GEOMETRY — Example Sheet 2

TKC Easter 2002

1. Prove that two points $w, z \in \mathbb{C}_\infty$ correspond to antipodal points in S^2 under stereographic projection if, and only if, $w = J(z)$ for the transformation $J(z) = -1/\overline{z}$.

   Show that a Möbius transformation $T : z \mapsto (az+b)/(cz+d)$ with $ad-bc = 1$ satisfies $J^{-1}TJ = T$ precisely when $d = \overline{a}$ and $c = -\overline{b}$.

   Which circles in $\mathbb{C}_\infty$ correspond to the great circles (that is, the geodesics) in S^2?

2. Prove that Möbius transformations of the extended complex plane $\mathbb{C}_\infty$ preserve cross-ratios. Let the points $u, v \in \mathbb{C}$ correspond under stereographic projection to points $\boldsymbol{P}, \boldsymbol{Q} \in S^2$. Show that the cross-ratio of the four points $u, v, -1/\overline{u}, -1/\overline{v}$ (in some order) is equal to $-\tan^2 \frac{1}{2} d(\boldsymbol{P}, \boldsymbol{Q})$, where $d(\boldsymbol{P}, \boldsymbol{Q})$ is the spherical distance between $\boldsymbol{P}$ and $\boldsymbol{Q}$.

3. If w, z are points in the upper half-plane, prove that the hyperbolic distance between them is $2\tanh^{-1}|(w-z)/(w-\overline{z})|$.

4. Let $J : z \mapsto 1/\overline{z}$ be inversion in the unit circle and recall that Möbius transformations map inverse points to inverse points.

   Show that, a Möbius transformation T maps the unit circle onto itself if and only if $J^{-1}TJ = T$. Deduce that a Möbius transformation

   $$T : z \mapsto \frac{az+b}{cz+d} \qquad \text{with} \qquad ad - bc = 1$$

   maps the unit disc $\mathbb{D}$ onto itself if and only if $d = \overline{a}$ and $c = \overline{b}$. Show that every such transformation is an isometry for the hyperbolic metric.

   Show that we can also write these Möbius transformations as

   $$z \mapsto \zeta \left(\frac{z - z_o}{1 - \overline{z_o} z} \right)$$

   for some $z_o \in \mathbb{D}$ and some $\zeta \in \mathbb{C}$ of modulus 1.

5. Let Γ be the hyperbolic circle $\{z \in \mathbb{D} : \rho(z, z_0) = \rho_o\}$ in the disc $\mathbb{D}$. Show that it is also an Euclidean circle and a spherical circle but that the Euclidean or spherical centre and radius can be different from the hyperbolic centre z_o and radius ρ_o.

6. Show that a hyperbolic circle with hyperbolic radius r has length $2\pi \sinh r$ and encloses a disc of hyperbolic area $4\pi \sinh^2 \frac{1}{2} r$. Sketch these as functions of r.

7. Show that two hyperbolic lines have a common orthogonal line if and only if they are ultraparallel. Prove that, in this case, the common orthogonal line is unique.

8. Fix a point P on the boundary of the unit disc $\mathbb{D}$. Describe the curves in $\mathbb{D}$ that are orthogonal to every hyperbolic line that passes through P.

9. Prove that a hyperbolic N-gon with interior angles $\alpha_1, \alpha_2, \ldots, \alpha_N$ has area $(N-2)\pi - \sum \alpha_j$. Show that, for every $N \geqslant 3$ and every α with $0 < \alpha < (1 - \frac{2}{N})\pi$, there is a regular N-gon with all angles equal to α.

10. Show that in a spherical, Euclidean or hyperbolic triangle, the angle bisectors are lines and they meet at a point.

11. Let ℓ and m be two fixed hyperbolic lines that cross at an angle α at a point $\boldsymbol{A}$. Another line n crosses ℓ at a (movable) point $\boldsymbol{B}$ and a fixed angle β. If n also crosses m at an angle θ, show that θ varies monotonically as the point $\boldsymbol{B}$ moves along the line ℓ.

    Deduce that there is a hyperbolic triangle with angles α, β, γ provided that $\alpha + \beta + \gamma < \pi$.

12. State the sine rule for hyperbolic triangles. Show that $a \leqslant b \leqslant c$ if and only if $\alpha \leqslant \beta \leqslant \gamma$.

13. *In this question we will show how to deduce the sine rule and second cosine rule for a hyperbolic triangle from the first cosine rule.*

    Use the cosine rule to show that

    $$\cos\alpha = \frac{\cosh b \cosh c - \cosh a}{\sqrt{\cosh^2 b - 1}\sqrt{\cosh^2 c - 1}} \qquad \text{and} \qquad \sin^2\alpha = \frac{D^2}{(\cosh^2 b - 1)(\cosh^2 c - 1)}$$

where $D^2 = 1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2\cosh a \cosh b \cosh c$. Deduce that

$$\frac{\sin^2 a}{\sinh^2 \alpha} = \frac{D^2}{(\cosh^2 a - 1)(\cosh^2 b - 1)(\cosh^2 c - 1)} \ .$$

Show that, since the right hand side is symmetric in a, b, c, this implies the hyperbolic sine rule.

In a similar way, show that

$$\cos\beta\cos\gamma + \cos\alpha = \frac{D^2 \cosh a}{(\cosh^2 a - 1)\sqrt{\cosh^2 b - 1}\sqrt{\cosh^2 c - 1}}$$

and deduce the second cosine rule:

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a \ .$$

Deduce that two hyperbolic triangles are congruent if and only if they have the same angles.

14. Let Δ be a triangle on a sphere of radius R, with angles α, β, γ and sides of length a, b, c. Prove a version of the cosine and sine rules for this triangle.

Show that, if we formally set R equal to the complex number i, then we obtain the hyperbolic cosine and sine rules. (Thus hyperbolic geometry is the geometry of a sphere with radius i and curvature $R^2 = -1$.)

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Supervisors can obtain an annotated version of this example sheet from DPMMS.