GEOMETRY — Example Sheet 3 — Extra questions for enthusiasts

TKC Easter 2002

1. Let ℓ be a hyperbolic line. Describe the set of points that lie within a distance r of ℓ. Is this set bounded by hyperbolic lines?
2. Given a regular dodecahedron D, show that we can choose a diagonal in each face so as to form the edges of a cube. Show that there are five such cubes inscribed in the dodecahedron and that the symmetry group of the dodecahedron permutes them. Show that only the identity and $J : \boldsymbol{x} \mapsto -\boldsymbol{x}$ map each cube to itself. Deduce that the symmetry group of the dodecahedron is isomorphic to the Cartesian product of the alternating group A_5 and the cyclic group C_2 of order 2.
3. The *real projective plane* $\mathbb{RP}^2$ or $\mathbb{P}(\mathbb{R}^3)$ is the set of all 1-dimensional vector subspaces of $\mathbb{R}^3$. A *line* in the real projective plane consists of all the 1-dimensional subspaces in a fixed 2-dimensional subspace of $\mathbb{R}^3$. Show that any two distinct lines meet at an unique point. Show that any invertible linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ induces a bijection from $\mathbb{RP}^2$ to itself that maps lines to lines.

   We can regard the real projective plane as the plane $\mathbb{R}^2$ with extra points added "at infinity". For the map
   $$\mathbb{R}^2 \to \mathbb{RP}^2 \ ; \quad (x, y) \mapsto \{\lambda(x, y, 1) : \lambda \in \mathbb{R}\}$$
   embeds the plane $\mathbb{R}^2$ as a subset of $\mathbb{RP}^2$ and sends each line in $\mathbb{R}^2$ into a line in $\mathbb{RP}^2$. Show that the image of this map is all of $\mathbb{RP}^2$ except for a single line (called the circle at infinity). Two parallel lines in the plane correspond to two lines in $\mathbb{RP}^2$ that meet at a point on this circle.

   We can also regard the real projective plane as a quotient of the unit sphere. For the map
   $$S^2 \to \mathbb{RP}^2 \ ; \quad (x, y, z) \mapsto \{\lambda(x, y, z) : \lambda \in \mathbb{R}\}$$
   sends two antipodal points of S^2 to each point of $\mathbb{RP}^2$. Show that each spherical line corresponds to a line in $\mathbb{RP}^2$. The metric on S^2 gives us a metric on $\mathbb{RP}^2$. What is the group of isometries for this metric? Prove versions of the cosine and sine rules for triangles in the real projective plane.
4. The *quaternions* $\mathcal{Q}$ consist of all 2×2 complex matrices
   $$q = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$
   with addition and multiplication as for the matrices. Every such quaternion q can be written as $q_0\mathbf{1} + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$ where
   $$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ; \ \boldsymbol{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ ; \ \boldsymbol{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ; \ \boldsymbol{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ .$$
   Show that these four elements, together with their additive inverses $-\mathbf{1}, -\boldsymbol{i}, -\boldsymbol{j}, -\boldsymbol{k}$ form a non-commutative group: the *Quaternion 8-group*. We can identify the subspace of $\mathcal{Q}$ spanned by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ with $\mathbb{R}^3$ by making $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ correspond to the standard basis vectors of $\mathbb{R}^3$. We can then write any quaternion q as $q_0\mathbf{1} + \boldsymbol{v}$ for a scalar q_0 and a vector $\boldsymbol{v} \in \mathbb{R}^3$. Prove that we then have
   $$(p_0\mathbf{1} + \boldsymbol{u})(q_0\mathbf{1} + \boldsymbol{v}) = (p_0q_0 - \boldsymbol{u} \cdot \boldsymbol{v})\mathbf{1} \ + \ (\boldsymbol{u} \times \boldsymbol{v}) \ .$$
   In particular, for two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ we have $\boldsymbol{uv} + \boldsymbol{vu} = -2(\boldsymbol{u} \cdot \boldsymbol{v})\mathbf{1}$.

   The *conjugate* of a quaternion $q = q_o\mathbf{1} + \boldsymbol{v}$ is $\overline{q} = q_0\mathbf{1} - \boldsymbol{v}$. Show that $q\overline{q} = ||q||^2\mathbf{1} = \overline{q}q$ where $||q||^2 = q_0^2 + ||\boldsymbol{v}||^2$. Prove that, for any unit vector $\boldsymbol{u} \in \mathbb{R}^3$, we have
   $$\boldsymbol{uxu} = \boldsymbol{x} - 2(\boldsymbol{x} \cdot \boldsymbol{u})\boldsymbol{u} \ .$$
   So the map $T_{\boldsymbol{u}} : \mathbb{R}^3 \to \mathbb{R}^3 \ ; \ \ \boldsymbol{x} \mapsto \boldsymbol{uxu}$ is reflection in the plane perpendicular to $\boldsymbol{u}$. By writing any isometry of S^2 as a composite of reflection, or otherwise, show that for each quaternion q with $||q|| = 1$ the map
   $$T_q : \mathbb{R}^3 \to \mathbb{R}^3 \ ; \ \ \boldsymbol{x} \mapsto q\boldsymbol{x}\overline{q}$$
   is an orientation preserving isometry of S^2. Hence show that
   $$T : S(\mathcal{Q}) \to \mathrm{SO}(3) \ ; \ \ q \mapsto T_q$$
   is a group homomorphism from the unit sphere $S(\mathcal{Q})$ (which is a 3-dimensional sphere S^3) onto $\mathrm{SO}(3)$ with kernel $\{-\mathbf{1}, \mathbf{1}\}$.

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Supervisors can obtain an annotated version of this example sheet from DPMMS.