4.4 THE COMPLEX PLANE

4.1 Meromorphic functions.

A *entire* function is an analytic function from the complex plane to itself. Suppose that \( f : \mathbb{C} \rightarrow \mathbb{C}_\infty \) is a meromorphic function. Then it will have a finite or infinite sequence of poles \((z_n)\). These are isolated so, if there are infinitely many, they must converge to \( \infty \). The following theorem shows that any such sequence of poles can occur.

**Theorem 4.1.1** Mittag-Leffler expansions

Let \((z_n)\) be a sequence of points in \( \mathbb{C} \) which is either finite or else converges to \( \infty \). For each \( n \) let \( p_n \) be a polynomial. Then there is a meromorphic function \( f : \mathbb{C} \rightarrow \mathbb{C}_\infty \) which has a pole at each \( z_n \) with principal part \( p_n((z-z_n)^{-1}) \) and no other poles. Any two such functions differ by an entire function.

**Proof:**

For any polynomial \( q_n \) the function \( p_n((z-z_n)^{-1}) - q_n(z) \) has the same principal part at \( z_n \) as \( p_n((z-z_n)^{-1}) \). We will show that we can choose the \( q_n \) so that the series \( \sum p_n((z-z_n)^{-1}) - q_n(z) \) converges locally uniformly. The function it converges to will then have the required properties. If two functions \( f_1 \) and \( f_2 \) have these properties then their difference has no poles and so is entire.

If there are only finitely many poles then we can take each \( q_n \) equal to 0. The finite sum \( \sum p_n((z-z_n)^{-1}) - q_n(z) \) clearly gives a rational function with the desired behaviour at each pole. From now on we will assume that the sequence \((z_n)\) is infinite and converges to \( \infty \). Let \((M_n)\) be a sequence of positive numbers with \( \sum M_n \) finite. For each \( n \) the function \( p_n((z-z_n)^{-1}) \) is analytic on the disc \( \{ z : |z| < |z_n| \} \), so its Taylor series converges uniformly on the disc \( \{ z : |z| \leq \frac{1}{2}|z_n| \} \). Take \( q_n \) to be a partial sum of this Taylor series with
\[
|p_n((z-z_n)^{-1}) - q_n(z)| \leq M_n \quad \text{for} \quad |z| \leq \frac{1}{2}|z_n|.
\]

For each \( R > 0 \) there are only finitely many \( n \) with \( |z_n| < R \). The finite sum \( \sum (p_n((z-z_n)^{-1}) - q_n(z) : |z_n| < R) \) therefore gives a rational function which has the correct principal parts at each \( z_n \) with \( |z_n| < R \) and no other poles. The sum \( \sum (p_n((z-z_n)^{-1}) - q_n(z) : |z_n| \geq R) \) converges uniformly on \( \{ z : |z| \leq \frac{1}{2}R \} \) by comparison with \( \sum M_n \). So it gives an analytic function on \( \{ z : |z| \leq \frac{1}{2}R \} \). Since \( R \) is arbitrary, the full series \( \sum p_n((z-z_n)^{-1}) - q_n(z) \) converges giving a meromorphic function with poles at each \( z_n \) having principal part \( p_n((z-z_n)^{-1}) \) and no other poles. \( \Box \)

**Exercises**

1. Give an example to show that the series \( \sum p_n((z-z_n)^{-1}) \) in the theorem need not converge.

2. Show that any sequence of points \((z_n)\) in \( \mathbb{D} \) with \( |z_n| \rightarrow 1^- \) as \( n \rightarrow \infty \) is the sequence of poles of a meromorphic function \( f : \mathbb{D} \rightarrow \mathbb{C}_\infty \).
4.2 Entire functions.

Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function. If \( f \) has no zeros then the monodromy theorem 2.3.2 shows that we may find an entire function \( g \) with \( f = \exp g \). If \( f \) has finitely many zeros \( z_1, z_2, \ldots, z_N \), each repeated according to its multiplicity, then

\[
f(z) = F(z) \prod_{n=1}^{N} (z - z_n)
\]

for an entire function \( F \) with no zeros. We wish to find a similar formula when \( f \) has infinitely many zeros. To do this we will need to consider functions defined by infinite products.

Let \((u_n)\) be a sequence of non-zero complex numbers. We will say that the infinite product \( \prod_{n=1}^{\infty} u_n \) converges to \( L \neq 0 \) if the sequence of partial products \( L_N = \prod_{n=1}^{N} u_n \) converges to \( L \) as \( N \to \infty \). For this to happen we must have \( u_n \to 1 \) so it is convenient to write \( u_n = 1 + a_n \). If \( a_n \to 0 \) then there will be a \( N_o \) with \(|a_n| < 1\) for \( n > N_o \). Let \( \log : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C} \) be the principal branch of the logarithm. Then

\[
L_N = L_{N_o} \prod_{n=N_o+1}^{N} (1 + a_n) = \exp \sum_{n=N_o+1}^{N} \log(1 + a_n)
\]

for \( N > N_o \). Hence the product \( \prod_{n=1}^{\infty} (1 + a_n) \) converges if, and only if, \( a_n \to 0 \) and the series \( \sum_{n=N_o}^{\infty} \log(1 + a_n) \) converges. This enables us to transfer results about series to products. For any sequence of complex numbers \( u_n \), including 0, we say that the product \( \prod u_n \) converges if there exists \( n_o \) with \( u_n \neq 0 \) for \( n > n_o \) and \( \prod_{n=n_o}^{\infty} u_n \) converges.

Note in particular that \( \log(1 + a) \) is asymptotic to \( a \) as \( a \to 0 \) so the series

\[
\sum_{n=N_o}^{\infty} \log(1 + a_n) \text{ converges absolutely if, and only if, the series } \sum |a_n| \text{ converges.}
\]

Suppose that \((a_n : \Omega \to \mathbb{C})\) is a sequence of analytic functions on the domain \( \Omega \) and that \( \sum M_n \) is a convergent series. If \(|a_n(z)| < M_n \) for \( z \in \Omega \), then the series \( \sum |a_n(z)| \) converges uniformly and \( a_n(z) \) converges uniformly to 0. Consequently the series \( \sum_{n=n_o}^{\infty} \log(1 + a_n(z)) \) will converge uniformly to an analytic function for \( n_o \) large enough. This proves that the product \( \prod (1 + a_n(z)) \) converges on \( \Omega \) to an analytic function which has zeros at the points where \((1 + a_n(z)) = 0\) for some \( n \).

If \((z_n)\) is an infinite sequence of points in \( \mathbb{C} \) which converges to \( \infty \) then the product

\[
\prod_{n=1}^{\infty} \left( 1 - \frac{z}{z_n} \right)
\]

need not converge. However, if \( \sum |z_n|^{-1} \) converges, then the product will converge to an entire function with zeros precisely at the points \( z_n \). To deal with sequences \((z_n)\) which have \( \sum |z_n|^{-1} \) divergent we need to introduce exponential factors into the product.

**Theorem 4.2.1** Weierstrass products

Let \((z_n)\) be a sequence of points in \( \mathbb{C} \) which is either finite or else tends to \( \infty \). Then there is an entire function \( f \) which has a zero at each point \( \zeta \) in the sequence with order equal to the number of times that it occurs in the sequence, and no other zeros. If \( g \) is another such function then \( f(z) = g(z) \exp h(z) \) for some entire function \( h \).

**Proof:**

Choose positive numbers \( M_n \) for which \( \sum M_n \) converges. The function \( z \mapsto \log \left( 1 - \frac{z}{z_n} \right) \) is analytic on \( \{z : |z| < |z_n|\} \) so its Taylor series

\[
-\frac{z}{z_n} - \frac{1}{2} \left( \frac{z}{z_n} \right)^2 - \frac{1}{3} \left( \frac{z}{z_n} \right)^3 - \ldots
\]
converges uniformly on \( \{ z : |z| \leq \frac{1}{2}|z_n| \} \). Hence we can choose natural numbers \( N(n) \) so that

\[
q_n(z) = \frac{z}{z_n} + \frac{1}{2} \left( \frac{z}{z_n} \right)^2 + \frac{1}{3} \left( \frac{z}{z_n} \right)^3 + \ldots + \frac{1}{N(n)} \left( \frac{z}{z_n} \right)^{N(n)}
\]
satisfies

\[
|\log \left( 1 - \frac{z}{z_n} \right) + q_n(z)| \leq M_n \quad \text{for} \quad |z| \leq \frac{1}{2}|z_n|.
\]

Therefore, the series

\[
\sum_{n=1}^{\infty} \left( \log \left( 1 - \frac{z}{z_n} \right) + q_n(z) \right)
\]

will converge locally uniformly. Hence,

\[
f(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{z_n} \right) \exp q_n(z)
\]

converges and gives an entire function \( f \) with the desired properties.

If \( g \) were another such function then \( g/f \) would be an entire function with no zeros and therefore equal to \( \exp h \) for some entire function \( h \). \( \square \)

**Corollary 4.2.2**

*Every meromorphic function \( f : \mathbb{C} \to \mathbb{C}_\infty \) is the quotient \( a/b \) of two entire functions \( a \) and \( b \).*

**Proof:**

The theorem enables us to construct an entire function \( b \) whose zeros are poles of \( f \). Then \( a = b f \) is also entire. \( \square \)

As an example, let us try to construct a entire function with zeros at the integer points. The series \( \sum n^{-2} \) converges so the proof of Weierstrass theorem shows that

\[
f(z) = z \prod_{n \neq 0} \left( 1 - \frac{z}{n} \right) e^{z/n}
\]

converges to the desired entire function. We can rewrite this series as

\[
f(z) = z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right).
\]

Because of the locally uniform convergence we can differentiate the product to obtain

\[
f'(z) = f(z) \left\{ \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z - n} + \frac{1}{n} \right) \right\}
\]

\[
= f(z) \left\{ \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{2z}{z^2 - n^2} \right) \right\}
\]

Hence \( f'(z) = f(z) \epsilon_1(z) = f(z) \pi \cot \pi z \). We also have \( f'(0) = 1 \) so we can solve this differential equation to obtain

\[
z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) = f(z) = \frac{\sin \pi z}{\pi}.
\]
Exercises

3. Show that the product

\[ g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \]

converges and satisfies

\[ g'(z) = g(z) \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right). \]

Deduce that \( g(z + 1) = -z g(z) e^\gamma \) for some constant \( \gamma \) and prove that

\[ \gamma = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n} - \log N. \]

(This is Euler’s constant.)
4.3 Quotients of the complex plane.

Theorem 4.3.1

The group $\text{Aut } \mathbb{C}$ consists of the maps $z \mapsto az + b$ for $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$.

Proof:

Suppose that $T : \mathbb{C} \rightarrow \mathbb{C}$ is conformal. Then we can consider it acting on $\mathbb{C}_\infty$ with an isolated singularity at $\infty$ and show that it has a removable singularity there. The set $U = T^{-1}(\mathbb{D})$ is open in $\mathbb{C}$ and $T$ maps every point of $\mathbb{C} \setminus U$ into $\{z \in \mathbb{C} : |z| \geq 1\}$. Hence the map $S : z \mapsto 1/T(z^{-1})$ is bounded on a neighbourhood of 0 and so must have a removable singularity there. Consequently $T$ extends to an analytic map $T : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$. We know from Theorem 3.2.1 that $T$ must be a rational function and the only ones which restrict to give a conformal map $\mathbb{C} \rightarrow \mathbb{C}$ are those of the form $z \mapsto az + b$ with $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$. \[\square\]

Suppose that $G$ is a subgroup of $\text{Aut } \mathbb{C}$ for which the quotient $\mathbb{C}/G$ is a Riemann surface. Then Theorem 2.3.6 shows that every element of $G \setminus \{I\}$ has no fixed points. The only maps $z \mapsto az + b$ which have this property are those with $a = 1$: the translations. Thus $G$ is a subgroup of the group of translations: $\{z \mapsto z + b : b \in \mathbb{C}\}$. The set $\Lambda = \{T(0) : T \in G\}$ is then an additive subgroup of $\mathbb{C}$ isomorphic to $G$. For $\mathbb{C}/G$ to be a Riemann surface we certainly need $0 \in \mathbb{C}$ to be isolated in $\Lambda = G(0)$ so there is a $\delta > 0$ with $|\lambda| > 2\delta$ for each $\lambda \in \Lambda \setminus \{0\}$. Conversely, if this is true, then the neighbourhood $U = \{z \in \mathbb{C} : |z - w| < \delta\}$ of any point $w \in \mathbb{C}$ has all the sets $T(U)$ for $T \in G$ disjoint, so $\mathbb{C}/G$ is a Riemann surface by Theorem 2.3.6.

We will often identify $G$ with $\Lambda$ and write $\mathbb{C}/\Lambda$ for $\mathbb{C}/G$. We have shown that this quotient is a Riemann surface if $\inf(|\lambda| : \lambda \in \Lambda \setminus \{0\}) > 0$. Any additive subgroup of $\mathbb{C}$ with this property is called a lattice in $\mathbb{C}$.

Theorem 4.3.2

A subset $\Lambda$ of $\mathbb{C}$ is a lattice if, and only if, it is of one of the three forms:

(a) $\{0\}$.
(b) $\mathbb{Z}\omega_1 = \{n\omega_1 : n \in \mathbb{Z}\}$ for some $\omega_1 \in \mathbb{C} \setminus \{0\}$.
(c) $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\}$ for some $\omega_1, \omega_2 \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$.

In these three cases we have:

(a) $\mathbb{C}/\{0\} = \mathbb{C}$.
(b) $\mathbb{C}/\mathbb{Z}\omega_1$ is conformally equivalent to the infinite cylinder $\mathbb{C} \setminus \{0\}$.
(c) $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ is a compact Riemann surface homeomorphic to a torus.

In case (c) we call $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ an analytic torus. There are many conformally different analytic tori.

Proof:

If $\Lambda = \{0\}$ then (a) holds and $\mathbb{C}/\{0\}$ is clearly $\mathbb{C}$. Otherwise we can choose $\omega \in \Lambda \setminus \{0\}$ with $|\omega|$ smallest. Let this be $\omega_1$. If $\Lambda = \mathbb{Z}\omega_1$ then (b) holds and the mapping $\mathbb{C}/\mathbb{Z}\omega_1 \rightarrow \mathbb{C} \setminus \{0\} : [z] \mapsto \exp(2\pi iz/\omega_1)$
is conformal. Otherwise we can choose \( \omega \in \Lambda \setminus \mathbb{Z} \omega_1 \) with \( |\omega| \) smallest. Let this be \( \omega_2 \).

Suppose that \( \omega_1, \omega_2 \) were not linearly independent over \( \mathbb{R} \). Then \( \omega_2 = x \omega_1 \) for some \( x \in \mathbb{R} \). We can write \( x = n + q \) with \( n \in \mathbb{Z} \) and \( 0 \leq q < 1 \). Then \( \omega_2 - n \omega_1 = q \omega_1 \in \Lambda \). The definition of \( \omega_1 \) implies that \( q \) must be zero and then this contradicts \( \omega_2 \notin \mathbb{Z} \omega_1 \). Hence \( \omega_1, \omega_2 \) are linearly independent.

If \( \Lambda = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \) then (c) holds. The space \( C/(\mathbb{Z} \omega_1 + \mathbb{Z} \omega_2) \) is easily seen to be homeomorphic to the space obtained by identifying the opposite sides of the fundamental parallelogram \( P = \{ x \omega_1 + y \omega_2 : 0 \leq x, y \leq 1 \} \). This is clearly a torus.

It remains to show that we cannot have any elements \( \omega \in \Lambda \setminus (\mathbb{Z} \omega_1 + \mathbb{Z} \omega_2) \). Suppose that we did, then \( \omega = x \omega_1 + y \omega_2 \) for some \( x, y \in \mathbb{R} \). We can choose \( n, m \in \mathbb{Z} \) with \( |x - n|, |y - m| \leq \frac{1}{2} \). Then

\[
|\omega - (n \omega_1 + m \omega_2)| = |(x - n) \omega_1 + (y - m) \omega_2|.
\]

The triangle inequality shows that this is less than

\[
\frac{1}{2} |\omega_1| + \frac{1}{2} |\omega_2| \leq |\omega_1|,
\]

and the inequality must be strict because \( \omega_1, \omega_2 \) are linearly independent over \( \mathbb{R} \). This contradicts the definition of \( \omega_1 \). \( \square \)

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**Exercises**

4. A function \( f : \mathbb{C} \rightarrow \mathbb{C} \) is *periodic with period* \( p \) if \( f(z + p) = f(z) \) for every \( z \in \mathbb{C} \). Show that the set of periods of an analytic function \( f \) is either a lattice in \( \mathbb{C} \) or else all of \( \mathbb{C} \).

5. Show that every analytic function \( f : \mathbb{C} \rightarrow \mathbb{C} \) which is periodic with a period \( p \neq 0 \) has a Fourier expansion \( f(z) = \sum_{n=-\infty}^{\infty} a_n \exp(2\pi i n z / p) \) convergent everywhere.

6. Show that for any subset \( E \) of \( \mathbb{C} \setminus \{0\} \) which has no accumulation points except possibly 0 or \( \infty \) there is a meromorphic function on \( \mathbb{C} \setminus \{0\} \) with poles precisely at the points of \( E \).