2. THE SPHERE

2.1 The geometry of the sphere

Let $S = S^2 = \{x \in \mathbb{R}^3 : ||x|| = 1\}$ be the unit sphere.

A plane through the origin cuts $S$ in a *great circle*. We call these *spherical lines*. 
Proposition 2.1  Geodesics on the sphere

For any two points $P, Q \in S^2$, the shortest path from $P$ to $Q$ follows the shorter arc of a spherical line through $P$ and $Q$. The length of this path is

$$d(P, Q) = \cos^{-1} P \cdot Q .$$

This gives a metric $d(\cdot, \cdot)$ on the sphere.
2.2 Spherical isometries

**Proposition 2.2**  Isometries of $S^2$

*Every isometry of $S^2$ is of the form $x \mapsto R(x)$ for an orthogonal linear map $R : \mathbb{R}^3 \to \mathbb{R}^3$.***
2.3 Spherical Triangles

**Proposition 2.3**  Gauss – Bonnet theorem for spherical triangles

For a triangle $\Delta$ on the unit sphere $S^2$ with area $A(\Delta)$ we have

$$\alpha + \beta + \gamma = \pi + A(\Delta).$$
Suppose that we have a polygon $P$ with $N$ sides each of which is an arc of a spherical line. (We will only consider the case where $N$ is at least 1 and the sides of the polygon do not cross one another, so $P$ is simply connected.) If the internal angles of the polygon are $\theta_1, \theta_2, \ldots, \theta_N$, then we can divide it into $N - 2$ triangles and obtain

$$\theta_1 + \theta_2 + \ldots + \theta_N = (N - 2)\pi + A(P).$$

Now consider dividing the entire sphere into a finite number of polygonal faces by drawing arcs of spherical lines on the sphere. Let the number of polygonal faces be $F$, the number of arcs of spherical lines (edges) be $E$, and the number of vertices of the polygons $V$. The Euler number for this subdivision is $F - E + V$.

**Proposition 2.4**  Euler’s formula for the sphere

Let the sphere be divided into $F$ simply connected faces by drawing $E$ arcs of spherical lines joining $V$ vertices on $S^2$. Then

$$F - E + V = 2.$$
Proposition 2.5   Spherical Cosine Rule I

For a spherical triangle $\Delta$

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$  

Proposition 2.6   The Spherical Sine rule

For a spherical triangle $\Delta$

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$
Proposition 2.7  Dual spherical triangles

Let $\Delta$ be a spherical triangle with angles $\alpha, \beta, \gamma$ and side lengths $a, b, c$. Then the dual triangle $\Delta^*$ has sides of length $a^* = \pi - \alpha, b^* = \pi - \beta, c^* = \pi - \gamma$ and angles $\alpha^* = \pi - a, \beta^* = \pi - b, \gamma^* = \pi - c$.

Corollary 2.8  Spherical Cosine Rule II

For a spherical triangle $\Delta$

$$\cos \alpha = - \cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a.$$
2.4 The Projective Plane