GEOMETRY — Example Sheet 2

TKC Lent 2009

1. Prove that two points \( w, z \in \mathbb{C}_\infty \) correspond to antipodal points in \( S^2 \) under stereographic projection if, and only if, \( w = J(z) \) for the transformation \( J(z) = -1/z \).

Show that any M"obius transformation \( T \) other than the identity has either one or two fixed points on \( \mathbb{C} \cup \{\infty\} \). Show that the M"obius transformation corresponding under stereographic projection to a non-trivial rotation has two antipodal fixed points.

Show that a M"obius transformation \( T : z \mapsto (az + b)/(cz + d) \) with \( ad - bc = 1 \) satisfies \( J^{-1}TJ = T \) precisely when \( d = \pi \) and \( c = -\bar{b} \).

2. Prove that M"obius transformations of the extended complex plane \( \mathbb{C}_\infty \) preserve cross-ratios. Let the points \( u, v \in \mathbb{C} \) correspond under stereographic projection to points \( P, Q \in S^2 \). Show that the cross-ratio of the four points \( u, v, -1/\pi, -1/\pi \) (in some order) is equal to \( -\tan^2 \frac{1}{2}d(P, Q) \), where \( d(P, Q) \) is the spherical distance between \( P \) and \( Q \).

3. Let \( J : z \mapsto 1/\bar{z} \) be inversion in the unit circle and recall that M"obius transformations map inverse points to inverse points.

Show that, a M"obius transformation \( T \) maps the unit circle onto itself if and only if \( J^{-1}TJ = T \). Deduce that a M"obius transformation

\[
T : z \mapsto \frac{az + b}{cz + d} \quad \text{with} \quad ad - bc = 1
\]

maps the unit disc \( \mathbb{D} \) onto itself if and only if \( d = \pi \) and \( c = -\bar{b} \). Show that every such transformation is an isometry for the hyperbolic metric.

Show that we can also write these M"obius transformations as

\[
z \mapsto \frac{z - z_0}{1 - z_\zeta \bar{z}}
\]

for some \( z_0 \in \mathbb{D} \) and some \( \zeta \in \mathbb{C} \) of modulus 1.

4. Let \( \Gamma \) be the hyperbolic circle \( \{ z \in \mathbb{D} : \rho(z, z_0) = \rho_0 \} \) in the disc \( \mathbb{D} \). Show that it is also an Euclidean circle and a spherical circle but that the Euclidean or spherical centre and radius can be different from the hyperbolic centre \( z_0 \) and radius \( \rho_0 \).

5. Show that a hyperbolic circle with hyperbolic radius \( r \) has length \( 2\pi \sinh r \) and encloses a disc of hyperbolic area \( 4\pi \sinh^2 \frac{1}{2}r \). Sketch these as functions of \( r \).

6. Show that two hyperbolic lines have a common orthogonal line if and only if they are ultraparallel.

Prove that, in this case, the common orthogonal line is unique.

7. Fix a point \( P \) on the boundary of the unit disc \( \mathbb{D} \). Describe the curves in \( \mathbb{D} \) that are orthogonal to every hyperbolic line that passes through \( P \).

8. Prove that a hyperbolic \( N \)-gon with interior angles \( \alpha_1, \alpha_2, \ldots, \alpha_N \) has area \( (N - 2)\pi - \sum \alpha_j \). Show that, for every \( N \geq 3 \) and every \( \alpha \) with \( 0 < \alpha < (1 - \frac{2}{N})\pi \), there is a regular \( N \)-gon with all angles equal to \( \alpha \).

9. Show that in a spherical, Euclidean or hyperbolic triangle, the angle bisectors are lines and they meet at a point.

10. Let \( \ell \) and \( m \) be two fixed hyperbolic lines that cross at an angle \( \alpha \) at a point \( A \). Another line \( n \) crosses \( \ell \) at a (movable) point \( B \) and a fixed angle \( \beta \). If \( n \) also crosses \( m \) at an angle \( \theta \), show that \( \theta \) varies monotonically as the point \( B \) moves along the line \( \ell \).

Deduce that there is a hyperbolic triangle with angles \( \alpha, \beta, \gamma \) provided that \( \alpha + \beta + \gamma < \pi \).

11. State the sine rule for hyperbolic triangles. Show that \( a \leq b \leq c \) if and only if \( a \leq \beta \leq \gamma \).

12. If \( w, z \) are points in the upper half-plane, prove that the hyperbolic distance between them is \( 2 \tanh^{-1} |(w - z)/(w - \bar{z})| \).
13. In this question we will show how to deduce the sine rule and second cosine rule for a hyperbolic triangle from the first cosine rule.

Use the cosine rule to show that
\[ \cos \alpha = \frac{\cosh b \cosh c - \cosh a}{\sqrt{\cosh^2 b - 1} \sqrt{\cosh^2 c - 1}} \quad \text{and} \quad \sin^2 \alpha = \frac{D^2}{(\cosh^2 b - 1)(\cosh^2 c - 1)} \]
where \( D^2 = 1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2 \cosh a \cosh b \cosh c \). Deduce that
\[ \sin^2 \alpha = \frac{D^2}{\sinh^2 a} \]
Show that, since the right hand side is symmetric in \( a, b, c \), this implies the hyperbolic sine rule.

In a similar way, show that
\[ \cos \beta \cos \gamma + \cos \alpha = \frac{D^2 \cosh a}{(\cosh^2 a - 1) \sqrt{\cosh^2 b - 1} \sqrt{\cosh^2 c - 1}} \]
and deduce the second cosine rule:
\[ \cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cosh a \cdot \cosh b \cosh c \].

Deduce that two hyperbolic triangles are congruent if and only if they have the same angles.

14. Let \( \Delta \) be a triangle on a sphere of radius \( R \), with angles \( \alpha, \beta, \gamma \) and sides of length \( a, b, c \). Prove a version of the cosine and sine rules for this triangle.

Show that, if we formally set \( R \) equal to the complex number \( i \), then we obtain the hyperbolic cosine and sine rules. (Thus hyperbolic geometry is the geometry of a sphere with radius \( i \) and curvature \( R^2 = -1 \).)

15. The quaternions \( \mathbb{Q} \) consist of all \( 2 \times 2 \) complex matrices
\[ q = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \]
with addition and multiplication as for the matrices. Every such quaternion \( q \) can be written as \( q_0 \mathbf{1} + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} \) where
\[ \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad ; \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad ; \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \]

Show that these four elements, together with their additive inverses \(-\mathbf{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}\) form a non-commutative group: the Quaternion 8-group. We can identify the subspace of \( \mathbb{Q} \) spanned by \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) with \( \mathbb{R}^3 \) by making \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) correspond to the standard basis vectors of \( \mathbb{R}^3 \). We can then write any quaternion \( q \) as \( q_0 + \mathbf{v} \) for a scalar \( q_0 \) and a vector \( \mathbf{v} \in \mathbb{R}^3 \). Prove that we then have
\[ (p_0 \mathbf{1} + \mathbf{u})(q_0 \mathbf{1} + \mathbf{v}) = (p_0 q_0 + \mathbf{u} \cdot \mathbf{v}) \mathbf{1} + (p_0 v + q_0 u) + (\mathbf{u} \times \mathbf{v}) \]
In particular, for two vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^3 \) we have \( \mathbf{u} \mathbf{v} + \mathbf{v} \mathbf{u} = -2(\mathbf{u} \cdot \mathbf{v}) \mathbf{1} \).

The conjugate of a quaternion \( q = q_0 \mathbf{1} + \mathbf{v} \) is \( \overline{q} = q_0 \mathbf{1} - \mathbf{v} \). Show that \( q \overline{q} = ||q||^2 \mathbf{1} = \mathbf{q} \overline{q} \) where \( ||q||^2 = q_0^2 + ||\mathbf{v}||^2 \). Prove that, for any unit vector \( \mathbf{u} \in \mathbb{R}^3 \), we have
\[ \mathbf{u} \mathbf{v} \mathbf{u} = \mathbf{x} - 2(\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \]
So the map \( T_u : \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; \ x \mapsto \mathbf{u} \mathbf{v} \mathbf{u} \) is reflection in the plane perpendicular to \( \mathbf{u} \). By writing any isometry of \( S^2 \) as a composite of reflection, or otherwise, show that for each quaternion \( q \) with \( ||q|| = 1 \) the map
\[ T_q : \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; \ x \mapsto q \overline{q} \]
is an orientation preserving isometry of \( S^2 \). Hence show that
\[ T : S(\mathbb{Q}) \rightarrow SO(3) ; \ q \mapsto T_q \]
is a group homomorphism from the unit sphere \( S(\mathbb{Q}) \) (which is a 3-dimensional sphere \( S^3 \)) onto \( SO(3) \) with kernel \( \{-1, 1\} \).

Please send any comment or corrections to t.k.carne@dpmms.cam.ac.uk.

Supervisors can obtain an annotated version of this example sheet from DPMMS.