday 11th June, 2007 1.30pm to 4.30pm

PAPER 11

RIEMANN SURFACES AND DISCRETE GROUPS

*Attempt **four** questions.*

*There are **six** questions in total.*

The questions carry equal weight.

This is a Mock examination, intended to give you some idea of the sort of questions you will face in the proper Part III examination. It has not been moderated by the examiners.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove the Schwarz - Pick Lemma.

The function $f : \mathbb{D}_R \to \mathbb{C}$ is analytic on some disc $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ with $R > 1$ and satisfies

$$A \leqslant |f(z)| \leqslant B \qquad \text{when} \qquad |z| = 1 \ .$$

The constants A and B satisfy $|f(0)| < A \leqslant B < \infty$. Prove that f has a zero at some point $z_o \in \mathbb{D}_R$ and that $|z_o| \geqslant |f(0)|/B$. Show that there are functions f for which we obtain equality in this inequality.

2 What is a *normal family* of analytic functions? Show that the set of all analytic functions from a plane domain D into the unit disc $\mathbb{D}$ is a normal family.

Let $g : \mathbb{H}^+ \to \mathbb{C}$ be a bounded analytic function on the upper half-plane $\mathbb{H}^+$ and suppose that $g(z) \to \ell$ as z tends to ∞ along the positive imaginary axis. Show that the functions $z \mapsto g(tz)$ for $t \geqslant 1$ form a normal family. Deduce that, for each $\varepsilon > 0$, we have $g(z) \to \ell$ as z tends to ∞ in the sector

$$S(\varepsilon) = \{w \in \mathbb{H}^+ : \varepsilon < \arg w < \pi - \varepsilon\} \ .$$

Let $h : \mathbb{D} \to \mathbb{C}$ be a bounded analytic function and let ω be a complex number of modulus 1. Suppose that $h(r\omega) \to \ell$ as $r \nearrow 1$. Show that $h(z) \to \ell$ as z tends to ω in the region

$$\Sigma(k) = \{z \in \mathbb{D} : \text{ there exists } r \in [0,1) \text{ with } \rho(z, r\omega) < k\} \ .$$

Here ρ is the hyperbolic metric on the unit disc and k is an arbitrary positive constant.

3 Write an essay on the proof of the Riemann Mapping Theorem for simply-connected surfaces. You should explain the proof in detail for hyperbolic Riemann surfaces.

4 Explain how to define the hyperbolic metric on any Riemann surface R that has the unit disc as its universal cover. Prove that the metric is well-defined and is a metric. Calculate the hyperbolic metric on the annulus $A = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Prove Picard's Great Theorem. (You may assume the existence of a universal cover for the 3-punctured sphere.)

5 Let G be a discrete group of Möbius transformations acting on the unit disc $\mathbb{D}$. Prove that G acts discontinuously and explain, briefly, why this means that the quotient $\mathbb{D}/G$ is a Riemann surface. Give an example for which the quotient map is not a covering map.

Prove that the quotient $\mathbb{D}/G$ by the group G is hyperbolic if, and only if, $\sum(\exp -\rho(0, T(0)) : T \in G)$ converges for the hyperbolic metric ρ on $\mathbb{D}$.

6 Prove the Poisson–Jensen inequality.

Show how to use the Poisson–Jensen inequality to characterise those sequences that are zeros of a bounded analytic function on the unit disc.

Suppose that (z_n) is the sequence of zeros of a bounded analytic function $f : \mathbb{D} \to \mathbb{C}$. Show that there is a Blaschke product B with zeros precisely at the points (z_n). Show that the Blaschke product converges locally uniformly on the complement of the set E, which is the closure of the set of points $\{1/\overline{z_n} : n \in \mathbb{N}\}$.

Is the complement of E always connected?

END OF PAPER