5.1 The Hyperbolic Plane.

**Lemma 5.1.1** Schwarz’ lemma

If $f : \mathbb{D} \to \mathbb{D}$ is an analytic map with $f(0) = 0$ then

$$|f(w)| \leq |w| \quad \text{for} \quad w \in \mathbb{D} \setminus \{0\} \quad \text{and} \quad |f'(0)| \leq 1.$$  

Moreover, if equality holds at any point then $f$ must be the map $z \mapsto \omega z$ for some $\omega$ of modulus 1.

**Proof:**

The map

$$g : \mathbb{D} \to \mathbb{C} \quad ; \quad w \mapsto \begin{cases} f(w)/w & \text{for} \ w \in \mathbb{D} \setminus \{0\} \\ f'(0) & \text{for} \ w = 0 \end{cases}$$

is analytic on $\mathbb{D} \setminus \{0\}$ and continuous at 0. So it has a removable singularity at 0 and hence is analytic on all of $\mathbb{D}$. The maximum modulus principle shows that, for $r < 1$,

$$|g(w)| \leq \sup(|g(z)| : |z| = r) = \sup(|f(z)|/r : |z| = r) \quad \text{for} \quad |w| \leq r.$$  

Hence,

$$|g(w)| \leq 1 \quad \text{for} \quad w \in \mathbb{D}$$

which is the first part of the lemma. Moreover, if there is equality at any point of $\mathbb{D}$ then the maximum modulus principle implies that $g$ is constant. The constant must be of modulus 1.  

This lemma will enable us to prove that the only conformal maps $f : \mathbb{D} \to \mathbb{D}$ are the M"obius transformations which do map $\mathbb{D}$ to itself. To discover which M"obius transformations these are, consider the map $J : C_\infty \to C_\infty ; z \mapsto \bar{z}^{-1}$. This is inversion in the unit circle, so $J(z) = z$ if, and only if, $z \in \partial \mathbb{D}$. Hence, a M"obius transformation $T : z \mapsto (az + b)/(cz + d)$ ($ad - bc = 1$) will map the unit circle onto itself if, and only if,

$$J T = T J \quad \iff \quad JTJ(z) = \frac{az + b}{cz + d} = T(z) \quad \text{for} \quad z \in C_\infty$$

$$\iff \quad d = \pm \bar{a} \quad ; \quad b = \pm \bar{a}.$$  

The + signs give the M"obius transformations

$$T : z \mapsto \frac{az + \bar{a}}{cz + \bar{a}} \quad \text{for} \quad |a|^2 - |c|^2 = 1$$

which map $\mathbb{D}$ onto $\mathbb{D}$. While the - signs give the M"obius transformations $T : z \mapsto (az - \bar{a})/(cz - \bar{a})$ for $|a|^2 - |c|^2 = -1$ which map $\mathbb{D}$ onto $\{z \in C_\infty : |z| > 1\}$. In particular we see that the maps (*) form a group of conformal maps from $\mathbb{D}$ onto $\mathbb{D}$. This group of maps is transitive on $\mathbb{D}$, so for every $z \in \mathbb{D}$ there is a map $T$ with $T(0) = z$. The following result shows that there are no other conformal maps from $\mathbb{D}$ onto itself.

**Theorem 5.1.2**

$$\text{Aut} \ \mathbb{D} \ = \ \left\{ z \mapsto \frac{az + \bar{a}}{cz + \bar{a}} : |a|^2 - |c|^2 = 1 \right\}.$$
Proof:

Suppose that \( h \in \text{Aut} \mathbb{D} \). Then \( h(0) \in \mathbb{D} \) so we can find \( a, c \in \mathbb{C} \) with \( |a|^2 - |c|^2 = 1 \) and \( h(0) = -\overline{c}/a \). Let \( T \) be the Möbius transformation \( z \mapsto (az + \overline{c})/(cz + \overline{a}) \). Then \( f = Th \in \text{Aut} \mathbb{D} \) and \( f(0) = Th(0) = 0 \). By Schwarz' lemma, \( |f'(0)| \leq 1 \). However, \( f^{-1} \) is also an automorphism of \( \mathbb{D} \) so \( |(f^{-1})'(0)| = |f'(0)|^{-1} \leq 1 \). Hence equality must hold and so \( f(z) = \omega z \) for some \( \omega \) of modulus 1. It follows that \( f \), and hence also \( h \), is a Möbius transformation which maps \( \mathbb{D} \) onto itself. \( \square \)

If \( T : z \mapsto (az + \overline{c})/(cz + \overline{a}) \) (with \( |a|^2 - |c|^2 = -1 \)) is in \( \text{Aut} \mathbb{D} \) then \( \tau(T) = \text{tr}(T)^2/4 = (\Re a)^2 \in [0, \infty) \). So \( T \) can be the identity, or elliptic, or hyperbolic, or parabolic, but not loxodromic.

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**Exercises**

1. Show that \( z \mapsto \omega(z - z_0)/(1 - \overline{z_0}z) \) is in \( \text{Aut} \mathbb{D} \) for \( \omega \) with \( |\omega| = 1 \) and \( z_0 \in \mathbb{D} \). Conversely every map in \( \text{Aut} \mathbb{D} \) is of this form.
2. Find \( \text{Aut} \mathbb{H}^+ \) for the upper half plane \( \mathbb{H}^+ = \{ z \in \mathbb{C} : \Im z > 0 \} \).
3. Consider \( \mathbb{D} \) as the subset

\[
\{ [z_1 : z_2] : |z_1|^2 - |z_2|^2 < 0 \} \text{ of } \mathbb{P}(\mathbb{C}^2).
\]

Show that an invertible linear map \( T : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \) with determinant 1 induces a conformal map \( T : \mathbb{D} \to \mathbb{D} \) if, and only if,

(a) \( T \) preserves the indefinite form

\[
\beta : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \overline{z_1}w_1 - \overline{z_2}w_2.
\]

(That is \( \beta(Tz, Tw) = \beta(z, w) \).)

and

(b) \( T(0) \in \mathbb{D} \). (That is \( |b| < |d| \).)

4. Let \( T \in \text{Aut} \mathbb{D} \). Show that

(a) if \( T \) is elliptic, it has exactly one fixed point in \( \mathbb{D} \).

(b) if \( T \) is hyperbolic, it has two fixed points both on \( \partial \mathbb{D} \).

(c) if \( T \) is parabolic, it has one fixed point on \( \partial \mathbb{D} \).

Find the conjugacy classes in \( \text{Aut} \mathbb{D} \).

5. Prove directly that a loxodromic Möbius transformation cannot map any disc in \( \mathbb{C}_\infty \) onto itself.

---

There is a metric on \( \mathbb{D} \), called the **hyperbolic metric** \( \rho \), which is invariant under all of the maps in \( \text{Aut} \mathbb{D} \). To define it, let \( \gamma : I \to \mathbb{D} \) be a smooth curve and let its length be

\[
L_\rho(\gamma) = \int_I \left| \frac{2}{1 - |z'|^2} \right| |dz| = \int_0^1 \left( \frac{2}{1 - |\gamma'(t)|^2} \right) |\gamma'(t)| \, dt.
\]

Then define \( \rho(z_0, z_1) \) to be the infimum of the lengths \( L_\rho(\gamma) \) for all smooth paths in \( \mathbb{D} \) from \( z_0 \) to \( z_1 \). This is certainly symmetric and satisfies the triangle inequality. We will see shortly that it is 0 if, and only if, \( z_0 = z_1 \) and then we will know that it is a metric.

If \( T : z \mapsto (az + \overline{c})/(cz + \overline{a}) \) (with \( |a|^2 - |c|^2 = -1 \)), then

\[
T'(z) = \frac{1}{(cz + \overline{a})^2} \quad \text{and} \quad 1 - |T(z)|^2 = \frac{1 - |z|^2}{|cz + \overline{a}|^2}
\]
so \( L_\rho(T_\gamma) = L_\rho(\gamma) \) and hence \( T \) is an isometry. Suppose that \( x \in [0, 1) \) and \( \gamma \) is a path in \( \mathbb{D} \) from 0 to \( x \). Then the path \( \bar{\gamma} \) is also a path in \( \mathbb{D} \) from 0 to \( x \) and, since,

\[
\left( \frac{2}{1 - |\bar{\gamma}(t)|^2} \right) |(\bar{\gamma})'(t)| \leq \left( \frac{2}{1 - |\gamma(t)|^2} \right) |\gamma'(t)|
\]

it has a shorter length than \( \gamma \). So the straight line path from 0 to \( x \) is the unique shortest path between these points and hence

\[
\rho(0, x) = \int_0^x \left( \frac{2}{1 - |t|^2} \right) dt = \log \left( \frac{1 + x}{1 - x} \right).
\]

The invariance of \( \rho \) under \( \text{Aut} \, \mathbb{D} \) enables us to deduce that

\[
\rho(z_0, z_1) = \log \left( \frac{1 + \frac{z_0 - z_1}{1 - \overline{z_0}z_1}}{1 - \frac{z_0 - z_1}{1 - \overline{z_0}z_1}} \right)
\]

and that the unique path from \( z_0 \) to \( z_1 \) with shortest length is the arc of a circle orthogonal to \( \partial \mathbb{D} \). In particular, \( \rho(z_0, z_1) = 0 \) if, and only if, \( z_0 = z_1 \) so \( \rho \) is indeed a metric.

**Theorem 5.1.3**  
Schwarz - Pick theorem

*Every analytic function \( f : \mathbb{D} \to \mathbb{D} \) is a contraction for the hyperbolic metric, so*

\[
\rho(f(z_0), f(z_1)) \leq \rho(z_0, z_1) \quad \text{for} \quad z_0, z_1 \in \mathbb{D}.
\]

**Proof:**

Since each \( T \in \text{Aut} \, \mathbb{D} \) is an isometry and the group acts transitively on \( \mathbb{D} \) we can assume that \( z_0 = f(z_0) = 0 \). Then the result is Schwarz’ lemma. \( \square \)

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**Exercises**

6. Show that the hyperbolic metric on \( \mathbb{D} \) is complete.

7. (This requires the definition of a Riemannian metric.) Show that the only Riemannian metrics on \( \mathbb{D} \) which are invariant under \( \text{Aut} \, \mathbb{D} \) are the multiples of the hyperbolic metric. Show that there are no Riemannian metrics on \( \mathbb{C}_\infty \) or \( \mathbb{C} \) which are invariant under \( \text{Aut} \, \mathbb{C}_\infty \) or \( \text{Aut} \, \mathbb{C} \).

8. Find the hyperbolic metric on the upper half plane \( \mathbb{H} \) for which any Möbius transformation mapping \( \mathbb{D} \) onto \( \mathbb{H} \) is an isometry.

9. Let \( z_1, z_2, w_1 \) and \( w_2 \) be four points in \( \mathbb{D} \). Show that there is an analytic function \( f : \mathbb{D} \to \mathbb{D} \) with \( f(z_1) = w_1 \) and \( f(z_2) = w_2 \) if, and only if, \( \rho(w_1, w_2) \leq \rho(z_1, z_2) \).

10. Let \( C \) be the unique circle through the two points \( z_0, z_1 \in \mathbb{D} \) which is orthogonal to \( \partial \mathbb{D} \). Then \( C \) meets \( \partial \mathbb{D} \) at the points \( w_0, w_1 \) with \( w_0, z_0, z_1, w_1 \) in that order on \( C \). Express the cross ratios of \( w_0, z_0, z_1, w_1 \) and of \( J(z_0), z_0, z_1, J(z_1) \) in terms of \( \rho(z_0, z_1) \).

11. Prove that

\[
\left| \frac{z_0 - z_1}{1 - \overline{z_0}z_1} \right| = \tanh \left( \frac{1}{2} \rho(z_0, z_1) \right).
\]

Does the left side of this equation define a metric on \( \mathbb{D} \)? Find similar formulae for \( \sinh \rho(z_0, z_1) \) and \( \cosh \rho(z_0, z_1) \).
5.2 The Poisson - Jensen Formula.

Let \( f : \mathbb{D} \rightarrow \mathbb{C} \) be a continuous map which is analytic on the open disc \( \mathbb{D} \) and never 0 on \( \partial \mathbb{D} \). Then \( f \) can only have finitely many zeros in \( \mathbb{D} \), say \( z_1, z_2, \ldots, z_N \) each repeated according to its multiplicity. Now, for \( z_1 \in \mathbb{D} \), the Möbius transformation \( T : z \mapsto (z - z_1)/(1 - z_1z) \) maps \( \mathbb{D} \) continuously onto itself and its only zero is \( z_1 \). Hence, \( f_1(z) = f(z)/T(z) \) is continuous on \( \mathbb{D} \), analytic on \( \mathbb{D} \) and has \( |f_1(z)| = |f(z)| \) for \( z \in \partial \mathbb{D} \). Also \( f_1 \) has one less zero in \( \mathbb{D} \) than \( f \). Repeating this argument we find that

\[
f(z) = \left\{ \prod_{n=1}^{N} \left( \frac{z - z_n}{1 - z_nz} \right) \right\} f_N(z)
\]

where \( |f_N(z)| = |f(z)| \) for \( z \in \partial \mathbb{D} \). Since \( f_N \) has no zeros in \( \mathbb{D} \) we can write \( f_N = \exp g \) for a continuous map \( g : \mathbb{D} \rightarrow \mathbb{C} \) which is analytic on \( \mathbb{D} \).

**Theorem 5.2.1** The Poisson - Jensen Formula

If \( f : \mathbb{D} \rightarrow \mathbb{C} \) is continuous, analytic on \( \mathbb{D} \) and never 0 on \( \partial \mathbb{D} \) then the zeros \( z_1, z_2, \ldots, z_N \) of \( f \) satisfy

\[
\log |f(z)| = \sum_{n=1}^{N} \log \left| \frac{z - z_n}{1 - \overline{z_n}z} \right| + \int_{0}^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \log |f(e^{i\theta})| \frac{d\theta}{2\pi}
\]

for \( z \in \mathbb{D} \) with \( |f(z)| \neq 0 \).

**Proof:**

As shown above, we can write

\[
f(z) = \left\{ \prod_{n=1}^{N} \left( \frac{z - z_n}{1 - \overline{z_n}z} \right) \right\} \exp g(z)
\]

with \( g : \mathbb{D} \rightarrow \mathbb{C} \) continuous and analytic on \( \mathbb{D} \). Hence,

\[
\log |f(z)| = \sum_{n=1}^{N} \log \left| \frac{z - z_n}{1 - \overline{z_n}z} \right| + \Re g(z).
\]

However, \( \Re g \in \mathcal{H}(\mathbb{D}) \) so Poisson’s formula gives

\[
\Re g(z) = \int_{0}^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \Re g(e^{i\theta}) \frac{d\theta}{2\pi} = \int_{0}^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \log |f(e^{i\theta})| \frac{d\theta}{2\pi}.
\]

\[\square\]

**Corollary 5.2.2**

Let \( f : \mathbb{D} \rightarrow \mathbb{C} \) be an analytic function with \( f(0) \neq 0 \) and let \( (z_n) \) be the sequence of zeros of \( f \) each repeated according to its multiplicity. For \( 0 \leq r < 1 \) we have

\[
\log |f(0)| = \sum \left( \log \frac{|z_n|}{r} : |z_n| < r \right) + \int_{0}^{2\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi}.
\]
Proof:

If \( f \) has no zeros on \( \{ z : |z| = r \} \) then we may obtain the desired result by applying the theorem to the function \( z \mapsto f(rz) \). The two sides of the equation are clearly continuous functions of \( r \), so equality must persist even when \( f \) has a zero on \( \{ z : |z| = r \} \).

A particularly important case is when \( f : \mathbb{D} \to \mathbb{C} \) is a bounded analytic function. Then

\[
- \sum \log |z_n| \leq \log ||f||_{\infty} - \log |f(0)|
\]

so \( \sum \log |z_n| \) converges provided that \( f(0) \neq 0 \). If \( f \) has a zero of order \( k \) at 0 then we may apply this result to \( f(z)/z^k \) to find that the series \( \sum \log |z_n| \) is still convergent when we sum over all of the zeros \( z_n \) of \( f \) in \( \mathbb{D} \setminus \{0\} \).

The converse of this is also true: if \((z_n)\) is a sequence of points in \( \mathbb{D} \setminus \{0\} \) with \( \sum \log |z_n| \) convergent, then there is a bounded analytic function with zeros precisely at the points \((z_n)\). We will prove this by constructing a product

\[
B(z) = \prod \omega_n \left( \frac{z - z_n}{1 - \overline{z_n}z} \right)
\]

where \( |\omega_n| = 1 \). For this to converge we must have each term converging to 1 as \( n \to \infty \) and \( |z_n| \to 1 \). Hence we must take \( \omega_n = -|z_n|/z_n \) when \( z_n \neq 0 \) (and \( \omega_n \) is arbitrary when \( z_n = 0 \)). For this reason we define a Blaschke product \( B \) for a discrete sequence \((z_n)\) in \( \mathbb{D} \) to be

\[
B(z) = \omega^k \prod \frac{-|z_n|}{z_n} \left( \frac{z - z_n}{1 - \overline{z_n}z} \right)
\]

where \( \omega \in \mathbb{C} \) is of modulus 1, 0 occurs \( k \) times in the sequence \((z_n)\), and the product is over the non-zero elements of the sequence. We often set \( \omega = 1 \) and call \( B \) the Blaschke product for \((z_n)\).

**Lemma 5.2.3** Blaschke products

For each discrete sequence \((z_n)\) in \( \mathbb{D} \) for which \( \sum 1 - |z_n| \) is convergent, the Blaschke product \( B \) converges to an analytic function \( B : \mathbb{D} \to \mathbb{D} \) with zeros at the points of the sequence and nowhere else.

**Proof:**

It is clear that the Blaschke product \( B \), provided that it converges, has the desired properties. We can assume that the sequence \((z_n)\) is infinite and does not contain 0. The condition \( \sum 1 - |z_n| < \infty \) certainly implies that the product \( B(0) = \prod |z_n| \) converges. Hence, it will suffice to show that

\[
\prod \frac{1}{z_n} \left( \frac{z - z_n}{1 - \overline{z_n}z} \right)
\]

converges (to \( B(z)/B(0) \)) locally uniformly on \( \mathbb{D} \). This would certainly be implied by the locally uniform convergence of the series

\[
\sum \left| 1 - \frac{1}{z_n} \left( \frac{z - z_n}{1 - \overline{z_n}z} \right) \right|
= \sum \left| \frac{z(1 - |z_n|^2)}{z_n(1 - \overline{z_n}z)} \right|
= \sum \left| \frac{1 - |z_n|^2}{|z_n||1 - \overline{z_n}z|} z \right|
\]

This last series clearly converges locally uniformly on \( \mathbb{D} \) by comparison with \( \sum 1 - |z_n| \).
Theorem 5.2.4

The sequence of zeros \((z_n)\) of any bounded analytic function \(f : D \to \mathbb{C}\), repeated according to their multiplicity, is discrete and has \(\sum 1 - |z_n|\) convergent. Conversely any sequence with these properties is the set of zeros of a bounded analytic function.

\[\square\]

Note that if \(f : D \to \mathbb{C}\) is bounded and \(B\) is the Blaschke product on the sequence of zeros of \(f\), then \(f/B\) is an analytic function \(g : D \to \mathbb{C} \setminus \{0\}\). Also, since all the partial products of the Blaschke product have modulus 1 on \(\partial D\), we have \(||g||_\infty \leq ||f||_\infty\).

Exercises

12. Let \((z_n)\) be a discrete sequence of points in \(D\). For each \(w \in D\) show that following conditions are equivalent.
   (a) The series \(\sum 1 - |z_n|\) converges.
   (b) The series \(\sum \exp -\rho(w, z_n)\) converges.
   (c) The Blaschke product for \((z_n)\) converges at \(w\).

13. A Blaschke product on a finite set of points in \(D\) is called a finite Blaschke product. (This includes the constant maps \(z \mapsto \omega\) for \(|\omega| = 1\).) Prove that a continuous function \(f : D \to \mathbb{C}\) is a finite Blashke product if, and only if, it is analytic on \(D\) and maps \(\partial D\) into itself.

What are the continuous maps \(f : D \to \mathbb{C}_\infty\) which are meromorphic on \(D\) and map \(\partial D\) into itself?

14. Let \(B\) be the Blaschke product for a sequence \((z_n)\) in \(D\) which satisfies \(\sum 1 - |z_n| < \infty\). Show that the Blaschke product converges not only on \(D\) but also on \(\{z \in \mathbb{C}_\infty : |z| > 1\}\) giving a meromorphic function with poles at the points \((J(z_n))\). Prove that \(JB(z) = BJ(z)\) for \(z \in D\).

If \(z \in \partial D\) is not the limit point of a sequence \((z_n)\) then prove that the Blaschke product converges at \(z\), is analytic on a neighbourhood, and satisfies \(|B(z)| = 1\).
5.3 Models for the Hyperbolic Plane

We have defined the hyperbolic plane as the unit disc \( \mathbb{D} \) with the hyperbolic metric \( \rho \). Set

\[
\kappa(w, z) = 2 \sinh \frac{1}{2} \rho(w, z) .
\]

Note that

\[
\tau = \tanh \frac{1}{2} \rho(w, z) = \frac{|w - z|}{1 - wz}
\]

satisfies

\[
\sinh \frac{1}{2} \rho(w, z) = \frac{\tau}{\sqrt{1 - \tau^2}} ; \quad \cosh \frac{1}{2} \rho(w, z) = \frac{1}{\sqrt{1 - \tau^2}} .
\]

Therefore,

\[
\kappa(w, z)^2 = \frac{4\tau^2}{1 - \tau^2} = \frac{4|w - z|^2}{|1 - wz|^2 - |w - z|^2} = \frac{4|w - z|^2}{(1 - |w|^2)(1 - |z|^2)}
\]

and consequently:

\[
\kappa(w, z) = \frac{2|w - z|}{\sqrt{(1 - |w|^2)(1 - |z|^2)}} .
\]

This is the chordal distance. It should be compared to the chordal metric on the Riemann sphere.

Note that the chordal distance is not a metric on the disc. For, suppose that \( z_1, z_2, z_3 \) are three points in order on a hyperbolic geodesic in \( \mathbb{D} \). Then \( \rho(z_1, z_3) = \rho(z_1, z_2) + \rho(z_2, z_3) \), so

\[
\kappa(z_1, z_3) = 2 \sinh \frac{1}{2} (\rho(z_1, z_2) + \rho(z_2, z_3))
\]

while

\[
\kappa(z_1, z_2) + \kappa(z_2, z_3) = \sinh \frac{1}{2} \rho(z_1, z_2) + \sinh \frac{1}{2} \rho(z_2, z_3) .
\]

Hence we cannot have \( \kappa(z_1, z_3) \leq \kappa(z_1, z_2) + \kappa(z_2, z_3) \) for all \( z_1, z_2, z_3 \). However, the chordal distance is a strictly increasing function of the hyperbolic distance and so the sets

\[
\{ z \in \mathbb{D} : \kappa(w, z) < r \} \quad \text{for } 0 < r < \infty
\]

are a basis for the neighbourhoods of \( w \).

The group \( \text{M"{o}b}(\mathbb{D}) \) is the group of orientation preserving isometries for the hyperbolic metric \( \rho \) or for the chordal distance \( \kappa \).

There are other models for the hyperbolic plane. Let \( \langle \, , \, \rangle \) be the indefinite inner product (or sesquilinear form)

\[
\langle w, z \rangle = \overline{w_0}z_0 - \overline{w_1}z_1 .
\]

Set

\[
\mathcal{D} = \{ [z] \in \mathbb{P}(\mathbb{C}^2) : \langle z, z \rangle > 0 \}
\]

\[
= \{ [z_0 : z_1] \in \mathbb{P}(\mathbb{C}^2) : |z_0|^2 - |z_1|^2 > 0 \} .
\]

Then the map

\[
\alpha : \mathcal{D} \to \mathbb{D} ; \quad [z_0 : z_1] \mapsto \frac{z_1}{z_0}
\]

\[
\alpha^{-1} : \mathbb{D} \to \mathcal{D} ; \quad z \mapsto [1 : z]
\]

is a bijection. So we may take \( \mathcal{D} \) as a model for the hyperbolic plane. The Study distance on \( \mathcal{D} \) is:

\[
d([w], [z]) = 2 \sqrt{-1 + \frac{|\langle w, z \rangle|^2}{\langle w, w \rangle \langle z, z \rangle}} .
\]
Note that
\[
\frac{|\langle w, z \rangle|^2}{\langle w, w \rangle \langle z, z \rangle} = \frac{|1 - \alpha(w)\alpha(z)|^2}{(1 - |\alpha(w)|^2)(1 - |\alpha(z)|^2)} = 1 + \frac{|\alpha(w) - \alpha(z)|^2}{(1 - |\alpha(w)|^2)(1 - |\alpha(z)|^2)} = 1 + \sinh^2 \frac{1}{2} \rho(\alpha(w), \alpha(z)).
\]

Hence,
\[d([w], [z]) = \kappa(\alpha(w), \alpha(z))\]
so the chordal distance on $D$ corresponds to the Study distance on $D$.

Let $[w]$ be a point in $\mathbb{P}(\mathbb{C}^2)$ with $\langle w, w \rangle \neq 0$. Then $[w]$ is a complex 1-dimensional subspace of $\mathbb{C}^2$ and its orthogonal complement:
\[ [w]^\perp = \{ z \in \mathbb{C}^2 : \langle w, z \rangle = 0 \} \]
is another point in $\mathbb{P}(\mathbb{C}^2)$. 