GEOMETRY — Example Sheet 3

1. Let $U$ be an open subset of $\mathbb{R}^2$ with the Riemannian metric

$$ds^2 = E \, dx_1^2 + 2F \, dx_1 \, dx_2 + G \, dx_2^2.$$ 

For any point $P \in U$, show that there is a $\lambda > 0$ and a neighbourhood $N$ of $P$ with

$$(E - \lambda) \, dx_1^2 + 2F \, dx_1 \, dx_2 + (G - \lambda) \, dx_2^2$$

a Riemannian metric on $N$.

[Hint: A real matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ is positive definite if and only if $a > 0$ and $ad - b^2 > 0$.]

If $U$ is path-connected, we define the distance between two points of $U$ as the infimum of the lengths of all curves in $U$ between those two points. Give an example where this distance is not realised as the length of any curve in $U$ between the two points.

2. Consider the Riemannian metric

$$ds^2 = \frac{dx_1^2 + dx_2^2}{1 - (x_1^2 + x_2^2)}$$

on the unit disc $\mathbb{D}$. Prove that diameters of the disc are length minimising curves and hence geodesics. Show that the distance between points is bounded but areas are unbounded.

3. Let $U = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1|, |x_2| < 1 \}$ and consider the two Riemannian metrics

$$\frac{dx_1^2}{(1 - x_1^2)^2} + \frac{dx_2^2}{(1 - x_2^2)^2} \quad \text{and} \quad \frac{dx_1^2}{(1 - x_1^2)^2} + \frac{dx_2^2}{(1 - x_1^2)^2}$$

on $U$. Prove that there is no isometry between the two spaces but that an area preserving diffeomorphism does exist.

[Consider the length of curves going out to the boundary.]

4. For the unit sphere $S$ in $\mathbb{R}^3$, find the unit normal at a point $x$, the tangent plane at $x$ and the intersection of planes parallel to the tangent plane with $S$.

5. Show that

$$r : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3 : (u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u)$$

is a surface parametrisation. Describe the image. What is the corresponding Riemannian metric?

6. Let $T$ denote the torus obtained by rotating the circle $\{(x, 0, z) \in \mathbb{R}^3 : (x - 2)^2 + z^2 = 1 \}$ about the $z$-axis. Describe a surface parametrisation for $T$ and hence calculate its area.

7. Prove directly that the hyperbolic lines satisfy the differential equations for geodesics in the hyperbolic plane.

8. For $a > 0$, let $C(a)$ be the cone:

$$C(a) = \{(x, y, z) \in \mathbb{R}^3 : z^2 = a(x^2 + y^2) \text{ and } z > 0 \}.$$ 

Find a parametrisation for $C(a)$ and hence find the geodesics on $C(a)$.

When $a = 3$, show that no (infinite) geodesic intersects itself. When $a > 3$, show that there are geodesics that intersect themselves.

9. Let $\sigma = (\sigma_1, \sigma_2) : (a, b) \to \{(x, y) \in \mathbb{R}^2 : y > 0 \}$ be a unit speed curve in the upper half-plane that does not intersect itself and maps the open interval $(a, b)$ homeomorphically onto its image. The surface of revolution $R$ is then obtained by rotating $\sigma$ about the $x$-axis. Show that

$$(s, t) \mapsto (\sigma_1(t), \sigma_2(t) \cos s, \sigma_2(t) \sin s)$$

is a surface parametrisation for part of $R$. Calculate the Riemannian metric and the second fundamental form. Hence show that the Gaussian curvature is given by

$$K = \frac{\sigma''_2(t)}{\sigma_2(t)}.$$
10. Using the formulae from the previous question, calculate the Gaussian curvature for a sphere, for the hyperboloid of one sheet:
\[ x^2 + y^2 - z^2 = +1 \]
and the hyperboloid of two sheets:
\[ x^2 + y^2 - z^2 = -1 \]
For the torus described in question 6, mark the points where the Gaussian curvature \( K \) satisfies \( K < 0; K = 0 \) and \( K > 0 \).

11. Let \( R \) be a surface in \( \mathbb{R}^3 \) that is closed and bounded. Explain why there is a point \( Q \) of \( R \) at a maximal distance \( d \) from the origin. By considering the sphere \( S \) centred on the origin and of radius \( d \), or otherwise, show that the Gaussian curvature of \( R \) is strictly positive at \( Q \). Hence the closed and bounded surface \( R \) can not have Gaussian curvature less than or equal to 0 at every point.

12. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a smooth function with
\[ f(0,0) = 0 \quad \frac{\partial f}{\partial x}(0,0) = 0 \quad \frac{\partial f}{\partial y}(0,0) = 0 \]
Let \( r \) be the surface parametrisation:
\[ r : (x,y) \mapsto (x,y,f(x,y)) \]
Show that the Riemannian metric at the origin is
\[ ds^2 = dx^2 + dy^2 \]
and the second fundamental form is
\[ \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} \]
(for a suitable choice of the unit normal) where all of the partial derivatives are evaluated at \( (0,0) \). Deduce that the Gaussian curvature at the origin is
\[ K = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial y^2} \right)^2 \]
Now suppose that \( g : \mathbb{R}^2 \to \mathbb{R} \) is another smooth function with \( g(0,0) = 0 \) and \( g(x,y) \geq f(x,y) \) for every \( (x,y) \in \mathbb{R}^2 \). Show that
\[ \frac{\partial g}{\partial x}(0,0) = 0 \quad \frac{\partial g}{\partial y}(0,0) = 0 \]
Show further that
\[ \frac{\partial^2 g}{\partial x^2} u^2 + 2 \frac{\partial^2 g}{\partial x \partial y} uv + \frac{\partial^2 g}{\partial y^2} v^2 \geq \frac{\partial^2 f}{\partial x^2} u^2 + 2 \frac{\partial^2 f}{\partial x \partial y} uv + \frac{\partial^2 f}{\partial y^2} v^2 \]
at \( (0,0) \) and deduce that
\[ \begin{pmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial y^2} \end{pmatrix} \geq \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \]
at \( (0,0) \).
Does this imply that the Gaussian curvature of the graph of \( g \) at the origin is greater than or equal to the Gaussian curvature of the graph of \( f \) at the origin.

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Supervisors can obtain an annotated version of this example sheet from DPMMS.