GEOMETRY AND GROUPS – Example Sheet 3

TKC Michaelmas 2012

1. Show that the only elements of $\mathrm{Isom}^+(\mathbb{E}^3)$ with order 2 are rotations about a straight line through an angle π. These are called *involutions*. Show that every orientation preserving Euclidean isometry $T \in \mathrm{Isom}^+(\mathbb{E}^3)$ can be written as the composite $R_2 \circ R_1$ for two involutions R_1, R_2.
2. Show from the formula
$$J(\boldsymbol{x}) = \boldsymbol{c} + \left(\frac{r^2}{||\boldsymbol{x} - \boldsymbol{c}||^2} \right) (\boldsymbol{x} - \boldsymbol{c}) \ .$$
for inversion in the sphere $S(\boldsymbol{c}, r)$ that inversion maps a sphere to another sphere.
3. Let J be inversion in a sphere Σ and Q inversion in the unit sphere S^2. Show that Σ is orthogonal to S^2 if and only if $J \circ Q = Q \circ J$.
4. Draw the set of points that lie within a fixed hyperbolic distance ρ_o of a geodesic α in the unit disc $\mathbb{D}$ and in the unit ball B^3.
5. Show that the translation length of the transformation $M_k : z \mapsto kz$ is $\log |k|$. Hence show how to find the translation length of the Möbius transformation
$$z \mapsto \frac{2z+1}{5z+3} \ .$$
6. Let R_1, R_2 be involutions with axes α_1, α_2 in $\mathbb{H}^3$ that do not meet either in $\mathbb{H}^3$ or on its boundary. Show that $R_2 \circ R_1$ is hyperbolic when both α_1 and α_2 lie in a hyperbolic plane.
7. Suppose that T is a Möbius transformation that maps the unit disc $\mathbb{D}$ onto itself. Then T also acts as an isometry of the hyperbolic 3-space B^3. How are fundamental sets for $G = \langle T \rangle$ acting on $\mathbb{D}$ related to fundamental sets for G acting on B^3?
8. Let Δ be a triangle in the hyperbolic plane $\mathbb{H}^2$ with vertices A, B, C, angles α, β, γ and sides with hyperbolic length a, b, c.

   Suppose first that $A = 0$ and the triangle is in the unit disc $\mathbb{D}$. Show that
$$\tanh \tfrac{1}{2} c = |B| \ ; \quad \tanh \tfrac{1}{2} b = |C| \ ; \quad \tanh \tfrac{1}{2} a = \left| \frac{C - B}{1 - \overline{B} C} \right| \ .$$
   Use this to find formulae for $\cosh a, \cosh b, \cosh c$ and $\sinh a, \sinh b, \sinh c$.

   Deduce that, in any hyperbolic triangle we have the *first hyperbolic cosine rule*:
$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha \ .$$
   Find the length of the hypotenuse of a right-angled hyperbolic triangle in terms of the other two side lengths.

   Now fix A, α, β and consider the angle γ as a function of c. Show that γ is a strictly decreasing function of c. Deduce that there is a hyperbolic triangle with angles α, β, γ if and only if $\alpha + \beta + \gamma < \pi$. Is this triangle unique up to hyperbolic isometry?
9. Let G be a discrete group of Möbius transformations. An *invariant disc* for G is a disc which every element of G maps into itself. Show that G can not have an invariant disc if it contains a loxodromic transformation. Show also that there is group G that contains no loxodromic transformations but still has no invariant disc. [*Hint: Look for groups G generated by two transformations.*]
10. Let C_0, C_1, C_2, C_3 be four circles with C_i tangent to C_{i+1} at the point z_i for $i \equiv 0, 1, 2, 3 \pmod 4$ and there are no other points of tangency. Prove that z_0, z_1, z_2, z_3 all lie on a circle.
11. Show that there is an isometry T of $\mathbb{H}^2$ taking the pair of points (a, b) to the pair (u, v) if, and only if, $\rho(a, b) = \rho(u, v)$. Is this still true for pairs of points in $\mathbb{H}^3$?
12. Let ℓ, ℓ' be two hyperbolic geodesics. Draw the points m that are equidistant from ℓ and ℓ'. Show that, in a hyperbolic triangle, the three angle bisectors meet at a point.
13. Give an example of an elliptic element of a Kleinian group with fixed points that do not lie in the limit set. Give an example of a Kleinian group for which the limit set is empty.

14. Let G be a Kleinian group with an invariant disc $\Delta \subset \mathbb{P}$. Show that the limit set of G is a subset of $\partial\Delta$.

15. The *Gaussian integers* are $\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\}$. Let G be the set of Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{Z}[i]$ and $ad - bc = 1$. Prove that G is a discrete group of Möbius transformations.

    For each point $w = \frac{p+iq}{r}$ with $p, q, r \in \mathbb{Z}$, find a parabolic transformation $T \in G$ that fixes w. Deduce that w is in the limit set for G and hence that the limit set is all of the Riemann sphere.

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