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FURTHER ANALYSIS

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CONTENTS

1. TOPOLOGICAL SPACES

Recall, from the Analysis course, that in a metric space (X, d) a subset U is open when, for each $x \in U$ there is an $r > 0$ with $B(x, r) \subset U$. We can use the notion of an open set to define convergence and continuity.

A subset V of X is a neighbourhood of x if there is an open set U with $x \in U \subset V$. This means that U is open if and only if U is a neighbourhood of each of its points.

A sequence (x_n) in X converges to a limit ℓ when, for each neighbourhood V of x, there is a natural number $N(V) \in \mathbb{N}$ with

$$x_n \in V \qquad \text{for} \quad n \geqslant N(V) \ .$$

A function $f : X \to Y$ between two metric spaces is continuous at $x_o \in X$ when $f^{-1}(V)$ is a neighbourhood of x_o for each neighbourhood V of $f(x_o)$. (Taking $V = B(f(x_o), \varepsilon)$ we see that this means that there is a $\delta > 0$ with $B(x_o, \delta) \subset f^{-1}(B(f(x_o), \varepsilon)$, that is

$$d(f(x), f(x_o)) < \varepsilon \quad \text{whenever} \quad d(x, x_o) < \delta \ .)$$

The function $f : X \to Y$ is continuous if it is continuous at each point of X. This is equivalent to demanding that

$$f^{-1}(U) \ \text{ is open in } X \ \text{ whenever } \ U \ \text{ is open in } \ Y \ .$$

In many contexts it is simpler to argue using open sets rather than the metric. There are also cases where we need to need to work with more general spaces than metric spaces. Hence we introduce the idea of topological spaces.

A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X that satisfies the three conditions:

(a) $\emptyset, X \in \mathcal{T}$;

(b) if $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$;

(c) if $\mathcal{U} \subset \mathcal{T}$, then the union $\bigcup \mathcal{U} \in \mathcal{T}$.

(Condition (b) implies that the intersection of finitely many sets in $\mathcal{T}$ is itself in $\mathcal{T}$. However, condition (c) means that any union of sets in $\mathcal{T}$, finite or infinite, is itself in $\mathcal{T}$.)

When $\mathcal{T}$ is a topology on X, we call $(X, \mathcal{T})$ a *topological space*. The sets in $\mathcal{T}$ are the *open* sets in X for the topology.

Examples of topologies.

1. Metric topology. All the open sets for a metric form a topology.

2. Discrete topology. The collection of all subsets of X form the discrete topology on X. This is also a metric topology.

3. Indiscrete topology. The collection $\{\emptyset, X\}$ forms the indiscrete topology. Provided that X has more than one point, it is not a metric topology.

4. Gate topology. Let $X = \{f : [0, 1] \to \mathbb{R}\}$. A subset U of X is open if, for each $f_o \in U$, there is a finite set $F \subset [0, 1]$ and an $\varepsilon > 0$ with

   $$\{f : [0, 1] \to \mathbb{R} : |f(t) - f_o(t)| < \varepsilon \ \text{ for all } \ t \in F\ \} \subset U \ .$$

   (These are the functions whose graphs pass through "gates" for each $t \in F$.)

We can use the open sets of a topology $\mathcal{T}$ on X to define convergence and continuity. First, we say that a subset F of X is *closed* when the complement $X \setminus F$ is open, that is $X \setminus F \in \mathcal{T}$. Note that subsets of X need not be either open or closed.

Proposition 1.1 Interior and closure
Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then there is a largest open set contained in A, called the interior A° of A. There is smallest closed set containing A, called the closure $\overline{A}$ of A. The boundary of A is $\partial A = \overline{A} \setminus A^\circ$ and is closed.

Proof:
The union $\bigcup \{U \in \mathcal{T} : U \subset A\}$ is a union of open sets so it is open. Clearly it is the largest open set contained in A, so it is the interior: A°.

Similarly, the intersection $\bigcap \{F : F \text{ is closed and } A \subset F\}$ is the smallest closed set containing in A, so it is the closure: $\overline{A}$.

The boundary $\partial A = \overline{A} \cap (X \setminus A^\circ)$ is the intersection of two closed sets, so it is closed. □

Note that $X \setminus A^\circ = \overline{(X \setminus A)}$.

A subset V of X is a *neighbourhood* of $x_o \in X$ when there is an open set U with $x_o \in U \subset V$.

A sequence (x_n) converges to a limit ℓ in X if, for each neighbourhood V of ℓ, there is natural number $N(V)$ with

$$x_n \in V \qquad \text{for all } n \geqslant N(V) .$$

We write $x_n \to \ell$ as $n \to \infty$ to mean that (x_n) converges to ℓ.

Examples. $x_n \to \ell$ as $n \to \infty$ if and only if

1. Metric: $d(x_n, \ell) \to 0$ as $n \to \infty$.
2. Discrete: $x_n = \ell$ for $n \geqslant N$.
3. Indiscrete: Every sequence converges to every value $\ell \in X$.
4. Gate: $x_n(t) \to \ell(t)$ for each $t \in [0, 1]$. This means that the sequence of functions (x_n) converge pointwise to ℓ.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two topological spaces and $f : X \to Y$ a map between them. Then f is *continuous at* $x_o \in X$ if, for each neighbourhood V of $f(x_o)$ in Y the inverse image

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

is a neighbourhood of x_o in X. We think of a neighbourhood of x_o as being a set containing all of the points sufficiently close to x_o. Hence this definition says roughly that "$f(x)$ is close to $f(x_o)$ provided that x is sufficiently close to x_o". We say that f is *continuous* if it is continuous at each point x_o in X.

Proposition 1.2 Open set and neighbourhoods
A set $A \subset X$ is open if and only if it is a neighbourhood of each point $x \in A$.

Proof:
Clearly an open set A is a neighbourhood of each point $x \in A$.

For the converse, suppose that A is a neighbourhood of each $x \in A$. Then there is an open set U_x with $x \in U_x \subset A$. The union $U = \bigcup_{x \in A} U_x$ is then open. It contains every point $x \in A$ and is also contained in A. So $A = U$ is open. □

Proposition 1.3 Continuity
A map $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X for every set U that is open in Y.

Proof:

Suppose first that f is continuous and that U is an open subset of Y. We need to show that $f^{-1}(U)$ is open in X. Let $x_o \in f^{-1}(U)$. Then $f(x_o) \in U$. By Proposition 1.2, U is neighbourhood of $f(x_o)$. Since f is continuous at x_o, we see that $f^{-1}(U)$ is a neighbourhood of x_o. This is true for all $x_o \in f^{-1}(U)$, so Proposition 1.2 shows that $f^{-1}(U)$ is open.

For the converse, suppose that $f^{-1}(U)$ is open in X whenever U is open in Y. Let $x_o \in X$ and let V be a neighbourhood of $f(x_o)$. Then there is an open set U with $f(x_o) \in U \subset V$. Consequently, $x_o \in f^{-1}(U) \subset f^{-1}(V)$. Since $f^{-1}(U)$ is open in X, this shows that $f^{-1}(V)$ is a neighbourhood of x_o. Thus f is continuous at x_o. □

Example. Any map $f : X \to Y$ is continuous when X has the discrete topology, or when Y has the indiscrete topology.

A map $f : X \to Y$ is a *homeomorphism* if it is continuous and it has an inverse that is also continuous. For example, the map

$$\exp : \mathbb{R} \to (0, \infty) \ ; \quad x \mapsto \exp x$$

is a homeomorphism when both $\mathbb{R}$ and $(0, \infty)$ have the topology coming from the Euclidean metric. However, the map

$$(\mathbb{R}, \text{ discrete }) \to (\mathbb{R}, \text{ Euclidean }) \ ; x \mapsto x$$

from $\mathbb{R}$ with the discrete topology to $\mathbb{R}$ with the usual Euclidean metric topology is not a homeomorphism even though it is continuous and has an inverse.

Topologies on Subsets, Quotients and Products

Let $(X, \mathcal{T})$ be a topological space. Let $j : S \hookrightarrow X$ be the inclusion map for a subset S of X. The *subset topology* on S consists of the intersections

$$S \cap U \qquad \text{for } U \in \mathcal{T} \ .$$

This is a topology on S and the inclusion map j is continuous.

Example: The interval $[0, 1)$ is not open in $\mathbb{R}$ (with the Euclidean topology) but it is open in $[0, 1]$ with the subset topology.

Let $q : X \to Q$ be a surjective map. We can then think of Q as the quotient of X by the equivalence relation

$$x \sim y \qquad \Leftrightarrow \qquad q(x) = q(y) \ .$$

So $Q = X/\sim$ and q is the quotient map. The *quotient topology* on Q consists of the sets $U \subset Q$ with $q^{-1}(U) \in \mathcal{T}$. This is a topology on Q and q is continuous.

Example: Consider the map

$$q : \mathbb{R} \to \mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\} \ ; \quad t \mapsto \exp it \ .$$

The quotient topology on $\mathbb{T}$ is then the same as the topology on $\mathbb{T}$ coming from the usual Euclidean metric.

Now let $(Y, \mathcal{U})$ be another topological space and consider the Cartesian product $X \times Y$. A set U is open in the *product topology* on $X \times Y$ if it is the union of some collection of sets of the form

$$A \times B \qquad \text{with } A \in \mathcal{T} \ \text{ and } \ B \in \mathcal{U} \ .$$

This is a topology on $X \times Y$ and the projection maps

$$\pi_X : X \times Y \to X \ ; \quad (x, y) \mapsto x$$
$$\pi_Y : X \times Y \to Y \ ; \quad (x, y) \mapsto y$$

are continuous. Note that there are open sets in $X \times Y$ that are not of the form $A \times B$ for $A \in \mathcal{T}$ and $B \in \mathcal{U}$. For example, the product topology on $\mathbb{R} \times \mathbb{R}$ is the usual Euclidean topology and the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : \ x < y \ \}$ is open.

We can define the Cartesian products of more than two spaces in a similar way. Suppose that $(X_\alpha, \mathcal{T}_\alpha)$ is a topological space for each α in an index set A. Then the product topology on

$$\prod_{\alpha \in A} X_\alpha$$

consists of arbitrary unions of sets of the form

$$\left\{ (x_\alpha) \in \prod_{\alpha \in A} X_\alpha \ : \ x_\alpha \in U_\alpha \ \text{ for all } \ \alpha \in A \right\}$$

where $U_\alpha \in \mathcal{T}_\alpha$ for each $\alpha \in A$ and $U_\alpha = X_\alpha$ for all but a finite number of indices $\alpha \in A$. *The final underlined condition only matters when we are considering the product of infinitely many spaces. For example, if we take each X_α to be $\mathbb{R}$ for $\alpha \in [0, 1]$, then the product is

$$\mathbb{R}^{[0,1]} \ = \ \{f : [0, 1] \to \mathbb{R}\}$$

and the product topology is the gate topology.*

Hausdorff Spaces

A topological space $(X, \mathcal{T})$ is *Hausdorff* if, for each pair of distinct points $x, y \in X$, there are disjoint open sets U, V with

$$x \in U \ , \quad y \in V \quad \text{and} \quad U \cap V = \emptyset \ .$$

For example, any metric space is Hausdorff because $B(x, \frac{1}{2}d(x, y))$ and $B(y, \frac{1}{2}d(x, y))$ are disjoint. The indiscrete topology on a set with more than one point is not Hausdorff. For analysis, where we are concerned with limits, virtually all the spaces of interest are Hausdorff.

Proposition 1.4 Unique limits in Hausdorff spaces
Let $(X, \mathcal{T})$ be a Hausdorff topological space. Then a sequence (x_n) in X can have at most one limit.

Proof:
Suppose that x_n converged to two different limits ℓ and m. Then we could find disjoint open sets U, V each containing one of the limits. The convergence means that there are natural numbers N, M with

$$x_n \in U \quad \text{for} \quad n \geqslant N \quad \text{and} \quad x_n \in V \quad \text{for} \quad n \geqslant M \ .$$

This is impossible for $n \geqslant \max(N, M)$. □

2. COMPACT SPACES

Definition and Examples

Let $(X, \mathcal{T})$ be a topological space. An *open cover* for X is a collection $\mathcal{U}$ of open sets in X with union $\bigcup \mathcal{U} = X$. A *subcover* of $\mathcal{U}$ is a subset of $\mathcal{U}$ that is also a cover of X. For example, the collection of all intervals $(x, x+2)$ for $x \in \mathbb{R}$ is an open cover for $\mathbb{R}$. The collection $\mathcal{V} = \{(x, x+2) : \ x \in \mathbb{Z}\ \}$ is a subcover. However, no proper subset of $\mathcal{V}$ remains a cover of $\mathbb{R}$.

The topological space $(X, \mathcal{T})$ is *compact* if every open cover of X has a finite subcover.

Example: Any finite topological space is compact, as is any indiscrete topological space. The example above shows that $\mathbb{R}$ is not compact.

Proposition 2.1 Heine - Borel Theorem
Each closed, bounded interval $[a, b]$ in the real line is compact.

Proof:
First we give a proof using the fact that non-empty subsets of $[a, b]$ have a supremum.

Let $\mathcal{U}$ be an open cover for the interval $[a, b]$ and set

$$J = \{t \in [a, b] : \ [a, t] \ \text{ is contained in a finite union of sets from } \mathcal{U}\ \} .$$

Then $a \in J$, so K has a supremum $t_o \in [a, b]$. The point t_o itself must lie in one of the open sets in $\mathcal{U}$, say $t_o \in U_o \in \mathcal{U}$. Since U_o is open, we must have a $\delta > 0$ with

$$\{x \in [a, b] : \ t_o - \delta \leqslant x \leqslant t_o + \delta\ \} \subset U_o .$$

Since $t_o = \sup J$, we can find a point $s \in J$ with $t_o - \delta < s \leqslant t_o$. There will be a finite subset $\mathcal{F}$ of $\mathcal{U}$ with $[a, s] \subset \bigcup \mathcal{F}$. This implies that $[a, t_o + \delta] \cap [a, b]$ is covered by $\mathcal{F}$ together with U_o. If $t_o \neq b$, then this is a contradiction. Therefore $t_o = b$ and $\mathcal{F} \cup \{U_o\}$ covers all of $[a, b]$. □

There are many variations on this proof. We will give another based on repeated bisection or "condensation of singularities". Suppose that $\mathcal{U}$ is an open cover of $I_0 = [a, b]$ but no finite subset of $\mathcal{U}$ covers I_0. Divide I_0 into two intervals:

$$[a, \tfrac{1}{2}(a+b)] \quad \text{and} \quad [\tfrac{1}{2}(a+b), b] .$$

At least one of these, say I_1, is not contained in any finite union of sets from $\mathcal{U}$. If we repeat the process we obtain a recursively defined sequence of intervals $I_n = [a_n, b_n]$. The left endpoints a_n form an increasing sequence bounded above by b, so they converge to a limit, say a_∞. Similarly, the right endpoints form a decreasing sequence (b_n) converging to a limit b_∞. Since

$$b_n - a_n = (b-a)/2^n ,$$

we see that $a_\infty = b_\infty$. Since $a_n \nearrow a_\infty$ and $b_n \searrow a_\infty$, we also see that $a_\infty \in I_n$ for each $n \in \mathbb{N}$.

Now the point a_∞ must lie within one of the sets in $\mathcal{U}$, say $a_\infty \in U_o \in \mathcal{U}$. Since U_o is open, there is a ball $B(a_\infty, \delta)$ that lies entirely within U_o. However, I_n has length less than δ for n sufficiently large, say $n \geqslant N$. Hence,

$$I_n \subset B(a_\infty, \delta) \subset U_o \qquad \text{for} \qquad n \geqslant N .$$

Thus I_n is covered by a single element of $\mathcal{U}$, which is a contradiction. □

(This second proof of Proposition 2.1 can be easily extended to show that subsets of $\mathbb{R}^N$ which are closed and bounded are compact. To do this, we consider the set as a subset of a cube $[-M, M]^N$. Then divide this cube into 2^N cubes of half the side length and repeat the process as above.)

Note that open intervals $(a, b) \subset \mathbb{R}$ are not compact, for the sets

$$(a+\varepsilon, b-\varepsilon) \qquad \text{for } 0 < \varepsilon < \tfrac{1}{2}(b-a)$$

form an open cover with no finite subcover. Similarly, unbounded subsets of $\mathbb{R}$ are not compact since there is no subcover of $(-n, n)$ for $n \in \mathbb{N}$.

Compactness for Subsets, Quotients and Products

This section is devoted to studying when subsets, quotients or products of compact sets are compact.

We will say that a subset K of a topological space $(X, \mathcal{T})$ is compact if it is compact for the subset topology. Suppose that $\mathcal{U}$ is a collection of open subsets of K with $\bigcup \mathcal{U} = K$. Each set $U \in \mathcal{U}$ is the intersection of K with some set $\widetilde{U}$ open in X. Let $\widetilde{\mathcal{U}}$ be the collection of these sets $\widetilde{U}$, one chosen for each $U \in \mathcal{U}$. Then $K \subset \bigcup \widetilde{\mathcal{U}}$. This means that K is compact if, whenever $\mathcal{V}$ is a collection of open sets in X with $K \subset \bigcup \mathcal{V}$, there is a finite subset $\mathcal{F}$ of $\mathcal{V}$ with $K \subset \bigcup \mathcal{F}$.

Proposition 2.2 Closed subsets of compact sets are compact.
Let K be a closed subset of the compact set X. Then K is also compact.

Proof:
Suppose that $\mathcal{V}$ is a collection of open sets in X with $K \subset \bigcup \mathcal{V}$. Then $\mathcal{V}$, together with the open set $X \setminus K$, covers all of X. Since X is compact, there is a finite subset $\mathcal{F}$ of $\mathcal{V}$ with

$$X \subset \left(\bigcup \mathcal{F}\right) \cup (X \setminus K) .$$

This certainly implies that $K \subset \bigcup \mathcal{F}$, so we do have a finite subcover for K as required. □

Subsets of compact set that are not closed need not be compact. Indeed, we have:

Proposition 2.3 Compact subsets of Hausdorff spaces are closed
Let $(X, \mathcal{T})$ be a Hausdorff topological space and K a compact subset of X. Then K is closed in X.

Proof:
Let $x \in X \setminus K$. For each $y \in K$, there are disjoint open sets $U(y), V(y)$ in X with

$$x \in U(y) \ , \quad y \in V(y) \ \text{ and } U(y) \cap V(y) = \emptyset \ .$$

The collection $\{V(y) : y \in K\}$ is then an open cover of K, so it has a finite subcover, say $\{V(y_1), V(y_2), \ldots, V(y_N)\}$. Let U be the intersection

$$U = U(y_1) \cap U(y_2) \cap \ldots \cap U(y_N)$$

of the corresponding sets $U(y)$. Then U is the intersection of finitely many open sets, so it is open in K. Moreover U is disjoint from K because $U(y_n) \cap V(y_n) = \emptyset$. This is true for each $x \in K \setminus S$, so we have shown that $X \setminus K$ is open. Consequently, K is closed. □

Note that the result may fail for spaces that are not Hausdorff. For example, every subset of an indiscrete space is compact.

We can now see exactly which subsets or $\mathbb{R}$ are compact.

Corollary 2.4 Compact subsets of $\mathbb{R}$.
A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof:
If $K \subset \mathbb{R}$ is closed and bounded, then it is a closed subset of the interval $[-M, M]$ for some M. The Heine - Borel theorem (2.1) shows that $[-M, M]$ is compact, and Proposition 2.2 shows that the closed subset is also compact.

Now suppose that $K \subset \mathbb{R}$ is compact. The collection $\{(-N, N) \cap K : N \in \mathbb{N}\}$ is an open cover for K, so it has a finite subcover. The union of a finite number of the sets $(-N, N) \cap K$ is clearly bounded, so K is bounded. Since $\mathbb{R}$ is Hausdorff, Proposition 2.3 shows that K must be closed. □

This result is not true in a general metric space. For example, consider the space $(0, 1)$ with the Euclidean metric. The set $(0, 1)$ itself is certainly closed and bounded in $(0, 1)$ but is not compact.

Proposition 2.5 Continuous images of compact sets are compact.
Let $f : X \to Y$ be a continuous map between topological spaces. If X is compact, then $f(X)$ is also compact.

Proof:
Let $\mathcal{U}$ be a collection of open sets in Y with $f(X) \subset \bigcup \mathcal{U}$. Then the sets

$$f^{-1}(U) \qquad \text{for } U \in \mathcal{U}$$

are open in X and cover X. Since X is compact, there is a finite subset $\mathcal{F}$ of $\mathcal{U}$ with

$$\bigcup_{U \in \mathcal{F}} f^{-1}(U) = X \ .$$

Therefore, $f(X) \subset \bigcup \mathcal{F}$. This proves that $f(X)$ is compact. □

This shows, in particular, that any quotient of a compact set is compact for the quotient topology. If $f : X \to Y$ is any continuous map and K is a compact subset of X, then the restriction of f gives a continuous map $f|_K : K \to f(K)$. So $f(K)$ is compact.

Corollary 2.6 Continuous real-valued functions on a compact set
Let $\phi : K \to \mathbb{R}$ be a continuous real-valued map on a compact topological space K. Then ϕ is bounded on K and attains its bounds.

This means that $s = \sup\{\phi(t) : t \in K\}$ exists and there is a point $x \in K$ with $\phi(x) = s$. Similarly, there exists a point $y \in K$ with $\phi(y) = j = \inf\{\phi(t) : t \in K\}$.

Proof:
Proposition 2.5 shows that $\phi(K)$ is a compact subset of $\mathbb{R}$. Hence, by Corollary 2.4 , it is closed and bounded. Being bounded means that $s = \sup\{\phi(t) : t \in K\}$ and $j = \inf\{\phi(t) : t \in K\}$ exist; being closed means that both s and j are points of $\phi(K)$. This means that there are points $x, y \in K$ with $\phi(x) = s$ and $\phi(y) = j$. □

Proposition 2.7 Tychonoff's theorem.
The Cartesian product of two compact topological spaces is compact.

Proof:

Let $(X, \mathcal{S}), (Y, \mathcal{T})$ be two compact topological spaces and let $\mathcal{U}$ be an open cover for $X \times Y$. Each open set in $X \times Y$ is the union of products

$$A \times B \qquad \text{with } A \in \mathcal{S} \text{ and } B \in \mathcal{T} \ .$$

Consider the collection $\mathcal{V}$ of all such products that are contained in any of the sets in the cover $\mathcal{U}$. Then $\mathcal{V}$ is itself an open cover of $X \times Y$. If we can show that $\mathcal{V}$ has a finite subcover, then each of the sets $A \times B$ in this subcover lies inside one of the sets of $\mathcal{U}$, so $\mathcal{U}$ will also have a finite subcover. Hence, it will be sufficient to prove that $\mathcal{V}$ has a finite subcover.

Suppose that $\mathcal{V}$ consists of all the products $A_t \times B_t$ for t in some index set T. Each A_t lies in $\mathcal{S}$ and each B_t lies in $\mathcal{T}$. For each $x \in X$, the sets $A_t \times B_t$ must cover $\{x\} \times Y$. Hence the sets $\{B_t : \ x \in A_t\}$ form an open cover for Y. Since Y is compact, there must be a finite set of indices, say $t(1), t(2), \ldots, t(N)$, with

$$x \in A_{t(n)} \text{ for } n = 1, 2, \ldots, N \ \text{ and } Y \subset B_{t(1)} \cup B_{t(2)} \cup \ldots \cup B_{t(N)} \ .$$

Set $W(x) = A_{t(1)} \cap A_{t(2)} \cap \ldots \cap A_{t(N)}$. Then $W(x)$ is open, it contains x and

$$W(x) \times Y \ \subset (A_{t(1)} \times B_{t(1)}) \cup (A_{t(2)} \times B_{t(2)}) \cup \ldots \cup (A_{t(N)} \times B_{t(N)}) \ .$$

In particular, $W(x) \times Y$ is covered by a finite number of the sets in $\mathcal{V}$.

The sets $W(x)$ for $x \in X$ form an open cover for X, so there is a finite subcover, say $X = W(x_1) \cup W(x_2) \cup \ldots \cup W(x_K)$. Then

$$X \times Y = (W(x_1) \times Y) \cup (W(x_2) \times Y) \cup \ldots \cup (W(x_K) \times Y) \ .$$

Each strip $(W(x_k) \times Y)$ is covered by finitely many sets form $\mathcal{V}$, so their union $X \times Y$ is also. □

The Cartesian product of infinitely many compact spaces is also compact. This is harder to prove and requires the axiom of choice.

Corollary 2.8 Compact subsets of $\mathbb{R}^N$.
A subset of $\mathbb{R}^N$ is compact if and only if it is closed and bounded.

Proof:

If $K \subset \mathbb{R}^N$ is closed and bounded, then it a subset of $B(0, M)$ for some M. Consequently, $K \subset [-M, M]^N$. By Tychonoff's theorem , this product $[-M, M]^N$ is compact. Hence the closed subset K is also compact.

Now suppose that $K \subset \mathbb{R}^N$ is compact. The collection $\{B(0, N) \cap K : N \in \mathbb{N}\}$ is an open cover for K, so it has a finite subcover. The union of a finite number of the balls $B(0, N) \cap K$ is clearly bounded, so K is bounded. Since $\mathbb{R}^N$ is Hausdorff, Proposition 2.3 shows that K must be closed. □

Compact Metric Spaces

A topological space X is *sequentially compact* if every sequence in X has a convergent subsequence. The Bolzano - Weierstrass theorem in Analysis 2 shows that any closed bounded subset of $\mathbb{R}^N$ is sequentially compact.

Proposition 2.9 Compact implies sequentially compact
Any compact metric space is sequentially compact.

Proof:

Let (x_n) be a sequence in the compact space K. Suppose that there exists a point $\ell \in X$ such that every ball $B(\ell, r)$ contains infinitely many terms of the sequence. Then we can recursively choose terms $x_{N(k)}$ of the sequence with

$$N(k) > N(k-1) \qquad \text{and} \qquad x_{N(k)} \in B(\ell, 1/k) \ .$$

This would imply that $(x_{N(k)})$ converges to ℓ as $n \to \infty$.

If there is no such point ℓ, then, for each $\ell \in X$, we can find an open ball B_ℓ about ℓ that contains only finitely many terms of (x_n). The balls $\{B_\ell : \ell \in X\}$ form an open cover for X, so there is a finite subcover, say $X = B_{\ell(1)} \cup B_{\ell(2)} \cup \ldots \cup B_{\ell(M)}$. Since each $B_{\ell(m)}$ contains only finitely many terms of (x_n), their union X can only contain finitely many terms. This is impossible. □

For subsets of $\mathbb{R}^N$ compactness and sequential compactness are equivalent.

Proposition 2.10 Compactness for subsets of $\mathbb{R}^N$.
For a subset X of $\mathbb{R}^N$, the following conditions are equivalent:

(a) X is compact.

(b) X is sequentially compact.

(c) X is closed and bounded.

Proof:

We have just proved that (a) ⇒ (b).

Suppose that (b) is true. If X is not closed in $\mathbb{R}^N$, then there is a point $\ell \in \mathbb{R}^N \setminus X$ with every ball $B(\ell, r)$ meeting X. Hence we can choose $x_n \in B(\ell, 1/n)$. This gives a sequence that converges to ℓ in $\mathbb{R}^N$. Every subsequence also converges to ℓ, so no subsequence can possibly converge in X. If X is not bounded, we can find a sequence (x_n) with $d(x_n, 0) > n$ for each $n \in \mathbb{N}$. No subsequence can converge, even in $\mathbb{R}^N$. Thus (c) is true.

Finally, Corollary 2.8 shows that (c) ⇒ (a) is true. □

*In all metric spaces compactness and sequential compactness are the same. Indeed, in the Analysis course you showed that a metric space (X, d) was sequentially compact if and only if it was complete and totally bounded. The argument also gives:

Proposition 2.11 Compactness for metric spaces
The following conditions on a metric space (X, d) are equivalent.

(a) K is compact.

(b) K is sequentially compact.

(c) K is complete and totally bounded

Proof:

(Complete means that every Cauchy sequence in X converges in X. Totally bounded means that, for each $\varepsilon > 0$, there is an ε-*net*, that is a finite set $\{x_1, x_2, \dots, x_N\} \subset X$ with $X \subset \bigcup_{n=1}^{N} B(x_n, \varepsilon)$.)

(a) $\Rightarrow$ (b) This is Proposition 2.9 .

(b) $\Rightarrow$ (c) Suppose that (x_n) is a Cauchy sequence in X. Since X is sequentially compact, there is a subsequence $(x_{n(k)})$ that converges to a limit, sat $\ell \in X$. For any $\varepsilon > 0$ we know that there is a natural number N with

$$d(x_n, x_m) < \varepsilon \qquad \text{for} \quad n, m \geqslant N$$

and a natural number K with

$$d(x_{n(k)}, \ell) < \varepsilon \qquad \text{for} \quad k \geqslant K \ .$$

Then, we can find $k \geqslant K$ with $n(k) \geqslant N$. For $m \geqslant N$ we obtain

$$d(x_m, \ell) \leqslant d(x_m, x_{n(k)}) + d(x_{n(k)}, \ell) < \varepsilon + \varepsilon \ .$$

Hence the entire sequence (x_n) converges to ℓ.

To show that X is totally bounded, choose $\varepsilon > 0$ and construct a sequence (x_n) as follows. First x_1 is any point of X. Suppose that $x_1, x_2, \dots, x_k$ have been chosen. If they form an ε-net, then we stop. Otherwise, we can find $x_{k+1} \in X$ with

$$d(x_{k+1}, x_j) \geqslant \varepsilon \qquad \text{for } j = 1, 2, \dots, k \ .$$

Eventually we must stop, for otherwise we would obtain an infinite sequence (x_n) with $d(x_n, x_m) \geqslant \varepsilon$ for all $n \neq m$ and such a sequence can have no convergent subsequence. When we stop we get an ε-net.

(c) $\Rightarrow$ (a) Let $\mathcal{U}$ be an open cover for X and suppose that it has no finite subcover. We construct decreasing subsets (X_k) that also also have no finite subcover from $\mathcal{U}$. First take $X_0 = X$. When $X_0, X_1, \dots, X_k$ have been chosen, we know that there is a $1/(k+1)$-net for X, so X is the union of a finite number of open balls each of radius $1/(k+1)$, say $B_1, B_2, \dots, B_N$. These balls cover X_k so at least one of the intersections $X_k \cap B_n$ does not have a finite subcover from $\mathcal{U}$. Set X_{k+1} equal to this intersection: $X_k \cap B_n$.

We now have sets $X = X_0 \supset X_1 \supset X_2 \supset \dots$ with each X_k being contained within a ball of radius $1/k$ for $k \geqslant 1$. Choose any points $x_k \in X_k$. Then $d(x_k, x_m) \leqslant 2/k$ for $m > k$ because $x_m \in X_m \subset X_k$ and X_k has diameter no bigger than $2/k$. Therefore the sequence (x_k) is a Cauchy sequence and must converge to a limit $\ell \in X$. We also have $d(x_k, \ell) \leqslant 2/k$.

Finally, ℓ lies in one of the sets of the cover $\mathcal{U}$, say $\ell \in U$. Since U is open, we have $B(\ell, r) \subset U$ for some $r > 0$. When $k > 4/r$ we have

$$X_k \subset B(x_k, 2/k) \subset B(\ell, 4/k) \subset B(\ell, r) \subset U \ .$$

This would mean that X_k was covered by a single set from $\mathcal{U}$, contradicting its definition. □

Note that for a subset X of $\mathbb{R}^N$, X is complete if and only if it is closed, and X is totally bounded if and only if it is bounded. Hence Proposition 2.11 implies Proposition 2.10 . *

Proposition 2.12 Uniform continuity
Let $f : X \to Y$ be a continuous map between two metric spaces. If X is compact, then f is uniformly continuous.

Proof:

Let $\varepsilon > 0$ be a fixed number. Since f is continuous at each $x \in X$, we know that there exists a $\delta(x) > 0$ with

$$d(f(x), f(y)) < \varepsilon \quad \text{whenever} \quad d(x, y) < \delta(x) \ .$$

We wish to show that f is uniformly continuous, so we need to show that we can choose $\delta(x)$ independently of x.

Let $U(x)$ be the open ball $B(x, \frac{1}{2}\delta(x))$ in X. The sets $\{U(x) : x \in X\}$ form an open cover for X, so there is a finite subcover, say $X = U(x_1) \cup U(x_2) \cup \ldots \cup U(x_N)$. Any point $z \in X$ lies in one of these balls, say $z \in U(x_n)$. For any point w with $d(w, z) < \frac{1}{2}\delta(x_n)$ we have

$$d(w, x_n) \leqslant d(w, z) + d(z, x_n) < \tfrac{1}{2}\delta(x_n) + \tfrac{1}{2}\delta(x_n) = \delta(x_n) \ .$$

Consequently,

$$d(f(w), f(z)) \leqslant d(f(w), f(x_n)) + d(f(z), f(x_n)) < \varepsilon + \varepsilon = 2\varepsilon \ .$$

Consequently,

$$d(f(w), f(z)) < 2\varepsilon \quad \text{whenever} \quad d(w, z) < \delta = \min\{\delta(x_1), \delta(x_2), \ldots, \delta(x_N)\}$$

with δ independent of w and z. □

3. CONNECTEDNESS

A topological space $(X, \mathcal{T})$ is *disconnected* if there are two non-empty, open sets $U, V \in \mathcal{T}$ with

$$U \cap V = \emptyset \qquad \text{and} \qquad U \cup V = X\ .$$

If X is not disconnected, then we say it is connected.

Note that the open sets U and V are complements of each other, so they are both closed sets as well. Thus X is connected if there are no proper subsets that are both open and closed in X. We can also rephrase the definition in terms of maps into the two-point space $\{0, 1\}$. We give $\{0, 1\}$ the topology of a subset of $\mathbb{R}$; this is the same as the discrete topology.

Proposition 3.1 Continuous maps into $\{0, 1\}$
A topological space $(X, \mathcal{T})$ is connected if and only if every continuous map $f : X \to \{0, 1\}$ is constant.

Proof:
The open subsets of $\{0, 1\}$ are $\emptyset$, $\{0\}$, $\{1\}$ and $\{0, 1\}$. Hence a map $f : X \to \{0, 1\}$ is continuous if and only if the sets $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$ are both open. Clearly

$$U \cap V = f^{-1}(\{0\} \cap \{1\}) = \emptyset \qquad \text{and} \qquad U \cup V = f^{-1}(\{0\} \cup \{1\}) = X\ .$$

Conversely, if we are given any two open sets U, V in X with $U \cap V = \emptyset$ and $U \cup V = X$, we can define

$$f : X \to \{0, 1\} \quad \text{by} \quad x \mapsto \begin{cases} 0 & \text{when } x \in U; \\ 1 & \text{when } x \in V; \end{cases}$$

and f is continuous.

The map f is constant precisely when one of the sets U, V is empty. □

Corollary 3.2 Intervals in $\mathbb{R}$ are connected.
A non-empty subset of $\mathbb{R}$ is connected if and only if it is an interval.

Proof:
Let J be an interval in $\mathbb{R}$. Suppose that $f : J \to \{0, 1\}$ is continuous and not constant. Then f also gives a continuous map from J into $\mathbb{R}$ that takes the values 0 and 1. By the Intermediate Value Theorem it must also take the value $\frac{1}{2}$. This is impossible since f only takes values 0 and 1.

For the converse, let J be any non-empty Set $a = \inf X$ and $b = \sup X$. We will prove that X is one of the intervals (a, b), $[a, b)$, $(a, b]$ or $[a, b]$. (When a or b is infinite we get the unbounded intervals: $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$ or $(-\infty, \infty)$.) Firstly, it is clear that $X \subset [a, b]$. Suppose that $c \in (a, b)$ did not lie in X. Then the sets $X \cap (-\infty, c)$ and $X \cap (c, \infty)$ would disconnect X. Therefore X contains (a, b). □

Proposition 3.3 Continuous images of connected sets are connected
Let $f : X \to Y$ be a continuous map between two topological spaces. If X is connected, then the image $f(X)$ is also connected.

Proof:
By replacing Y by $f(X)$, we may assume that f is surjective. If $f(X)$ were disconnected, then there would be two non-empty open sets U, V in $f(X)$ with $U \cap V = \emptyset$ and $U \cup V = X$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open in X and disconnect it. □

Proposition 3.4 Unions of connected sets
For each α in some index set A suppose that S_α is a connected subsets of the topological space $(X,\mathcal{T})$. If there is a point $x_o \in \bigcap_{\alpha\in A} S_\alpha$ then the union $\bigcup_{\alpha\in A} S_\alpha$ is also connected.

Proof:

Let $f : \bigcup_{\alpha\in A} S_\alpha \to \{0,1\}$ be a continuous map. Then each restriction $f|S_\alpha$ is also continuous and, since S_α is connected, it is constant. This constant must be $f(x_o)$ for each $\alpha \in A$. Therefore f is constant on all of $\bigcup S_\alpha$. □

Suppose that X is a topological space and $x_o \in X$. Then there are certainly some connected subsets of X that contain x_o, for example $\{x_o\}$. Let

$$C = \bigcup \{S \subset X : S \text{ is connected, and } x_o \in S\} .$$

The Proposition shows that C itself is connected. It is therefore the unique largest subset of X that contains x_o and is connected. We say that C is the *component of X containing x_o*. The Proposition also shows that two components are either disjoint or identical. Hence they partition X into disjoint subsets — the *components of X*.

Example: The components of $\mathbb{Q}$ as a subset of $\mathbb{R}$ are the singletons.

There are some strange sets that are still connected. Consider for example the set

$$X = \mathbb{T} \cup \{re^{i\theta} : \theta \in \mathbb{R} \text{ and } r = 1 + e^{\theta}\} .$$

This consists of the unit circle $\mathbb{T}$ and a spiral S that approaches $\mathbb{T}$. Each of $\mathbb{T}$ and S is the continuous image of $\mathbb{R}$, so is certainly connected. Let $f : X \to \{0,1\}$ be a continuous map. Then f must be constant on S since S is connected. Each point $e^{i\theta} \in \mathbb{T}$ is the limit of points in S, for example

$$(1 + e^{(\theta-2n\pi)})e^{i\theta} \to e^{i\theta} \qquad \text{as} \qquad n \to +\infty .$$

Hence, $f(e^{i\theta})$ must have the same value as f on S. Thus f is constant on all of X.

Nonetheless, the two parts $\mathbb{T}$ and S of X can not be connected by any continuous path $\gamma : [0,1] \to X$. For such a path from $\gamma(0) \in S$ to $\gamma(1) \in \mathbb{T}$ can not be continuous at the point $t_o = \sup\{t \in [0,1] : \gamma(t) \in S\}$. We say that X is connected but not path-connected.

A topological space $(X,\mathcal{T})$ is *path-connected* if, for each pair of points $x_0, x_1 \in X$ there is a continuous map

$$\gamma : [0,1] \to X \qquad \text{with } \gamma(0) = x_0 \text{ and } \gamma(1) = x_1 .$$

Proposition 3.5 Path-connected implies connected
Every path-connected space is connected.

Proof:

Suppose that U, V disconnected X. Choose $x_0 \in U$ and $x_1 \in V$ and find a continuous path γ from one to the other. Now $[0,1]$ is connected, so its image $\gamma([0,1])$ is also connected. Since $\gamma(0) = x_0 \in U$, we must have all of $\gamma([0,1])$ inside U. A contradiction. □

However, for sufficiently nice spaces, connectedness and path-connectedness do correspond. We will need the following result.

Proposition 3.6 For open subsets of $\mathbb{R}^N$, connected implies path-connected
An open subset U of $\mathbb{R}^N$ is connected if and only if it is path-connected.

Proof:

We already know that path-connected spaces are connected. So we may assume that U is connected and prove that it is path-connected.

Let $x_o \in U$ and define $W = \{x \in U :$ there exists a continuous path γ in U from x_o to x $\}$. Suppose that $x_1 \in W$ with a continuous path γ in U from x_o to x_1. Since U is open in $\mathbb{R}^N$, there is a ball $B(x_1, r)$ about x_1 within U. Hence, for each $y \in B(x_1, r)$, we can find a path in U from x_o to y by first following γ and then going along a radius of $B(x_1, r)$ from x_1 to y. This shows that $y \in W$. Thus $B(x_1, r) \subset W$ and W is open in U.

Similarly, suppose that $x_1 \notin W$ and $B(x_1, r) \subset U$. If any $y \in B(x_1, r)$ were in W, then we could find a path in U from x_o to x_1 by going from x_o to y in U and then along a radius in $B(x_1, r)$ to x_1. This would contradict $x_1 \notin W$. Therefore, $B(x_1, r) \subset U \setminus W$.

We have now shown that U is the disjoint union of the two open sets W and $U \setminus W$. Since U is connected, one of these must be empty. Thus $W = U$, which shows that U is path connected. $\square$

The proof actually gives us more than path-connectedness. Within any ball $B(x_1, r)$ we can join any point y to the centre x_1 by a straight line segment. Hence we can join any two points in a connected open subset of $\mathbb{R}^N$ by a piecewise linear path. We could also join x_1 to y by a path made up of line segments parallel to the co-ordinate axes. So we can join any two points in a connected open subset of $\mathbb{R}^N$ by a piecewise linear path with each segment parallel to one of the co-ordinate axes.

4. CURVES

A *domain* in the complex plane $\mathbb{C}$ is an open, connected subset of $\mathbb{C}$. A function $f : D \to \mathbb{C}$ on a domain $D \subset \mathbb{C}$ is *analytic* if it is complex differentiable at each point of D. (It is more common to call such a function *holomorphic* or *regular*.) It is a much stronger condition on a function to be complex differentiable than to be real differentiable. Indeed, we will later show that any complex differentiable function on a domain D can be written locally as a power series. The reason for this is that we can apply the fundamental theorem of calculus when we integrate f along a curve in D that starts and ends at the same point. This will show that, for suitable curves, the integral is 0 — a result we call Cauchy's theorem. This theorem has many important consequences and is the key to the rest of the course.

We wish to integrate functions along curves in D. First consider integrals. If $\phi : [a, b] \to \mathbb{C}$ is a continuous function, then the Riemann integral

$$I = \int_a^b \phi(t)\ dt$$

exists. If I has argument θ, then

$$|I| = Ie^{-i\theta} = \int_a^b \phi(t)e^{-i\theta}\ dt \leqslant \int_a^b |\phi(t)|\ dt$$

so we have the inequality

$$\left| \int_a^b \phi(t)\ dt \right| \leqslant \int_a^b |\phi(t)|\ dt\ .$$

A *continuously differentiable curve* in D is a map $\gamma : [a, b] \to D$ defined on a compact interval $[a, b] \subset \mathbb{R}$ that is continuously differentiable at each point of $[a, b]$. (At the endpoints a, b we demand a one-sided derivative.) The image $\gamma([a, b])$ will be denoted by $[\gamma]$. For such a curve γ we define the integral of the continuous function $f : D \to \mathbb{C}$ along γ by

$$\int_\gamma f(z)\ dz\ =\ \int_a^b f(\gamma(t))\gamma'(t)\ dt\ .$$

We can also define the *length* of γ to be the integral

$$\int_a^b |\gamma'(t)|\ dt\ .$$

Then we have the important inequality:

$$\left| \int_\gamma f(z)\ dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\ dt \right| \leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|\ dt \leqslant L(\gamma). \sup\{|f(z)| : z \in [\gamma]\}\ .$$

Example: The straight-line curve $[w_0, w_1]$ between two points of $\mathbb{C}$ is given by

$$[0, 1] \to \mathbb{C}\ ;\quad t \mapsto (1 - t)w_o + tw_1\ .$$

This has length $|w_1 - w_0|$. The circle $C(z_o, r)$ is given by

$$[0, 1] \to \mathbb{C}\ ;\quad t \mapsto z_o + re^{2\pi it}$$

and has length $2\pi r$. A *piecewise continuously differentiable curve* is a map $\gamma : [a, b] \to D$ for which there is a subdivision

$$a = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = b$$

with each of the restrictions $\gamma| : [t_n, t_{n+1}] \to D$ $(n = 0, 1, \ldots, N)$ being a continuously differentiable curve. The integral along γ is then

$$\int_\gamma f(z)\, dz \;=\; \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f(\gamma(t))\gamma'(t)\, dt$$

and the length is $L(\gamma) = \sum L(\gamma|[t_n, t_{n+1}])$. We clearly have

$$\left| \int_\gamma f(z)\, dz \right| \leqslant L(\gamma). \sup\{|f(z)| : z \in [\gamma]\} \; .$$

From now on, we will suppose, tacitly, that all the curves we consider are piecewise continuously differentiable

It is possible to re-parametrise a curve $\gamma : [a, b] \to D$. Suppose that $h : [c, d] \to [a, b]$ is a continuously differentiable, strictly increasing function with a continuously differentiable inverse $h^{-1} : [a, b] \to [c, d]$. Then $\gamma \circ h : [c, d] \to D$ is a curve and the substitution rule for integrals shows that

$$\int_{\gamma\circ h} f(z)\, dz = \int_c^d f(\gamma(h(s))\gamma'(h(s))h'(s)\, ds = \int_a^b f(\gamma(t))\gamma'(t)\, dt = \int_\gamma f(z)\, dz$$

and similarly that $L(\gamma \circ h) = L(\gamma)$. Sometimes it is useful to reverse the orientation of the curve. For any curve $\gamma : [a, b] \to D$, the *reversed curve* $-\gamma$ is given by

$$-\gamma : [-b, -a] \to D \; ; \quad t \mapsto \gamma(-t) \; .$$

This traces out the same image as γ but in the reverse direction.

Proposition 4.1 Fundamental Theorem of Calculus
Let $f : D \to \mathbb{C}$ be an analytic function. If f is the derivative of another analytic function $F : D \to \mathbb{C}$, then

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a))$$

for any piecewise continuously differentiable curve $\gamma : [a, b] \to D$.

We call $F : D \to \mathbb{C}$ an antiderivative of f if $F'(z) = f(z)$ for all $z \in D$.

Proof:

The fundamental theorem of calculus show that

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt = \int_a^b F'(\gamma(t))\gamma'(t)\, dt = \int_a^b (F \circ \gamma)'(t)\, dt = F(\gamma(b)) - F(\gamma(a))$$

for any continuously differentiable curve γ. The result follows for piecewise continuously differentiable curves by adding the results for each continuously differentiable section. □

A curve $\gamma : [a, b] \to D$ is *closed* if $\gamma(b) = \gamma(a)$. In this case, the Proposition shows that

$$\int_\gamma f(z)\, dz = 0$$

provided that f is the derivative of a function $F : D \to \mathbb{C}$. This is our first form of Cauchy's theorem.

For the sake of variety, we use many different names for curves, such as paths or routes. Closed curves are sometimes called cycles or contours.

Example: Let A be the domain $\mathbb{C} \setminus \{0\}$ and γ the closed curve

$$\gamma : [0,1] \to A \ ; \quad t \mapsto e^{2\pi i t}$$

that traces out the unit circle in a positive direction. Let $f(z) = z^n$ for $n \in \mathbb{Z}$. Then

$$\int_\gamma z^n \, dz = \int_0^1 e^{2n\pi i t} \, 2\pi i e^{2\pi i t} \, dt = \begin{cases} 2\pi i & \text{when } n = -1; \\ 0 & \text{otherwise.} \end{cases}$$

This agrees with the Proposition. For each function $f(z) = z^n$ with $n \neq -1$ there is a function $F(z) = z^{n+1}/(n+1)$ with $F'(z) = f(z)$ on A, so the integral around γ should be 0. However, for $n = -1$ there is no such function $F : A \to \mathbb{C}$ and so the integral can be non-zero. Indeed, if there were such a function F it would have to be $F(z) = \log z + \textit{constant}$ and there is no continuous way to choose a branch of the logarithm on all of A. This example is of crucial importance and the study of the complex logarithm is at the centre of complex analysis.

Winding Numbers

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve that does not pass through 0. A *continuous choice of the argument on* γ is a continuous map $\theta : [a, b] \to \mathbb{R}$ with $\gamma(t) = |\gamma(t)|e^{i\theta(t)}$ for each $t \in [a, b]$. The change $\theta(b) - \theta(a)$ measures the angle about 0 turned through by γ. We call $(\theta(b) - \theta(a))/2\pi$ the *winding number* $n(\gamma, 0)$ *of* γ *about* 0. Suppose that ϕ is another continuous choice of the argument on γ. Then $\theta(t) - \phi(t)$ must be an integer multiple of 2π. Since $\theta - \phi$ is continuous on the connected interval $[a, b]$, we see that there is an integer k with $\phi(t) - \theta(t) = 2k\pi$ for all $t \in [a, b]$. Hence $\theta(b) - \theta(a) = \phi(b) - \phi(a)$ and the winding number is well defined.

When γ is a piecewise continuously differentiable curve, we can give a continuous choice of $\theta(t)$ explicitly and hence find an expression for the winding number. Let

$$h(t) = \int_{\gamma|[a,t]} \frac{1}{z} \, dz = \int_a^t \frac{\gamma'(t)}{\gamma(t)} \, dt$$

for $t \in [a, b]$. The chain rule shows that

$$\frac{d}{dt}\left(e^{-h(t)}\gamma(t)\right) = -h'(t)e^{-h(t)}\gamma(t) + e^{-h(t)}\gamma'(t) = -\frac{\gamma'(t)}{\gamma(t)}e^{-h(t)}\gamma(t) \ + \ e^{-h(t)}\gamma'(t) = 0 \ .$$

Hence $e^{-h(t)}\gamma(t)$ is constant. Therefore,

$$\gamma(t) = e^{h(t)}\gamma(a) = e^{\Re h(t)} e^{i\Im h(t)}\gamma(a) \ .$$

This means that $\theta(t) = \arg \gamma(a) + \Im h(t)$ gives a continuous choice of the argument of $\gamma(t)$. Consequently, the total angle turned through by γ is

$$\Im\left(\int_\gamma \frac{1}{z} \, dz\right) \ .$$

If γ is piecewise continuously differentiable, we can apply this argument to each section of γ and so find that the final formula still holds.

The formula is particularly important when γ is a closed curve. Then $\gamma(b) = \gamma(a)$, so $e^{h(b)} = 1$ and we must have $h(b) = 2N\pi i$ for some integer N. The number N counts the number of times γ winds positively around 0. We have the formula:

$$N = \frac{h(b)}{2\pi i} = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} \, dz \ .$$

We can also consider how many times a closed curve γ winds around any point w_o that does not lie on γ. By translating w_o to 0 we see that this is

$$n(\gamma; w_o) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - w_o} \, dz \ ,$$

which is called the *winding number of* γ *about* w_o.

Example: The curve $\gamma : [0, 1] \to \mathbb{C};\ t \mapsto z_o + re^{2\pi i t}$ has winding number

$$n(\gamma; w_o) = \begin{cases} 1 & \text{when } |w_o - z_o| < r; \\ 0 & \text{when } |w_o - z_o| > r. \end{cases}$$

It is not defined when $|w_o - z_o| = r$.

Lemma 4.2
Let γ *be a piecewise continuously differentiable closed curve taking values in the disc* $B(z_o, R)$*. Then* $n(\gamma; w_o) = 0$ *for all points* $w_o \notin B(z_o, R)$*.*

Proof:
By translating and rotating the curve, we may assume that $w_o = 0$ and z_o is a positive real number no smaller than R. For z in the disc $B(z_o, R)$, we can find an unique real number $\phi(z) \in (-\pi, \pi)$ with $z = |z|e^{i\phi(z)}$. (This is the principal branch of the argument of z.) The map $\phi : B(1, 1) \to \mathbb{R}$ is then continuous. Hence, $t \mapsto \phi(\gamma(t))$ is a continuous choice of the argument on γ. So

$$n(\gamma; 0) = \frac{\phi(\gamma(b)) - \phi(\gamma(a))}{2\pi} \ .$$

Since $\gamma(b) = \gamma(a)$, this winding number must be 0. □

The winding number $n(\gamma; w)$ is unchanged if we perturb γ by a sufficiently small amount.

Proposition 4.3 Winding numbers under perturbation
Let $\alpha, \beta : [a, b] \to \mathbb{C}$ *be two closed curve and* w *a point not on* $[\alpha]$*. If*

$$|\beta(t) - \alpha(t)| < |\alpha(t) - w| \qquad \text{for each } t \in [a, b]$$

then $n(\beta; w) = n(\alpha; w)$*.*

Proof:
By translating the curves, we may assume that $w = 0$. Then $|\beta(t) - \alpha(t)| < |\alpha(t)|$ for $t \in [a, b]$. This certainly implies that $\beta(t) \neq 0$, so the winding number $n(\beta; 0)$ exists. Write

$$\beta(t) = \alpha(t) \left(1 + \frac{\beta(t) - \alpha(t)}{\alpha(t)} \right) = \alpha(t)\gamma(t) \ .$$

Since the argument of a product is the sum of the arguments, this implies that

$$n(\beta; 0) = n(\alpha; 0) + n(\gamma; 0) \ .$$

However the inequality in the proposition shows that γ takes values in the disc $B(1, 1)$ so the lemma proves that $n(\gamma; 0) = 0$. □

Proposition 4.4 Winding number constant on each component
Let γ *be a piecewise continuously differentiable closed curve in* $\mathbb{C}$*. The winding number* $n(\gamma; w)$ *is constant for* w *in each component of* $\mathbb{C} \setminus [\gamma]$ *and is 0 on the unbounded component.*

Proof:

The image $[\gamma]$ is a compact subset of $\mathbb{C}$, so it is bounded, say $[\gamma] \subset B(0, R)$. The complement $U = \mathbb{C} \setminus [\gamma]$ is open, so each component of the complement is also open. One component contains $\mathbb{C} \setminus B(0, R)$, so it is the unique unbounded component that contains all points of sufficiently large modulus.

Let $w_o \in U = \mathbb{C} \setminus [\gamma]$. Then there is a disc $B(w_o, r) \subset U$. For w with $|w - w_o| < r$ we have

$$|(\gamma(t) - w) - (\gamma(t) - w_o)| = |w - w_o| < r \leqslant |\gamma(t) - w_o| .$$

Proposition 4.3 then shows that $n(\gamma; w) = n(\gamma; w_o)$. So the function $w \mapsto n(\gamma; w)$ is continuous (indeed constant) at w_o. It follows that $w \mapsto n(\gamma; w)$ is a continuous integer-valued function on U. It must therefore be constant on each component of U.

Lemma 4.2 shows that $n(\gamma; w) = 0$ for w outside the disc $B(0, R)$. So the winding number must be 0 on the unbounded component of U. □

Homotopy

Let $\gamma_0, \gamma_1 : [a, b] \to D$ be two piecewise continuously differentiable closed curves in the domain D. A homotopy from γ_0 to γ_1 is a family of piecewise continuously differentiable closed curves γ_s for $s \in [0, 1]$ that vary continously from γ_0 to γ_1. This means that the map

$$h : [0, 1] \times [a, b] \to D \ ; \quad (s, t) \mapsto \gamma_s(t)$$

is continuous. More formally, we define a *homotopy* to be a continuous map $h : [0, 1] \times [a, b] \to D$ with

$$h_s : [a, b] \to D \ ; \quad t \mapsto h(s, t)$$

being a piecewise continuously differentiable closed curve in D for each $s \in [0, 1]$. We then say that the curves h_0 and h_1 are homotopic and write $h_0 \simeq h_1$. This gives an equivalence relation between closed curves in D.

Example: Suppose that $\gamma_0, \gamma_1 : [0, 1] \to D$ are closed paths in the domain D and that, for each $t \in [0, 1]$, the line segment $[\gamma_0(t), \gamma_1(t)]$ lies within D. Then the map

$$h : [0, 1] \times [0, 1] \to D \ ; \quad (s, t) \mapsto (1 - s)\gamma_0(t) + s\gamma_1(t)$$

is continuous and defines a homotopy from γ_0 to γ_1. We sometimes call such a homotopy a *linear* homotopy.

A closed curve γ in D is *null-homotopic* if it is homotopic in D to a constant curve. The domain D is *simply-connected* if every closed curve in D is null-homotopic. For example, a disc $B(z_o, r)$ is simply-connected since there is a linear homotopy from any curve γ in the disc to z_o.

A domain $D \subset \mathbb{C}$ is called a *star with centre* z_o if, for each point $w \in D$ the entire line segment $[z_o, w]$ lies within D. A domain D is a *star domain* if it is a star with some centre z_o. Clearly every disc is a star domain but such domains as $\mathbb{C} \setminus \{0\}$ are not. Every star domain is simply-connected because a curve is linearly homotopic to the constant curve at the centre.

Proposition 4.5 Winding number and homotopy
If two closed curves γ_0 and γ_1 are homotopic in a domain D and $w \in \mathbb{C} \setminus D$, then $n(\gamma_0; w) = n(\gamma_1; w)$.

Proof:

By translating the curves and the domain, we may assume that $w = 0$.

Let $h : [0,1] \times [a,b] \to D$ be the homotopy with $\gamma_0 = h_0$ and $\gamma_1 = h_1$. Since $[0,1] \times [a,b]$ is a compact subset of D, there is an $\varepsilon > 0$ with $|h_s(t)| > \varepsilon$ for each $(s,t) \in [0,1] \times [a,b]$. The homotopy h is uniformly continuous. Hence there is a $\delta > 0$ with

$$|h_s(t) - h_u(t)| < \varepsilon \qquad \text{whenever} \quad |s-u| < \delta \ .$$

This means that

$$|h_s(t) - h_u(t)| < |h_u(t)| \qquad \text{whenever} \quad |s-u| < \delta \ .$$

Hence Proposition 4.4 shows that

$$n(h_s; 0) = n(h_u; 0) \qquad \text{whenever} \quad |s-u| < \delta \ .$$

This clearly establishes the result. □

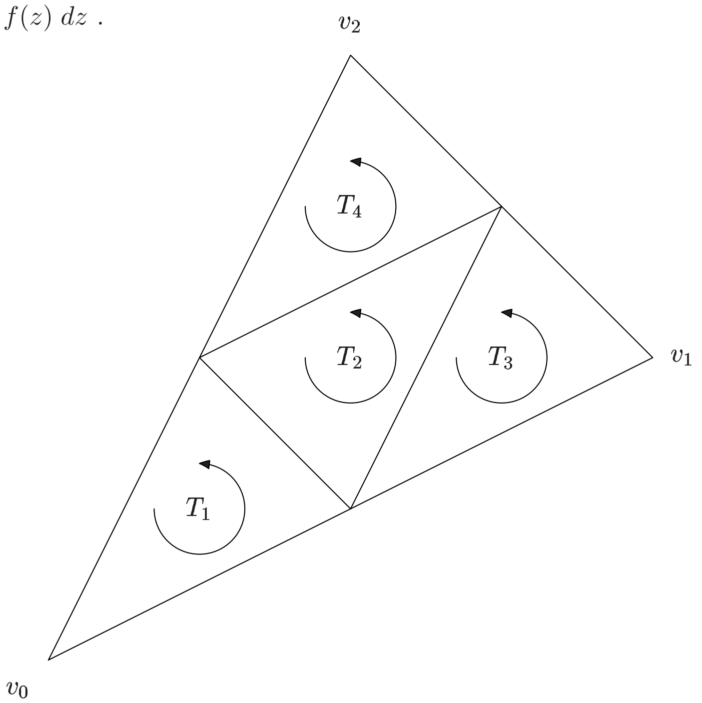
Let T be a closed triangle that lies inside the domain D. Let v_0, v_1, v_2 be the vertices labelled in anti-clockwise order around T. Then the edges $[v_0, v_1]$, $[v_1, v_2]$, $[v_2, v_0]$ are straight-line paths in D. The three sides taken in order give a closed curve $[v_0, v_1] + [v_1, v_2] + [v_2, v_0]$in D that we denote by ∂T.

Proposition 5.1 Cauchy's theorem for triangles
Let $f : D \to \mathbb{C}$ be an analytic function and T a closed triangle that lies within D. Then

$$\int_{\partial T} f(z)\, dz = 0\ .$$

This proof is due to Goursat and relies on repeated bisection. It underlies all the stronger versions of Cauchy's theorem that we will prove later.

Proof:
Set $I = \displaystyle\int_{\partial T} f(z)\, dz$.



Subdivide T into four similar triangles T_1, T_2, T_3, T_4 as shown. Then we have

$$\sum_{k=1}^{4} \int_{\partial T_k} f(z)\, dz = \int_{\partial T} f(z)\, dz$$

because the integrals along the sides of T_k in the interior of T cancel. At least one the integrals

$$\int_{\partial T_k} f(z)\, dz$$

must have modulus at least $\frac{1}{4}|I|$. Choose one of the triangles with this property and call it T'. Repeating this procedure we obtain sequence of triangles $(T^{(n)})$ nested inside one another with

$$\left| \int_{\partial T^{(n)}} f(z)\, dz \right| \geqslant \frac{|I|}{4^n}\ .$$

Let $L(\gamma)$ denote the length of a path γ and set $L = L(\partial T)$. Then each T_k has $L(\partial T_k) = \frac{1}{2}L$. Therefore, $L(\partial T^{(n)}) = L/2^n$.

The triangle T is a compact subset of $\mathbb{C}$ with $T^{(n)}$ closed subsets. If the intersection $\bigcap_{n\in\mathbb{N}} T^{(n)}$ of these sets were empty, then the complements $T \setminus T^{(n)}$ would form an open cover of T with no finite subcover. Therefore, we must have $\bigcap_{n\in\mathbb{N}} T^{(n)}$ non-empty. Choose a point $z_o \in \bigcap_{n\in\mathbb{N}} T^{(n)}$.

The function f is differentiable at z_o. So, for each $\varepsilon > 0$, there is a $\delta > 0$ with

$$\left| \frac{f(z) - f(z_o)}{z - z_o} - f'(z_o) \right| < \varepsilon$$

whenever $z \in B(z_o, \delta)$. This means that

$$f(z) = f(z_o) + f'(z_o)(z - z_o) + \eta(z)(z - z_o)$$

with $|\eta(z)| < \varepsilon$ for $z \in B(z_o, \delta)$. For n sufficiently large, we have $T^{(n)} \subset B(z_o, \delta)$, so

$$\left| \int_{\partial T^{(n)}} f(z)\ dz \right| = \left| \int_{\partial T^{(n)}} f(z_o) + f'(z_o)(z - z_o) + \eta(z)(z - z_o)\ dz \right| .$$

The integrals

$$\int_{\partial T^{(n)}} f(z_o)\ dz \quad \text{and} \quad \int_{\partial T^{(n)}} f'(z_o)(z - z_o)\ dz$$

can be evaluated explicitly and are both zero, so

$$\left| \int_{\partial T^{(n)}} f(z)\ dz \right| \leqslant \int_{\partial T^{(n)}} \varepsilon |z - z_o|\ dz \leqslant \varepsilon L(\partial T^{(n)}) \sup\{|z - z_o| : z \in \partial T^{(n)}\} \leqslant \varepsilon L(\partial T^{(n)})^2 = \varepsilon \frac{L^2}{4^n} .$$

This gives

$$|I| = \left| \int_{\partial T} f(z)\ dz \right| \leqslant 4^n \left| \int_{\partial T^{(n)}} f(z)\ dz \right| \leqslant \varepsilon L^2 .$$

This is true for all $\varepsilon > 0$, so we must have $I = 0$. □

We can use this proposition to prove Cauchy's theorem for discs. The proof actually works for any star domain.

Theorem 5.2 Cauchy's theorem for a star domain
Let $f : D \to \mathbb{C}$ be an analytic function on a star domain $D \subset \mathbb{C}$ and let γ be a piecewise continuously differentiable closed curve in D. Then

$$\int_\gamma f(z)\ dz = 0 .$$

Proof:

Let D be the star domain with centre z_o then each line segment $[z_o, z]$ to a point $z \in D$ lies within D. By Proposition 4.1 we need only show that there is an anti-derivative F of f, that is a function with $F'(z) = f(z)$ for $z \in D$. Define $F : D \to \mathbb{C}$ by

$$F(w) = \int_{[z_o, w]} f(z)\ dz .$$

Then Cauchy's theorem for the triangle with vertices z_o, w and $w + h$ gives

$$F(w + h) - F(w) = \int_{[w, w+h]} f(z)\ dz .$$

Consequently,

$$|F(w + h) - F(w) - f(w)h| = \left| \int_{[w, w+h]} f(z) - f(w)\ dz \right| \leqslant |h|. \sup\{|f(z) - f(w)| : z \in [w, w + h]\} .$$

The continuity of f at w shows that $\sup\{|f(z) - f(w)| : z \in [w, w + h]\}$ tends to 0 as h tends to 0. Hence F is differentiable at w and $F'(w) = f(w)$. □

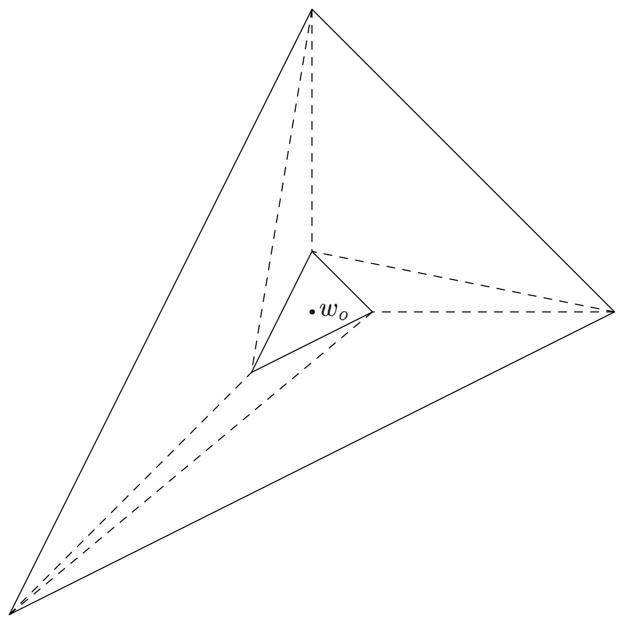
We wish to apply Theorem 5.2 under slightly weaker conditions on f. We want to allow there to be a finite number of exceptional points in D where f is not necessarily differentiable but is continuous. Later we will see that such a function must, in fact, be differentiable at each exceptional point.

Proposition 5.1' Cauchy's theorem for triangles
Let $f : D \to \mathbb{C}$ be a continuous function that is complex differentiable at every point except $w_o \in D$. Let T be a closed triangle that lies within D. Then

$$\int_{\partial T} f(z)\, dz = 0 .$$

Proof:
If $w_o \notin T$, then this result is simply Proposition 5.1 . Hence, we may assume that $w_o \in T$. Let T^ε be the triangle obtained by enlarging T with centre w_o by a factor $\varepsilon < 1$. Then we can divide $T \setminus T^\varepsilon$ into triangles that lie entirely within $T \setminus \{w_o\}$. The integral around each of these triangles is 0 by Proposition 5.1. Adding these results we see that

$$\int_{\partial T} f(z)\, dz = \int_{\partial T^\varepsilon} f(z)\, dz .$$



Since f is continuous on D, there is a constant K with $|f(z)| \leqslant K$ for every $z \in T$. Therefore,

$$\left| \int_{\partial T} f(z)\, dz \right| = \left| \int_{\partial T^\varepsilon} f(z)\, dz \right| \leqslant L(\partial T^\varepsilon)K = \varepsilon L(\partial T)K .$$

This is true for every $\varepsilon > 0$, so we must have $\displaystyle\int_{\partial T} f(z)\, dz = 0$ as required. □

This proposition allows us to extend Cauchy's Theorem 5.2 to functions that fail to be differentiable at one point (or, indeed, at a finite number of points).

Theorem 5.2' Cauchy's theorem for a star domain
Let $f : D \to \mathbb{C}$ be a continuous function on a star domain $D \subset \mathbb{C}$ that is complex differentiable at every point except $w_o \in D$. Let γ be a piecewise continuously differentiable closed curve in D. Then

$$\int_\gamma f(z)\, dz = 0 .$$

Proof:

We argue exactly as in the proof of Theorem 5.2. Let z_o be a centre for the star domain D and define $F(z)$ to be the integral of f along the straight line path $[z_o, z]$ from z_o to z. The previous proposition shows that

$$F(z+h) - F(z) = \int_{[z,z+h]} f(z)\, dz\ .$$

So F is differentiable with $F'(z) = f(z)$ for each $z \in D$. Now Proposition 4.1 gives the result. □

The crucial application of this corollary is the following. Suppose that $f : D \to \mathbb{C}$ is an analytic function on a disc $D = B(z_o, R) \subset \mathbb{C}$ and $w_o \in D$. Then we can define a new function $g : D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z) - f(w_o)}{z - w_o} & \text{for } z \neq w_o; \\ f'(w_o) & \text{for } z = w_o. \end{cases}$$

This is certainly complex differentiable at each point of D except w_o. At w_o we know that f is differentiable, so g is continuous. We can now apply Theorem 5.2' to g and obtain

$$0 = \int_\gamma g(z)\, dz = \int_\gamma \frac{f(z) - f(w_o)}{z - w_o}\, dz$$

for any closed curve γ in D that does not pass through w_o. Now

$$0 = \int_\gamma g(z)\, dz = \int_\gamma \frac{f(z) - f(w_o)}{z - w_o}\, dz = \int_\gamma \frac{f(z)}{z - w_o}\, dz - f(w_o) \int_\gamma \frac{1}{z - w_o}\, dz\ .$$

So we obtain

$$f(w_o) n(\gamma; w_o) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w_o}\, dz\ . \qquad (*)$$

This applies, in particular, when γ is the boundary of a circle contained in D.

Theorem 5.3 Cauchy's Representation Formula
Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D \subset \mathbb{C}$ and let $\overline{B(z_o, R)}$ be a closed disc in D. Then

$$f(w) = \frac{1}{2\pi i} \int_{C(z_o,R)} \frac{f(z)}{z - w}\, dz \qquad \text{for } w \in D(z_o, R)$$

when $C(z_0, R)$ is the circular path $C(z_0, R) : [0, 2\pi] \to \mathbb{C}\ ;\ \ t \mapsto z_o + Re^{it}$.

Proof:

This follows immediately from formula $(*)$ above since the winding number of $C(z_o, R)$ about any $w \in B(z_o, R)$ is 1. □

Cauchy's representation formula is immensely useful for proving the local properties of analytic functions. These are the properties that hold on small discs rather then the global properties that require we study a function on its entire domain. The next chapter will use the representation formula frequently but, as a first example:

Example: Let $f : D \to \mathbb{C}$ be an analytic function on a domain D. For $z_o \in D$ there is a closed disc $\overline{B(z_o, R)}$ within D and Cauchy's representation formula gives

$$f(z_o) = \frac{1}{2\pi i} \int_{C(z_o,R)} \frac{f(z)}{z - z_o}\, dz = \int_0^{2\pi} f(z_o + Re^{i\theta})\, \frac{d\theta}{2\pi}\ .$$

So the value of f at the centre of the circle is the average of the values on the circle C.

Theorem 5.4 Liouville's theorem
Any bounded analytic function $f : \mathbb{C} \to \mathbb{C}$ defined on the entire complex plane is constant.

Proof:

Let w, w' be any two points of $\mathbb{C}$ and let M be an upper bound for $|f(z)|$ for $z \in \mathbb{C}$. Then Cauchy's representation formula gives

$$f(w) = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f(z)}{z-w} \, dz \qquad \text{for each } r > |w| \ .$$

Hence,

$$f(w) - f(w') = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f(z)}{z-w} - \frac{f(z)}{z-w'} \, dz = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f(z)(w-w')}{(z-w)(z-w')}$$

for $r > \max\{|w|, |w'|\}$. Consequently,

$$|f(w) - f(w')| \leqslant \frac{L(C(0,r))}{2\pi} \sup \left\{ \frac{|f(z)||w-w'|}{|z-w||z-w'|} : |z| = r \right\} \leqslant r \left(\frac{M|w-w'|}{(r-|w|)(r-|w'|)} \right) \ .$$

The right side tends to 0 as $r \nearrow +\infty$, so the left side must be 0. Thus $f(w) = f(w')$. □

Corollary 5.5 The Fundamental Theorem of Algebra
Every non-constant polynomial has a zero in $\mathbb{C}$*.*

Proof:

Suppose that $p(z) = z^N + a_{N-1}z^{N-1} + \ldots a_1 z + a_0$ is a polynomial that has no zero in $\mathbb{C}$. Then $f(z) = 1/p(z)$ is an analytic function. As $z \to \infty$ so $f(z) \to 0$. Hence f is bounded. By Liouville's theorem, p must be constant. □

By dividing a polynomial by $z - z_o$ for each zero z_o we see that the total number of zeros of p, counting multiplicity, is equal to the degree of p.

Homotopy form of Cauchy's Theorem.

Let $f : D \to \mathbb{C}$ be an analytic function on a domain D. We wish to study how the integral

$$\int_\gamma f(z) \, dz$$

varies as we vary the closed curve γ in D. Recall that two closed curves $\beta, \gamma : [a, b] \to D$ are *linearly homotopic in* D if, for each $t \in [a, b]$ the line segment $[\beta(t), \gamma(t)]$ is a subset of D.

Theorem 5.6 Homotopy form of Cauchy's Theorem.
Let $f : D \to \mathbb{C}$ *be an analytic map on a domain* $D \subset \mathbb{C}$*. If the two piecewise continuously differentiable closed curves* α*,* β *are homotopic in* D*, then*

$$\int_\alpha f(z) \, dz = \int_\beta f(z) \, dz \ .$$

Proof:

Let $h : [0,1] \times [a,b] \to D$ be the homotopy. So each map $h_s : [a,b] \to D \; ; \quad t \mapsto h(s,t)$ is a piecewise continuously differentiable closed curve in D, $h_0 = \alpha$ and $h_1 = \beta$. This means that h is piecewise continuously differentiable on each "vertical" line $\{s\} \times [a,b]$. Initially we will assume that h is also continuously differentiable on each "horizontal" line $[0,1] \times \{t\}$. For any rectangle

$$Q = \{(s,t) \in [0,1] \times [a,b] : s_1 \leqslant s \leqslant s_2 \text{ and } t_1 \leqslant t \leqslant t_2\}$$

let ∂Q denote the boundary of Q positively oriented. Then h is piecewise continuously differentiable on each segment of the boundary, so $h(\partial Q)$ is a piecewise continuously differentiable closed curve in D. If we divide Q into two smaller rectangles Q_1, Q_2 by drawing a horizontal or vertical line ℓ then the segments of the integrals $\int_{h(\partial Q_1)} f(z)\, dz$ and $\int_{h(\partial Q_2)} f(z)\, dz$ along ℓ cancel, so

$$\int_{h(\partial Q)} f(z)\, dz = \int_{h(\partial Q_1)} f(z)\, dz + \int_{h(\partial Q_2)} f(z)\, dz \; .$$

For the original rectangle $R = [0,1] \times [a,b]$ the image of the horizontal sides $[0,1] \times \{a\}$ and $[0,1] \times \{b\}$ are the same since each h_s is closed. Hence

$$\int_{h(\partial R)} f(z)\, dz = \int_\beta f(z)\, dz - \int_\alpha f(z)\, dz \; .$$

We need to show that this is 0.

Define $\rho(z) = \inf\{|z-w| : w \in \mathbb{C} \setminus D\}$ to be the distance from $z \in D$ to the complement of D. Since D is open, $\rho(z) > 0$ for each $z \in D$. Moreover, ρ is continuous since $|\rho(z) - \rho(z')| \leqslant |z - z'|$. Hence, ρ attains a minimum value on the compact set $h(R)$, say

$$\rho(h(s,t)) \geqslant r > 0 \qquad \text{for every } s \in [0,1],\; t \in [a,b] \; .$$

This means that each disc $B(h(s,t), r)$ is contained in D.

Furthermore, Proposition 2.12 shows that h is uniformly continuous. So there is a $\delta > 0$ with

$$|h(u,v) - h(s,t)| \leqslant r \qquad \text{whenever} \quad ||(u,v) - (s,t)|| < \delta \; . \tag{$*$}$$

Suppose that Q is a rectangle in R with diameter less than δ and P_o a point in Q. Then $h(Q) \subset B(h(P_o), r)$ and the disc $B(h(P_o), r)$ is a subset of D. Cauchy's theorem for star domains (5.2) can now be applied to this disc to see that

$$\int_{h(\partial Q)} f(z)\, dz = 0 \; .$$

We can divide R into rectangles $(Q_n)_{n=1}^N$ each with diameter less than δ. So

$$\int_{h(\partial R)} f(z)\, dz = \sum_{n=1}^{N} \int_{h(\partial Q_n)} f(z)\, dz = 0$$

as required.

It remains to deal with the case where the homotopy h is not continuously differentiable on each horizontal line. Choose a subdivision

$$0 = s(0) < s(1) < \ldots < s(N-1) < s(N) = 1$$

of $[0,1]$ with $|s(k+1) - s(k)| < \delta$ for $k = 0, 1, \ldots, N-1$. Then equation $(*)$ above shows that $|h(s(k), t) - h(s(k+1), t)| < r$ for each $t \in [a,b]$. Hence the entire line segment $[h(s(k), t), h(s(k+1), t)]$ lies in the disc $B(h(s(k), t), r)$ and hence in D. So $h_{s(k)}$ and $h_{s(k+1)}$ are **LINEARLY** homotopic in D. We can certainly apply the above argument to linear homotopies, so we see that

$$\int_{h_{s(k)}} f(z)\, dz = \int_{h_{s(k+1)}} f(z)\, dz \; .$$

Adding these results gives

$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz \; .$$

□

Corollary 5.7 Cauchy's Theorem for null-homotopic curves
Let $f : D \to \mathbb{C}$ be an analytic map on a domain D and γ a piecewise continuously differentiable closed curve in D that is null-homotopic in D. Then

$$\int_\gamma f(z)\, dz = 0 \ .$$

□

If the domain D is simply connected, then any closed curve in D is null-homotopic, so Cauchy's theorem will apply.

6. POWER SERIES

A *power series* is an infinite sum of the form $\sum_{n=0}^{\infty} a_n(z-z_o)^n$. Recall that a power series converges on a disc.

Proposition 6.1 Radius of convergence
For the sequence of complex numbers (a_n) define $R = \sup\{r : a_n r^n \to 0 \text{ as } n \to \infty\}$. Then the power series $\sum a_n z^n$ converges absolutely on the open disc $B(z_o, R)$ and diverges outside the corresponding closed disc $\overline{B(z_o, R)}$. Indeed, the power series converges uniformly on each disc $B(z_o, r)$ with r strictly less than R.

We call R the *radius of convergence of the power series $\sum a_n(z-z_o)^n$. It can take any value from 0 to $+\infty$ including the extreme values. The series may converge or diverge on the circle $\partial B(z_o, R)$.*

Proof:
It is clear that if $\sum a_n(z-z_o)^n$ converges then the terms $a_n(z-z_o)^n$ must tend to 0 as $n \to \infty$. Therefore, $a_n r^n \to 0$ as $n \to \infty$ for each $r \leqslant |z-z_o|$. Hence $R \geqslant |z-z_o|$ and we see that the power series diverges for $|z-z_o| > R$.

Suppose that $|z-z_o| < R$. Then we can find r with $|z-z_o| < r < R$ and $a_n r^n \to 0$ as $n \to \infty$. This means that there is a constant K with $|a_n|r^n \leqslant K$ for each $n \in \mathbb{N}$. Hence

$$\sum |a_n||z-z_o|^n \leqslant \sum K \left(\frac{|z-z_o|}{r}\right)^n .$$

The series on the right is a convergent geometric series, and $\sum a_n z^n$ converges, absolutely, by comparison with it. Also, this convergence is uniform on $B(z_o, r)$. □

We wish to prove that a power series can be differentiated term-by-term within its disc of convergence.

Proposition 6.2 Power series are differentiable.
Let R be the radius of convergence of the power series $\sum a_n(z-z_o)^n$. The sum $s(z) = \sum_{n=0}^{\infty} a_n(z-z_o)^n$ is complex differentiable on the disc $B(z_o, R)$ and has derivative $t(z) = \sum_{n=1}^{\infty} n a_n (z-z_o)^{n-1}$.

Proof:
We may assume that $z_o = 0$. For a fixed point w with $|w| < R$, we can choose r with $|w| < r < R$. We will consider h satisfying $|h| < r - |w|$ so that $|w+h| < r$.

First note that

$$(w+h)^n - w^n - nw^{n-1}h = \int_{[w,w+h]} nz^{n-1} - nw^{n-1}\, dz = \int_{[w,w+h]} \int_{[w,z]} n(n-1)u^{n-2}\, du\, dz .$$

Since $|u^{n-2}| \leqslant r^{n-2}$ for $|u| < r$, this implies that

$$|(w+h)^n - w^n - nw^{n-1}| \leqslant |h| \sup \left\{ \left| \int_{[w,z]} n(n-1)u^{n-2}\, du \right| : u \in [w, w+h] \right\} \leqslant |h|^2 n(n-1) r^{n-2} .$$

Hence,

$$\begin{aligned} |s(w+h) - s(w) - t(w)h| &= \left| \sum_{n=0}^{\infty} a_n \left((w+h)^n - w^n - nw^{n-1}h \right) \right| \\ &\leqslant \sum_{n=0}^{\infty} |a_n||(w+h)^n - w^n - nw^{n-1}h| \\ &\leqslant \left(\sum_{n=0}^{\infty} n(n-1)|a_n| r^{n-2} \right) |h|^2 . \end{aligned}$$

The series $\sum n(n-1)|a_n|r^{n-2}$ converges by comparison with $\sum |a_n|s^n$ for any s with $r < s < R$. Therefore, s is differentiable at w and $s'(w) = t(w)$. □

The derivative of the power series s is itself a power series, so s is twice differentiable. Repeating this shows that s is infinitely differentiable, that is we can differentiate it as many times as we wish.

Corollary 6.3 Power series are infinitely differentiable
Let R be the radius of convergence of the power series $\sum a_n(z-z_o)^n$. Then the sum

$$s(z) = \sum_{n=0}^{\infty} a_n(z-z_o)^n$$

is infinitely differentiable on $B(z_o, R)$ with

$$s^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n(z-z_o)^{n-k} .$$

In particular, $s^{(k)}(z_o) = k!a_k$, so the power series is the Taylor series for s.

□

Cauchy Transforms

Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise continuously differentiable path in $\mathbb{C}$ and $\phi : [\gamma] \to \mathbb{C}$ a continuous function on $[\gamma]$. Then the integral

$$\Phi(w) = \frac{1}{2\pi i} \int_\gamma \frac{\phi(z)}{z-w} \, dz$$

exists for each $w \in \mathbb{C} \setminus [\gamma]$. This is the *Cauchy transform of ϕ*. We will show that it defines a function analytic everywhere except on $[\gamma]$.

Proposition 6.4 Cauchy transforms have power series
Let Φ be the Cauchy transform of a continuous function $\phi : [\gamma] \to \mathbb{C}$. For $z_o \in \mathbb{C} \setminus [\gamma]$ let R be the radius of the largest disc $B(z_o, R)$ that lies within $\mathbb{C} \setminus [\gamma]$. Then

$$\Phi(w) = \sum_{n=0}^{\infty} a_n(w-z_o)^n \qquad \text{for} \quad |w-z_o| < R$$

where the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{\phi(z)}{(z-z_o)^{n+1}} \, dz .$$

Proof:

We may assume, by translating γ, that $z_o = 0$. The formula for the sum of a geometric series shows that

$$\frac{1}{z-w} = \frac{1}{z} + \frac{w}{z^2} + \ldots + \frac{w^{N-1}}{z^N} + \frac{w^N}{z^N(z-w)} \ .$$

Integrating this gives

$$\Phi(w) = a_0 + a_1 w + \ldots + a_{N-1}w^{N-1} + E_N(w)$$

where

$$a_k = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)}{z^{k+1}}\, dz \qquad \text{and} \qquad E_N(w) = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)w^N}{z^N(z-w)}\, dz \ .$$

Let $||\phi||_\infty = \sup\{|\phi(z)| : z \in [\gamma]\}$. For $z \in [\gamma]$ we have $|z| \geqslant R$ and $|z-w| \geqslant R - |w|$, so

$$|E_N(w)| \leqslant \frac{L(\gamma)}{2\pi}\frac{||\phi||_\infty}{(R-|w|)}\left(\frac{|w|}{R}\right)^N \ .$$

This shows that, for $|w| < R$,

$$\left|\Phi(w) - \sum_{n=0}^{N-1} a_n w^n\right| = |E_N(w)| \to 0 \qquad \text{as } N \to \infty \ .$$

Therefore the power series $\sum a_n w^n$ converges on $B(0,R)$ to Φ. □

Corollary 6.5 Cauchy transforms are infinitely differentiable
The Cauchy transform Φ of a continuous function $\phi : [\gamma] \to \mathbb{C}$ is infinitely differentiable on $\mathbb{C} \setminus [\gamma]$ with

$$\Phi^{(n)}(z_o) = \frac{n!}{2\pi i}\int_\gamma \frac{\phi(z)}{(z-z_o)^{n+1}}\, dz \ .$$

Proof:

We know that Φ is given by a power series $\Phi(z) = \sum_{n=0}^\infty a_n(z-z_o)^n$ on the disc $B(z_o,R)$. By Corollary 6.3 this power series is infinitely differentiable. Moreover,

$$\Phi^{(n)}(z_o) = n!a_n = \frac{n!}{2\pi i}\int_\gamma \frac{\phi(z)}{(z-z_o)^{n+1}}\, dz$$

as required. □

If we apply these results to the Cauchy representation formula we obtain the following theorem.

Theorem 6.6 Analytic functions have power series
Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D \subset \mathbb{C}$. For each point $z_o \in D$, let R be the radius of the largest disc $B(z_o,R)$ that lies within D. Then

$$f(z) = \sum_{n=0}^\infty a_n(z-z_o)^n \qquad \text{for} \quad |z-z_o| < R$$

where the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_o)^{n+1}}\, dz$$

for C_r the circle of radius r $(0 < r < R)$ about z_o. Therefore, f is infinitely differentiable on D and we have representation formulae

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_{C_r} \frac{f(z)}{(z-w)^{n+1}}\, dz$$

for w with $|w-z_o| < r$.

Proof:

For $0 < r < R$, let C_r be the circle of radius r with centre z_o. The Cauchy representation formula (Theorem 5.3) shows that f is the Cauchy transform

$$f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} \, dz$$

for $w \in B(z_o, r)$. Hence, f must be given by a power series $\sum_{n=0}^{\infty} a_n (w - z_o)^n$ on this disc $B(z_o, r)$. The coefficients a_n must be

$$a_n = \frac{f^{(n)}(z_o)}{n!} \ ,$$

which is independent of r. This holds for all $r < R$, so the series $\sum_{n=0}^{\infty} a_n (w - z_o)^n$ must converge on all of $B(z_o, R)$.

Also Corollary 6.5 shows that the Cauchy transform satisfies

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z - w)^{n+1}} \, dz \ .$$

□

This theorem has many useful consequences. Our first will be a partial converse of Cauchy's theorem.

Proposition 6.7 Morera's theorem
Let $f : D \to \mathbb{C}$ be a continuous function on a domain $D \subset \mathbb{C}$. If, for every closed triangle $T \subset D$, the integral $\int_{\partial T} f(z) \, dz$ is 0, then f is analytic.

Proof:

Let $z_o \in D$ and choose $R > 0$ so that $B(z_o, R) \subset D$. Then we can define a function $F : B(z_o, R) \to \mathbb{C}$ by

$$F(z) = \int_{[z_o, z]} f(z) \, dz \ .$$

Since f is continuous, the fundamental theorem of calculus shows that F is complex differentiable at each point of $B(z_o, R)$ with $F'(z) = f(z)$ (compare Theorem 5.2). Now F is analytic on the disc $B(z_o, R)$ and so the previous theorem shows that it is twice continuously differentiable. Thus $f'(z) = F''(z)$ exists. □

Note that the result fails if we do not insist that f is continuous. For example the function $f : \mathbb{C} \to \mathbb{C}$ that is 0 except at at a single point is not analytic.

The Local Behaviour of Analytic Functions

The power series expansion for an analytic function is very useful for describing the local behaviour of analytic functions. A key result is that the zeros of an non-constant analytic function are isolated. This means that if $f : D \to \mathbb{C}$ is a non-constant analytic function and $f(z_o) = 0$, then there is a neighbourhood V of z_o on which f has no other zeros.

Theorem 6.8 Isolated Zeros
The zeros of a non-constant analytic function are isolated.

Proof:

Let $f : D \to \mathbb{C}$ be an analytic function. For each $z \in D$ we know that there is a power series

$$f(w) = \sum_{n=0}^{\infty} a_n (w-z)^n$$

that converges to $f(w)$ on some disc $B(z, R)$. The coefficients a_n are given by $f^{(n)}(z)/n!$. If all the coefficients a_n are 0, then f is zero on the entire disc $B(z, R)$. Conversely, if f is zero on some neighbourhood V of z, then each derivative $f^{(n)}(z)$ is 0 and so each coefficient a_n is 0.

Let A be the set: $\{z \in D :$ there is a neighbourhood V of z with $f(w) = 0$ for all $w \in V\}$. This is clearly open. However, we have shown that $A = \{z \in D : f^{(n)}(z) = 0 \text{ for all } n = 0, 1, 2, \dots\}$. If $z \in B = D \setminus A$, then there is a natural number n with $f^{(n)}(z) \neq 0$. Since $f^{(n)}$ is continuous, $f^{(n)}(w) \neq 0$ on some neighbourhood of z. Therefore, B is also open. Since D is connected, one of the two sets A, B must be empty. If B is empty, then f is constantly 0 on D. If A is empty, we will show that the zeros of f are isolated.

Let $f : D \to \mathbb{C}$ be a non-constant analytic function with $f(z) = 0$ for some $z \in D$. Since f is not constant, the set B can not be all of D and must therefore be empty. This means that at least one of the coefficients of the power series

$$f(w) = \sum_{n=0}^{\infty} a_n (w-z)^n \qquad \text{for} \quad w \in B(z, r)$$

is non-zero. Let a_N be the first such coefficient. Then

$$f(w) = (w-z)^N \left(\sum_{n=N}^{\infty} a_n (w-z)^{n-N} \right) .$$

Since the power series $\sum a_n (w-z)^n$ converges on $B(z, r)$, so does $\sum a_n (w-z)^{n-N}$ and it gives an analytic function $F : B(z, r) \to \mathbb{C}$. Note that $F(z) = a_N \neq 0$. Since F is continuous, there is an r with $0 < r < R$ and $F(w) \neq 0$ for $w \in B(z_o, r)$. This means that $f(w) = (w - z_o)^N F(w)$ is not 0 on $B(z_o, r)$ except at z_o. Thus z_o is an isolated zero. □

Corollary 6.9 Identity Theorem
Let $f, g : D \to \mathbb{C}$ be two analytic functions on a domain D. If the set $E = \{z \in D : f(z) = g(z)\}$ contains a non-isolated point, then $f = g$ everywhere on D.

Proof:

E is the set of zeros of the analytic function $f - g$. □

This corollary gives us the *principle of analytic continuation*: If $f : D \to \mathbb{C}$ is an analytic function on a (non-empty) domain D and f extends to an analytic function $F : \Omega \to \mathbb{C}$ on some larger domain Ω, then F is unique. For, if $\widetilde{F} : \Omega \to \mathbb{C}$ were another extension of f, then F and $\widetilde{F}$ would agree on D and hence on all of Ω. However, there may not be any extension of f to a larger domain.

Let $f : D \to \mathbb{C}$ be a non-constant analytic function on a domain $D \subset \mathbb{C}$. For any point $z_o \in D$, we know that $f(z)$ is represented by a power series

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z_o)^n$$

on some disc $B(z_o, R)$. Clearly $a_0 = f(z_o)$. Since the zeros of $f - f(z_o)$ are isolated, there must be a first coefficient (after a_0) that is non-zero, say a_N. We call N the *degree of f at z_o* and write it $\deg(f; z_o)$. We can write f as

$$f(w) = f(z_o) + (w - z_o)^N g(w)$$

for $w \in B(z_o, R)$ and some analytic function $g : B(z_o, R) \to \mathbb{C}$ with $g(z_o) \neq 0$. Indeed, we can define a function F on all of D by

$$F(w) = \begin{cases} \frac{f(w)-f(z_o)}{(w-z_o)^N} & \text{when } w \in D \setminus \{z_o\}; \\ g(w) & \text{when } w \in B(z_o, R). \end{cases}$$

These definitions agree on $B(z_o, R) \setminus \{z_o\}$ and so do define an analytic function $F : D \to \mathbb{C}$ with $f(w) = f(z_o) + (w - z_o)^N F(w)$ on all of D.

Locally Uniform Convergence

Let f_n and f be functions from a domain D into $\mathbb{C}$. We say that $f_n \to f$ *locally uniformly* on D if, for each $z_o \in D$, there is a neighbourhood V of z_o in D with $f_n(z) \to f(z)$ uniformly for $z \in V$.

Example: Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then the partial sums

$$S_N(z) = \sum_{n=0}^{N} a_n z^n$$

converge locally uniformly on $B(0, R)$ to $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This was proven in Proposition 6.1 .

Suppose that $f_n \to f$ on the domain D. Then, for each $z_o \in D$, there is an open disc $\Delta(z_o)$ in D, centred on z_o, with $f_n(z) \to f(z)$ uniformly on $\Delta(z_o)$. If K is any compact subset of D, then K is covered by these sets $\Delta(z_o)$ for $z_o \in K$. Hence, there is a finite subcover. This shows that $f_n \to f$ uniformly on the compact set K. We will use this particularly when K is the image $[\gamma]$ of a curve γ.

Suppose that each of the functions f_n is continuous on D. The uniform limit of continuous functions is continuous, so f is continuous on each $\Delta(z_o)$ and hence on all of D. We will now prove the the locally uniform limit of analytic functions is analytic.

Proposition 6.10 Locally uniform convergence of analytic functions
Let $f_n : D \to \mathbb{C}$ be a sequence of analytic functions on a domain D that converges locally uniformly to a function f. Then f is analytic on D. Moreover, the derivatives $f_n^{(k)}$ converge locally uniformly on D to $f^{(k)}$.

Proof:
Let $z_o \in D$. Then there is a disc $\Delta = B(z_o, r)$ on which f_n converge uniformly to f. The functions f_n are continuous so the uniform limit f is also continuous on Δ. Also, the uniform convergence implies that

$$\int_\gamma f_n(z)\, dz \to \int_\gamma f(z)\, dz$$

for any closed curve γ in Δ. Since f_n is analytic, Cauchy's theorem for the disc Δ implies that $\int_\gamma f_n(z)\, dz = 0$. Therefore, $\int_\gamma f(z)\, dz = 0$. Morera's theorem now shows that f is analytic on Δ. Since z_o is arbitrary, f is analytic on all of D.

Now let $C(z_o, s)$ be the circle of radius $s < r$ about z_o. For $|w| < s$ Cauchy's representation formula (5.3) gives

$$f_n^{(k)}(w) = \frac{k!}{2\pi i} \int_{C(z_o,s)} \frac{f_n(z)}{(z-w)^{k+1}}\, dz$$

and a similar formula for f, which we now know is analytic. Therefore,

$$\begin{aligned} |f_n^{(k)}(w) - f^{(k)}(w)| &= \left| \frac{k!}{2\pi i} \int_{C(z_o,s)} \frac{f_n(z) - f(z)}{(z-w)^{k+1}}\, dz \right| \\ &\leqslant \frac{k!}{2\pi} L(C(z_o, s)) \sup \left\{ \left| \frac{f_n(z) - f(z)}{(z-w)^{k+1}} \right| : |z - z_o| = s \right\} \\ &\leqslant \frac{k!s}{(s - |w - z_o|)^k} \sup\{|f_n(z) - f(z)| : |z - z_o| = s\} \end{aligned}$$

and we see that $f_n^{(k)}(w) \to f^{(k)}(w)$ uniformly on any disc $D(z_o, t)$ with $t < s$. □

This theorem gives us an alternative proof of Proposition 6.2 , which showed that a power series could be differentiated term by term inside its radius of convergence. For suppose that $s(z) = \sum a_n(z - z_o)^n$ is a power series with radius of convergence $R > 0$. Then the partial sums

$$S_N(z) = \sum_{n=0}^{N} a_n(z - z_o)^n$$

converge locally uniformly to s on $B(z_o, R)$. Each S_N is a polynomial and so is certainly analytic. Therefore s is analytic on $B(z_o, R)$. Moreover,

$$s'(z) = \lim_{N\to\infty} S'_N(z) = \lim_{N\to\infty} \sum_{n=0}^{N} na_n(z - z_o)^{n-1} = \sum_{n=0}^{\infty} na_n(z - z_o)^{n-1} \ .$$

Isolated Singularities

Let D be a domain and z_o a point of D. We are concerned about an analytic function $f : D\setminus\{z_o\} \to \mathbb{C}$ that is not defined at the point z_o. We call z_o an *isolated singularity of* f. It is defined and analytic at every point of some disc $B(z_o, R)$ except the centre z_o. We will study the behaviour of f as we approach the singular point.

The simplest possibility for f is that we can extend it to a function analytic on all of D, even at the point z_o. If this is the case, we say that f has a *removable singularity at* z_o. Usually we replace f by the analytic extension:

$$F(z) = \begin{cases} f(z) & \text{when } z \in D \setminus \{z_o\}; \\ w_o & \text{when } z = z_o. \end{cases}$$

Since F is to be continuous, the value w_o it takes at z_o must be $\lim_{z\to z_o} f(z)$ and F is unique. We will now show that f has a removable singularity at z_o if and only if the limit $\lim_{z\to z_o} f(z)$ exists.

Example: The function

$$s : \mathbb{C} \setminus \{0\} \to \mathbb{C} \ ; \quad z \mapsto \frac{\sin z}{z}$$

has a removable singularity at 0. For the power series for the sine function shows that

$$s(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!} \ .$$

So we can extend s to 0 by sending 0 to 1. This extension is given by a power series and so is analytic on all of $\mathbb{C}$.

Proposition 6.11 Removable singularities
The analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ has a removable singularity at $z_o \in D$ if and only if there is a finite limit $w_o \in \mathbb{C}$ with $f(z) \to w_o$ as $z \to z_o$.

Proof:
If f has a removable singularity at z_o, then there is an analytic extension $F : D \to \mathbb{C}$. This extension is continuous, so $f(z) = F(z) \to F(z_o)$ as $z \to z_o$.

For the converse, suppose that $f(z) \to w_o$ as $z \to z_o$. Then we can define

$$F : D \to \mathbb{C} \ ; \quad z \mapsto \begin{cases} f(z) & \text{when } z \in D \setminus \{z_o\}; \\ w_o & \text{when } z = z_o. \end{cases}$$

This is certainly continuous at z_o and analytic elsewhere on D. Therefore, we can apply Cauchy's theorem to any triangle T within D using Proposition 5.1' and obtain $\int_{\partial T} f(z)\, dz = 0$. Morera's theorem now shows that F is analytic on all of D. □

When we proved Cauchy's theorem we considered a function $f : D \to \mathbb{C}$ that was analytic except at one point z_o where it was continuous. The last proposition shows that such a function is actually analytic even at z_o. So the exceptional point is no different from any other.

It is useful to strengthen the last proposition a little.

Corollary 6.12 Riemann's Removable Singularity Criterion
The analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ *has a removable singularity at* $z_o \in D$ *if and only if* $\lim_{z \to z_o} (z - z_o)f(z) = 0$.

Note that when f is bounded in a neighbourhood of z_o, then the limit $\lim(z - z_o)f(z)$ certainly exists and is 0 and so there must be a removable singularity at z_o.

Proof:
The function $g(z) = (z - z_o)f(z)$ is analytic on $D \setminus \{z_o\}$ and tends to 0 as $z \to z_o$. Hence the previous proposition tells us that g has a removable singularity at z_o. Let $G : D \to \mathbb{C}$ be the analytic extension of g. We certainly have $G(z_o) = \lim_{z \to z_o} g(z) = 0$. Hence

$$f(z) = \frac{G(z) - G(z_o)}{z - z_o} \to G'(z_o) \qquad \text{as} \qquad z \to z_o \ .$$

Therefore, the previous proposition shows that f has a removable singularity at z_o. $\square$

So far we have only considered functions $f : D \to \mathbb{C}$ taking values in the finite complex plane $\mathbb{C}$. However, in the Algebra and Geometry course you considered functions taking values in the Riemann sphere (or extended complex plane) $\mathbb{C}_\infty$. The Riemann sphere consists of the complex plane $\mathbb{C}$ and one extra point ∞. You saw that the extra point ∞ behaved in the same way as the finite points in $\mathbb{C}$ and that the Möbius transformations $z \mapsto (az + b)/(cz + d)$ permuted the points of $\mathbb{C}_\infty$. We now wish to explain what it means for a function $f : D \to \mathbb{C}_\infty$ that takes values in the Riemann sphere to be analytic.

Let $f : D \to \mathbb{C}_\infty$ be a function defined on a domain $D \subset \mathbb{C}$ and $z_o \in D$. If $f(z_o) \in \mathbb{C}$, then f is complex differentiable at z_o if the limit $\lim_{z \to z_o} \dfrac{f(z) - f(z_o)}{z - z_o}$ exists and is a point of $\mathbb{C}$. If $f(z_o) = \infty$, we use the Möbius transformation $J : w \mapsto 1/w$ to send ∞ to a finite point and then ask if $J \circ f$ is complex differentiable at z_o. Thus we say that f *is complex differentiable at the point* z_o *with* $f(z_o) = \infty$ if $z \mapsto 1/f(z)$ is complex differentiable at z_o. (It is not useful to define a value for $f'(z_o)$ at points where $f(z_o) = \infty$.) We call a point z_o where $f(z_o) = \infty$ and f is complex differentiable a *pole* of f. A function $f : D \to \mathbb{C}_\infty$ that is is not identically ∞ but is complex differentiable at each point of D is *meromorphic on* D. Since the zeros of a non-constant analytic function are isolated, the poles of a meromorphic function are also isolated. Thus a meromorphic function is analytic on its domain except for a set of poles each of which is isolated. For example, if $f : D \to \mathbb{C}$ is an analytic function and is not identically 0, then $z \mapsto 1/f(z)$ is meromorphic. This implies that each rational function is meromorphic on $\mathbb{C}$.

Suppose that $f : D \to \mathbb{C}$ is a meromorphic function and has a pole at z_o. The function f is certainly continuous at z_o so there is a neighbourhood V of z_o with $|f(z)| > 1$ for $z \in V$. Now the function $g : z \mapsto 1/f(z)$ is complex differentiable and finite at each point of V and it has a zero at z_o. Since f is not identically ∞, g can not be identically 0. Therefore, the zero at z_o is isolated. This means that we can write $g(z) = (z - z_o)^N G(z)$ for some natural number $N \geqslant 1$ and some function G that is analytic near z_o and has $G(z_o) \neq 0$. Therefore $f(z) = (z - z_o)^{-N} F(z)$ where $F(z) = 1/G(z)$ is analytic near z_o and has $F(z_o) \neq 0, \infty$. This show how the meromorphic function f behaves near a pole. We write $N = \deg(f; z_o)$ and call z_o a *pole of order* N *for* f.

We will say that an analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ has a *pole* at $z_o \in D$ if there is a meromorphic function $F : D \to \mathbb{C}_\infty$ that extends f and F has a pole at z_o. This is similar to f having a removable singularity at z_o except that the correct value to put for $f(z_o)$ is ∞.

Proposition 6.13 Poles as isolated singularities
The analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ *has a pole at* z_o *if and only if* $f(z) \to \infty$ *as* $z \to z_o$.

Proof:

If f has an extension F with a pole at z_o, then $f(z) = F(z) \to F(z_o) = \infty$ as $z \to z_o$.

For the converse, suppose that $f(z) \to \infty$ as $z \to z_o$. There is a neighbourhood V of z_o with $|f(z)| > 1$ for $z \in V \setminus \{z_o\}$. Hence, $g : z \mapsto 1/f(z)$ is bounded, analytic on $V \setminus \{z_o\}$ and has $g(z) \to 0$ as $z \to z_o$. Proposition 6.11 shows that g has a removable singularity at z_o so there is a function $G : V \to \mathbb{C}$ extending g. Now the function

$$F : z \mapsto \begin{cases} f(z) & \text{when } z \in D \setminus \{z_o\}; \\ 1/G(z) & \text{when } z \in V. \end{cases}$$

is well-defined and gives a meromorphic extension of f. □

There remain some isolated singularities that are neither removable singularities nor poles. We call these *essential singularities*. Functions behave very dramatically near an essential singularity.

Example: The function $f : z \mapsto \exp(1/z)$ has an essential singularity at 0. For real values of t we have

$$\exp(1/t) \to \infty \quad \text{as} \quad t \searrow 0+ \qquad \text{while} \qquad \exp(1/t) \to 0 \quad \text{as} \quad t \nearrow 0-$$

so the limit $\lim_{z \to 0} f(z)$ can not exist either as a finite complex number or as ∞. Therefore, f can not have either a removable singularity or a pole at 0.

Proposition 6.14 Weierstrass - Casorati Theorem
An analytic function takes values arbitrarily close to any complex number on any neighbourhood of an essential singularity.

Proof:

Let $f : D \setminus \{z_o\} \to \mathbb{C}$ be an analytic function with an isolated singularity at z_o. Suppose that there is some neighbourhood of z_o on which f does not take values arbitrarily close to $w_o \in \mathbb{C}$. Say

$$|f(z) - w_o| > \varepsilon \qquad \text{for} \quad 0 < |z - z_o| < R\ .$$

Then the function $g : z \mapsto 1/(f(z) - w_o)$ is bounded by $1/\varepsilon$ for $0 < |z - z_o| < R$. Therefore, g has a removable singularity at z_o by Corollary 6.12 . Consequently, $f(z) = w_o + 1/g(z)$ will have a removable singularity or a pole at z_o.

A similar argument applies for $w_o = \infty$. Suppose that

$$|f(z)| > K \qquad \text{for} \quad 0 < |z - z_o| < R\ .$$

Then $g : z \mapsto 1/f(z)$ is bounded by $1/K$ for $0 < |z - z_o| < R$. Therefore, g has a removable singularity at z_o and f will have a removable singularity or a pole. □

(In fact much more is true. Picard showed that in every neighbourhood of an essential singularity the function takes each value $w \in \mathbb{C}_\infty$ with at most two exceptions. The example $z \mapsto \exp(1/z)$ takes every value except 0 and ∞.)

7. ANALYTIC FUNCTIONS ON AN ANNULUS

Let $A = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$ be an annulus or ring-shaped domain and let $f : A \to \mathbb{C}$ be an analytic function. We have seen that $\int_\gamma f(z)\, dz$ can be non-zero, for example when $f(z) = 1/z$. In this section we want to study what values the integral can take.

Proposition 7.1 Cauchy's theorem on an annulus
For each analytic function $f : A \to \mathbb{C}$ there is a constant K_f with

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = n(\gamma; 0) K_f$$

for every closed, piecewise continuously differentiable path γ in A.

Note that this is certainly true when f is analytic on the entire disc $\{z : |z| < R_2\}$ because of Cauchy's theorem. In this case $K_f = 0$. Also, it is true for $f(z) = 1/z$ because of the definition of the winding number $n(\gamma; 0)$. In this case, $K_f = 1$.

Proof:
Let S be the strip $\{w = u + iv \in \mathbb{C} : \log R_1 < u < \log R_2\}$, which is a star with any point as a centre. The exponential mapping $\exp : S \to A;\ w \mapsto e^w$ maps S onto A. Cauchy's theorem for star domains (5.2) shows that the analytic function $\phi : S \to \mathbb{C}\ ;\ \ \phi(w) = f(e^w)e^w$ has an antiderivative Φ. Now $e^{w+2\pi i} = e^w$ so $\phi(w + 2\pi i) = \phi(w)$ and hence $\Phi'(w + 2\pi i) = \Phi'(w)$. Hence, there is a constant K_f with

$$\Phi(w + 2\pi i) = \Phi(w) + 2\pi i K_f\ .$$

Let C_r be the circle $C_r : [0, 2\pi] \to A$, $t \mapsto re^{it}$ for $R_1 < r < R_2$. Then

$$\int_{C_r} f(z)\, dz = \int_0^{2\pi} f(re^{it}) ire^{it}\, dt = i \int_0^{2\pi} \phi(\log r + it)\, dt = \Phi(\log r + 2\pi i) - \Phi(\log r) = 2\pi i K_f$$

so we can determine K_f from this integral.

Consider first the case where $K_f = 0$. Then we have $\Phi(w + 2n\pi i) = \Phi(w)$ for each $n \in \mathbb{Z}$. So we can define a function $F : A \to \mathbb{C}$ unambiguously by $F(z) = \Phi(w)$ for any w with $z = e^w$. The derivative of this satisfies $F'(e^w)e^w = \Phi'(w) = \phi(w)w = f(e^w)e^w$. Hence, $F'(z) = f(z)$ and f has an antiderivative on A. Consequently,

$$\int_\gamma f(z)\, dz = 0$$

for any closed curve γ in A by Proposition 4.1 .

Now suppose that $K_f \neq 0$. Then we can replace f by the function

$$g(z) = f(z) - \frac{K_f}{z}\ .$$

This has

$$K_g = \frac{1}{2\pi i} \int_{C_r} g(z)\, dz = \frac{1}{2\pi i} \int_{C_r} g(z)\, dz - \frac{K_f}{2\pi i} \int_{C_r} \frac{1}{z}\, dz = K_f - n(C_r; 0)K_f = 0\ .$$

Therefore, we can apply the previous argument to g and obtain

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \frac{1}{2\pi i} \int_\gamma g(z)\, dz + \frac{K_f}{2\pi i} \int_\gamma \frac{1}{z}\, dz = 0 + n(\gamma; 0)K_f$$

as required. □

We can also apply this result to an annulus $A = \{z \in \mathbb{C} : R_1 < |z - z_o| < R_2\}$ centred at some other point z_o. Then we have

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = n(\gamma; z_o)K_f$$

for any closed curve γ in A. This result is particularly useful when $R_1 = 0$. Then we call the constant K_f the *residue of* f *at* z_o and denote it by $\mathrm{Res}(f; z_o)$.

Proposition 7.2 Analytic functions on an annulus
For each analytic function $f : A \to \mathbb{C}$ there are analytic functions

$$F_1 : \{z : |z| > R_1\} \to \mathbb{C} \qquad \textit{and} \qquad F_2 : \{z : |z| < R_2\} \to \mathbb{C}$$

with $f(z) = F_2(z) - F_1(z)$ for each $z \in A$.

Proof:
We proceed as in the proof of the Cauchy Representation Theorem (5.3). Let w be a fixed point in A and set

$$g(z) = \frac{f(z) - f(w)}{z - w} \qquad \text{for } z \in A \setminus \{w\} .$$

Then $g(z) \to f'(w)$ as $z \to w$, so g has a removable singularity at w (Proposition 6.11). If we set $g(w) = f'(w)$ then we obtain a function g analytic on all of the annulus A. For any closed curve γ in $A \setminus \{w\}$ we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz - \frac{f(w)}{2\pi i}\int_\gamma \frac{1}{z-w}\,dz = \frac{1}{2\pi i}\int_\gamma g(z)\,dz ,$$

which gives

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz = n(\gamma; 0)f(w) + \frac{1}{2\pi i}\int_\gamma g(z)\,dz .$$

We can apply this when γ is the circle C_r for $r \neq |w|$. For this the previous proposition shows that

$$\frac{1}{2\pi i}\int_\gamma g(z)\,dz = K_g$$

is independent of r. Hence

$$\begin{aligned} \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\,dz &= K_g \qquad \text{when } R_1 < r < |w| \\ \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\,dz &= f(w) + K_g \qquad \text{when } |w| < r < R_2 . \end{aligned} \tag{$*$}$$

Let

$$F_1(w) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\,dz \qquad \text{for } R_1 < r < |w| .$$

Corollary 6.5 shows that F_1 is an analytic function of w on $\{w : r < |w|\}$. Since $f(z)/(z-w)$ is analytic on the annulus $\{z : R_1 < |z| < |w|\}$ the value of $F_1(w)$ is independent of $r \in (R_1, |w|)$. This means that F_1 is an analytic function on $\{w : R_1 < |w|\}$. Similarly,

$$F_2(w) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\,dz \qquad \text{for } |w| < r < R_2$$

gives an analytic function on $\{w : |w| < R_2\}$.

Finally, equations $(*)$ shows that

$$f(w) = F_2(w) - F_1(w) .$$

□

We already know that analytic functions on discs have power series expansions. The last proposition gives similar expansions for analytic functions on an annulus.

Corollary 7.3 Laurent expansions
For each analytic function $f : A = \{z \in \mathbb{C} : R_1 < |z - z_o| < R_2\} \to \mathbb{C}$ *there are coefficients* a_n *for* $n \in \mathbb{Z}$ *with*

$$f(w) = \sum_{n=-\infty}^{\infty} a_n (w - z_o)^n \qquad \text{for } w \in A \ .$$

This series converges locally uniformly on the annulus A. Moreover,

$$n(\gamma; 0) a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_o)^{n+1}} \, dz$$

for every $n \in \mathbb{Z}$ *and any piecewise continuously differentiable closed curve* γ *in* A*.*

Proof:
By translating A we may ensure that $z_o = 0$. Then we know that $f(w) = F_2(w) - F_1(w)$ for analytic functions $F_1 : \{w : R_1 < |w|\} \to \mathbb{C}$ and $F_2 : \{w : |w| < R_2\} \to \mathbb{C}$. The function F_2 is analytic on a disc, so it has a power series expansion $F_2(w) = \sum_{n=0}^{\infty} b_n w^n$ that converges locally uniformly on $\{w : |w| < R_2\}$.

The argument for F_1 is similar but the disc is centred on ∞ in $\mathbb{C}_\infty$ rather than on 0. Hence we must begin by using a Möbius transformation to move ∞ to 0. First note that

$$F_1(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} \, dz \qquad \text{has} \qquad |F_1(w)| \leqslant \frac{r \sup\{|f(z)| : |z| = r\}}{|w| - r}$$

so $F_1(w) \to 0$ as $w \to \infty$. Let $G(z) = F_1(1/z)$ then $G(z) \to 0$ as $z \to 0$. Therefore G has a removable singularity at 0 and so gives us an analytic function $G : \{z : |z| < 1/R_1\} \to \mathbb{C}$. This has a power series expansion $G(z) = \sum_{n=1}^{\infty} c_n z^n$ that converges locally uniformly on $\{z : |z| < 1/R_1\}$. (The constant term is 0 since $G(0) = 0$.) Thus $F_1(w) = \sum_{n=1}^{\infty} c_n w^{-n}$ and the series converges locally uniformly on $\{w : R_1 < |w|\}$.

Putting these power series together we obtain

$$f(w) = \sum_{n=0}^{\infty} b_n w^n \; - \; \sum_{n=1}^{\infty} c_n w^{-n} \ .$$

Both parts of this sum converge locally uniformly on the annulus A. This gives the Laurent series we wanted.

Since the Laurent series for f converges uniformly on the compact set $[\gamma]$, we see that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_o)^{n+1}} \, dz = \sum_{k=-\infty}^{\infty} a_k \frac{1}{2\pi i} \int_\gamma (z - z_o)^{k-n-1} \, dz \ .$$

We can easily evaluate the integrals $\int_\gamma (z - z_o)^m \, dz$ and see that they are 0 except when $m = -1$. Hence,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_o)^{n+1}} \, dz = a_n n(\gamma; z_o) \ .$$

□

Laurent Series about isolated singularities

Let z_o be a point in the domain D and let $f : D \setminus \{z_o\} \to \mathbb{C}$ be an analytic function. So f has an isolated singularity at z_o. There will be a disc $B(z_o, R)$ that lies within D. So f is analytic on the annulus $A = \{z : 0 < |z - z_o| < R\}$ and has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

on this annulus. Corollary 7.3 shows that the residue of f at z_o is $\text{Res}(f; z_o) = a_{-1}$.

Proposition 7.4 Laurent series for isolated singularities
Let $f : D \setminus \{z_o\} \to \mathbb{C}$ be an analytic function with an isolated singularity at z_o and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

be its Laurent expansion that converges for $0 < |z - z_o| < R$. Then

(a) f has a removable singularity at z_o if and only if $a_n = 0$ for $n < 0$.

(b) f has a pole at z_o of order N if and only if $a_n = 0$ for $n < -N$ and $a_{-N} \neq 0$.

(c) f has an essential singularity at z_o if and only if $a_n \neq 0$ for infinitely many negative values of n.

Proof:
(a) Suppose that f has a removable singularity at z_o. then there is an analytic function $F : D \to \mathbb{C}$ extending f. For γ a closed curve in the annulus A we have

$$a_n = \frac{1}{2\pi i} \int_C \frac{F(z)}{(z - z_o)^{n+1}}\, dz$$

and Cauchy's theorem shows that this is 0 for $n < 0$. Conversely, if $a_n = 0$ for $n < 0$, then the Laurent series reduces to a power series and defines an analytic extension of f.

(b) Suppose that f has a pole of order N at z_o. Then $f(z) = (z - z_o)^{-N} G(z)$ for some function G analytic near z_o and with $G(z_o) \neq 0$. The Laurent series for G is

$$G(z) = \sum_{n=-\infty}^{\infty} a_{n-N} (z - z_o)^n \ .$$

This has a removable singularity at z_o, so part (a) implies that $a_n = 0$ for $n < -N$. We also have $a_{-N} = G(z_o) \neq 0$. Conversely, if $a_n = 0$ for $n < -N$ and $a_{-N} = 0$, then

$$f(z) = (z - z_o)^{-N} \sum_{n=0}^{\infty} a_{n-N} (z - z_o)^n$$

so f has a pole of order N at z_o.

(c) The singularity is essential if and only if it is neither removable nor a pole. Similarly, the Laurent series has $a_n \neq 0$ for infinitely many negative n if and only if there is no integer N with $a_n = 0$ for $n < -N$. Thus (a) and (b) imply (c). □

Laurent series give us a quick proof of the Residue theorem at least for simply connected domains. Suppose that f has an isolated singularity at z_o and has Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_o)^n$. The part

$$P(z) = \sum_{n=-\infty}^{-1} a_n(z-z_o)^n$$

of this series is called the *principal part of f at z_o*. The principal part is a power series in $1/(z-z_o)$ and converges for z sufficiently close to z_o. Therefore, it must converge for all $z \in \mathbb{C} \setminus \{z_o\}$. The difference $f - P$ is analytic at z_o.

Theorem 7.5 Residue theorem for simply connected domains
Let D be a simply connected domain in $\mathbb{C}$ and f a function that is analytic on D except for isolated singularities at the points $z_1, z_2, \ldots, z_K$. For any piecewise continuously differentiable closed curve γ in $D \setminus \{z_1, z_2, \ldots, z_K\}$ we have

$$\frac{1}{2\pi i}\int_\gamma f(z)\, dz = \sum_{k=1}^{K} n(\gamma; z_k)\mathrm{Res}(f; z_k) \ .$$

Proof:

Let P_k be the principal part of f at z_k and let

$$g(z) = f(z) - \sum_{k=1}^{K} P_k(z) \ .$$

Then g is analytic on all of D including the points z_k. Hence Cauchy's theorem for simply connected domains (Corollary 5.7) shows that $\int_\gamma g(z)\, dz = 0$. Therefore,

$$\int_\gamma f(z)\, dz = \sum_{k=1}^{K}\int_\gamma P_k(z)\, dz \ .$$

Now Corollary 7.3 shows that the residue of f at z_k is the coefficient of $1/(z-z_k)$ in the Laurent expansion of f about z_k. This is same for the principal part P_k. Hence Corollary 7.3 gives

$$\frac{1}{2\pi i}\int_\gamma P_k(z)\, dz = n(\gamma; z_k)\mathrm{Res}(f; z_k)$$

and the proof is complete. □

We will give a different proof of the residue theorem in the next section when we have proved a stronger form of Cauchy's theorem.

8. THE HOMOLOGY FORM OF CAUCHY'S THEOREM

Let D be a domain in $\mathbb{C}$. A *chain* in D is a finite collection $\gamma_n : [a_n, b_n] \to D$ (for $n = 1, 2, 3, \ldots, N$) of piecewise continuously differentiable curves in D. We will write $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_N$ for this collection. The empty chain will be written as 0. We can add two chains and obtain another chain. Let $[\Gamma]$ be the union of the images $[\gamma_n] = \gamma_n([a_n, b_n])$. The integral of a continuous function $f : D \to \mathbb{C}$ around Γ is then defined to be the sum

$$\int_\Gamma f(z)\, dz = \sum_{n=1}^{N} \int_{\gamma_n} f(z)\, dz\ .$$

In particular, the *winding number* $n(\Gamma; w)$ of a chain Γ about any point $w \notin [\Gamma]$ is

$$n(\Gamma; w) = \sum_{n=1}^{N} n(\gamma_n; w) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - w}\, dz\ .$$

A *cycle* in D is a chain $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_N$ where each point $w \in \mathbb{C}$ occurs the same number of times as an initial point $\gamma_n(a_n)$ as it does as an final point $\gamma_n(b_n)$. This means that a cycle consists of a finite number of closed curves, each of which may be made up from a number of the curves γ_n. The winding number $n(\Gamma; w)$ of a cycle Γ is therefore an integer. Two cycles Γ and Γ' are *homologous in D* if

$$n(\Gamma; w) = n(\Gamma'; w) \qquad \text{for each } w \in \mathbb{C} \setminus D\ .$$

We write this as $\Gamma \sim \Gamma'$. In particular, a cycle Γ is *homologous to* 0 (or *null-homologous*) in D if $n(\Gamma; w) = 0$ for every $w \notin D$.

Example: Let γ be a closed curve homotopic in D to a constant curve. Proposition 4.5 shows that

$$n(\gamma; w) = 0 \qquad \text{for each } w \notin D\ .$$

Therefore, each closed curve homotopic to a constant is homologous to 0.

However, there are cycles homologous to 0 that are not made up of closed curves homotopic to a constant. For example, consider the domain $D = \mathbb{C} \setminus \{0, 1\}$ and the cycle $\gamma_0 + \gamma_1 + \gamma_\infty$ where

$$\gamma_0 : [0,1] \to D\ ;\ t \mapsto \tfrac{1}{3}e^{2\pi i t}\ ;\ \gamma_1 : [0,1] \to D\ ;\ t \mapsto 1 + \tfrac{1}{3}e^{2\pi i t}\ . \quad \text{and} \quad \gamma_\infty : [0,1] \to D\ ;\ t \mapsto 3e^{-2\pi i t}\ .$$

It is easy to check that $n(\Gamma; w) = 0$ for $w = 0, 1$, so Γ is homologous to 0. However, none of the components γ_w of Γ is homotopic to a constant curve in D.

Let us consider, informally, a simple closed curve γ in a domain $D \subset \mathbb{C}$. The Jordan curve theorem tells us that $[\gamma]$ divides $\mathbb{C}$ into two components. The inside J of $[\gamma]$ where $n(\gamma; w) = \pm 1$ and the outside of $[\gamma]$ where $n(\gamma; w) = 0$. It is clear that γ is homologous to 0 when $J \subset D$. This is when γ is the boundary in $\mathbb{C}$ of a region $J \subset D$. More generally, a cycle Γ is homologous to 0 when there is a finite collection of regions $J_n \subset D$ with Γ being the sum of the boundaries of each J_n. The Algebraic Topology course in Part 2 explains this more carefully.

Our aim is to prove the most general possible form of Cauchy's theorem. It is convenient to simultaneously prove a corresponding representation formula.

Theorem 8.1 Homology form of Cauchy's theorem
The following conditions on a cycle Γ in a domain $D \subset \mathbb{C}$ are equivalent.

(a) *Γ is homologous to 0 in D.*

(b) *For each analytic function $f : D \to \mathbb{C}$ and each point $w \in D \setminus [\Gamma]$*

$$n(\Gamma; w) f(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - w}\, dz\ .$$

(c) *For each analytic function $f : D \to \mathbb{C}$*

$$\int_\Gamma f(z)\, dz = 0\ .$$

Proof: (Following J.D. Dixon.)

(b) $\Rightarrow$ (c)
If we apply (b) to the function $z \mapsto (z-w)f(z)$ for any $w \in D \setminus [\Gamma]$ we obtain (c).

(c) $\Rightarrow$ (a)
If $w \notin D$, then $z \mapsto 1/(z-w)$ is analytic on D. So (c) implies that $n(\Gamma; w) = 0$ and therefore (a) is true.

(a) $\Rightarrow$ (b)
As in the earlier proof of the representation theorem for discs, we consider the difference quotient

$$h(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{when } z \neq w; \\ \\ f'(w) & \text{when } z = w. \end{cases}$$

For $w \in D$ define

$$H(w) = \frac{1}{2\pi i}\int_\Gamma h(z,w)\, dz\ .$$

We will later prove that $H : D \to \mathbb{C}$ is analytic. For the present we postpone this and instead show how it leads to a proof that (a) $\Rightarrow$ (b).

Let $E = \{w \in \mathbb{C} : n(\Gamma; w) = 0\}$. (This is the "exterior" of Γ.) The set $[\Gamma]$ is compact and hence bounded, say $[\Gamma] \subset B(0,R)$. Then Proposition 4.4 shows that $\mathbb{C} \setminus B(0,R) \subset E$. The Cauchy transform:

$$J(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\, dz$$

is analytic on E and satisfies

$$|J(w)| \leqslant \frac{L(\Gamma)}{2\pi(|w|-R)} \sup\{|f(z)| : z \in [\Gamma]\}\ .$$

So $J(w) \to 0$ as $w \to \infty$. Moreover, if $w \in D \cap E$, then

$$H(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w} - \frac{f(w)}{z-w}\, dz = J(w) - f(w)n(\Gamma; w) = J(w)\ .$$

Therefore, we can define a function K by

$$K(w) = \begin{cases} H(w) & \text{when } w \in D; \\ J(w) & \text{when } w \in E; \end{cases}$$

because the two definitions agree on $D \cap E$. Condition (a) shows that $n(\Gamma; w) = 0$ for all $w \notin D$. So $D \cup E = \mathbb{C}$ and $K : \mathbb{C} \to \mathbb{C}$. Since H and J are both analytic, the function K is analytic. It is also bounded, since $J(w) \to 0$ as $w \to \infty$. Therefore Liouville's theorem implies that K is identically 0.

In particular, $H(w) = 0$ for $w \in D \setminus [\Gamma]$, so

$$0 = H(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\, dz - \frac{f(w)}{2\pi i}\int_\Gamma \frac{1}{z-w}\, dz\ ,$$

which proves condition (b).

It remains to prove that $H : D \to \mathbb{C}$ is analytic. We will do this in stages, using the lemmas below. The first lemma is simply a topological result.

Lemma 8.2
Let (K,d) be a compact metric space and $\phi : [\Gamma] \times K \to \mathbb{C}$ a continuous map. Then

$$\Phi : K \to \mathbb{C}\ ; \quad x \mapsto \int_\Gamma \phi(z,x)\, dz$$

is continuous.

Proof:

The product $[\Gamma] \times K$ is compact, so ϕ is uniformly continuous on it. This means that, for $\varepsilon > 0$, there is a $\delta > 0$ with

$$|\phi(z, x) - \phi(z_o, x_o)| < \varepsilon \qquad \text{whenever} \qquad |z - z_o| < \delta \text{ and } d(x, x_o) < \delta \ .$$

Integrating this gives

$$|\Phi(x) - \Phi(x_o)| \leqslant L(\Gamma) \sup\{|\phi(z, x) - \phi(z, x_o)| : z \in [\Gamma]\} \leqslant L(\Gamma)\varepsilon$$

when $d(x, x_o) < \delta$. □

We can apply this lemma to prove that h is continuous.

Lemma 8.3

The function $\qquad h : D \times D \to \mathbb{C}; \qquad h(z, w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w} & \text{when } z \neq w; \\ f'(w) & \text{when } z = w \end{cases}$ *is continuous.*

Proof:

The function $h(z, w)$ is certainly continuous at points where $z \neq w$. We need to prove that it is also continuous at a point (a, a).

Let $a \in D$ and choose a closed disc $\overline{B(a, 2r)}$ lying within D. Let C be the boundary circle of this disc. For $z, w \in B(a, 2r)$ the Cauchy representation formula for a disc (5.3) gives

$$h(z, w) = \frac{1}{(z - w)} \frac{1}{2\pi i} \int_C f(u) \left(\frac{1}{u - z} - \frac{1}{u - w} \right) \, du = \frac{1}{2\pi i} \int_C f(u) \left(\frac{1}{(u - z)(u - w)} \right) \, du \ .$$

The function

$$\phi : (u, (z, w)) \mapsto f(u) \left(\frac{1}{(u - z)(u - w)} \right)$$

is certainly continuous on $[C] \times (\overline{B(a, r)} \times \overline{B(a, r)})$. Hence the previous lemma shows that h is continuous on $\overline{B(a, r)} \times \overline{B(a, r)}$. □

We now know that h is continuous. For each $z \in D$, the function $w \mapsto h(z, w)$ is complex differentiable at each $w \neq z$ and continuous at $w = z$. So it has a removable singularity at $w = z$ and must be analytic on all of D. Hence, the following lemma will complete the proof.

Lemma 8.4
Let Γ be a cycle in a domain $D \subset \mathbb{C}$ and $h : [\Gamma] \times D \to \mathbb{C}$ a continuous map. Suppose that, for each $z \in [\Gamma]$ the map $w \mapsto h(z, w)$ is analytic on D. Then the integral

$$H(w) = \frac{1}{2\pi i} \int_\Gamma h(z, w) \, dz$$

is also analytic on D.

Proof:

Let a be a point in D, as above, and let $\overline{B(a,2r)}$ be a closed disc about a that lies within D. Its boundary is C. The product $[\Gamma] \times \overline{B(a,2r)}$ is compact, so $|h|$ has a finite supremum $||h||_\infty$ on this set.

For each $z \in [\Gamma]$ we know that $w \mapsto h(z,w)$ is analytic, so we can use Cauchy's representation formula for derivatives (6.6) to see that

$$\frac{\partial h}{\partial w}(z,w) = \frac{1}{2\pi i}\int_C \frac{h(z,u)}{(u-w)^2}\, du$$

for $w \in B(a,2r)$. The map $(u,(z,w)) \mapsto h(z,u)/(u-w)^2$ is certainly continuous on $[C] \times ([\Gamma] \times \overline{B(a,r)})$, so the first lemma shows that $\partial h(z,w)/\partial w$ is continuous on $[\Gamma] \times B(a,r)$.

Similarly,

$$\begin{aligned}\left|h(z,w) - h(z,w_o) - (w-w_o)\frac{\partial h}{\partial w}(z,w_o)\right| &= \left|\frac{1}{2\pi i}\int_C h(z,u)\left(\frac{1}{u-w} - \frac{1}{u-w_o} - \frac{w-w_o}{(u-w_o)^2}\right)\, du\right| \\ &\leqslant \left|\frac{1}{2\pi i}\int_C h(z,u)\left(\frac{(w-w_o)^2}{(u-w)(u-w_o)^2}\right)\, du\right| \\ &\leqslant \frac{L(C)}{2\pi}||h||_\infty |w-w_o|^2 \sup\left\{\frac{1}{|u-w||u-w_o|^2} : u \in [C]\right\} .\end{aligned}$$

So we have

$$\left|h(z,w) - h(z,w_o) - (w-w_o)\frac{\partial h}{\partial w}(z,w_o)\right| \leqslant \frac{2||h||_\infty}{r^2}|w-w_o|^2$$

for $w, w_o \in B(a,r)$.

Since $h(z,\cdot)$ and $\partial h(z,\cdot)/\partial w$ are both continuous, we can integrate this last inequality to obtain

$$\left|H(w) - H(w_o) - \frac{w-w_o}{2\pi i}\int_\Gamma \frac{\partial h}{\partial w}(z,w_o)\, dz\right| \leqslant \frac{2||h||_\infty L(\Gamma)}{r^2}|w-w_o|^2$$

for $w, w_o \in B(a,r)$. This proves that H is complex differentiable on $B(a,r)$ with

$$H'(w_o) = \int_\Gamma \frac{\partial h}{\partial w}(z,w_o)\, dz .$$

□

This result is stronger than our earlier versions of Cauchy's theorem. For the last example showed that a closed curve that is homotopic to a constant curve in D is homologous to 0. So the theorem certainly implies that $\int_\gamma f(z)\, dz = 0$ when γ is such a curve.

The Residue Theorem

Let D be a domain in $\mathbb{C}$ and f a function that is analytic on D except for isolated singularities at the points $z_1, z_2, \ldots, z_K$. This means that, for each $k = 1, 2, 3, \ldots, K$, there is a closed disc $\overline{B(z_k, R_k)}$ that lies within D and contains only the singularity at z_k. Then f is analytic on $B(z_k, R_k) \setminus \{z_k\}$ and has a residue $\mathrm{Res}(f; z_k)$ at z_k.

Theorem 8.5 Residue theorem
Let D be a domain in $\mathbb{C}$ and f a function that is analytic on D except for isolated singularities at the points $z_1, z_2, \ldots, z_K$. For any cycle Γ in $D \setminus \{z_1, z_2, \ldots, z_K\}$ that is homologous to 0 in D we have

$$\frac{1}{2\pi i}\int_\Gamma f(z)\, dz = \sum_{k=1}^{K} n(\Gamma; z_k)\mathrm{Res}(f; z_k) .$$

Proof:

Let C_k denote the positively oriented circle bounding the disc $B(z_k, r_k)$. Then $n(C_k; w) = 1$ if $w \in B(z_k, r_k)$ and $n(C_k; w) = 0$ for any $w \notin \overline{B(z_k, r_k)}$. In particular, $n(C_k; z_k) = 1$ but $n(C_k; z_j) = 0$ for any $j \neq k$. Hence the cycle

$$\Delta = \Gamma - \sum_{k=1}^{K} n(\Gamma; z_k) C_k$$

is homologous to 0 in $D \setminus \{z_1, z_2, \ldots, z_K\}$. Now the homology form of Cauchy's theorem (Theorem 8.1) shows that

$$0 = \int_\Delta f(z)\ dz = \int_\Gamma f(z)\ dz - \sum_{k=1}^{K} n(\Gamma; z_k) \int_{C_k} f(z)\ dz\ .$$

Finally, Proposition 7.1 shows that $\displaystyle\int_{C_k} f(z)\ dz = 2\pi i \mathrm{Res}(f; z_k)$. □

A closed curve $\gamma; [a, b] \to \mathbb{D}$ is *simple* if it does not cross itself, so $\gamma(s) = \gamma(t)$ for two distinct points s, t only when s and t are the endpoints a and b. The Jordan curve theorem shows that such a curve divides the plane into two connected components: the inside and the outside of γ. However, we will not prove this. It is usual to apply the residue theorem when the cycle Γ is a simple closed curve bounding a region in D. However, we will make a slightly more general definition: A cycle Γ *bounds a domain* Ω if the winding number $n(\Gamma; w)$ is 1 for all points $w \in \Omega$ and either 0 or undefined for all points not in Ω. It is clear that a cycle Γ in D that bounds a domain $\Omega \subset D$ is homologous to 0 in D.

Consequently, we can restate the residue theorem as:

Theorem 8.5' Residue theorem
Let D be a domain and f a function that is analytic on D except for isolated singularities. Let Γ be a cycle that bounds a subdomain Ω of D and does not pass through any singularity. Then

$$\frac{1}{2\pi i} \int_\Gamma f(z)\ dz = \sum n(\Gamma; w) \mathrm{Res}(f; w)$$

where the sum is over all the singularities in Ω.

Proof:

The set $\overline{\Omega}$ is compact since it is bounded by $[\Gamma]$. For each $w \in \overline{\Omega}$ there is an open neighbourhood that contains at most one singularity of f, because the singularities are isolated. These open neighbourhoods form an open cover for $\overline{\Omega}$ so there is a finite subcover. Hence there can be only a finite number of singularities within Ω. Now we can apply Theorem 8.5 . □

9. THE ARGUMENT PRINCIPLE

Let $f : D \to \mathbb{C}$ be an analytic map and Γ a cycle in D. Then $f \circ \Gamma$ is also a cycle. If $w \in \mathbb{C} \setminus [f \circ \Gamma]$ then the winding number $n(f \circ \Gamma; w)$ is given by

$$n(f \circ \Gamma; w) = \frac{1}{2\pi i} \int_{f \circ \Gamma} \frac{1}{z - w}\, dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z) - w}\, dz \ .$$

The integrand $f'(z)/(f(z) - w)$ is meromorphic with poles at the points z_k where $f(z_k) = w$. Near such a point we have

$$f(z) = w + (z - z_k)^N F(z)$$

where $N = \deg(f; z_k)$ and F is analytic on a neighbourhood of z_k with $F(z_k) \neq 0$. Hence,

$$\frac{f'(z)}{f(z) - w} = \frac{N}{z - z_k} + \frac{F'(z)}{F(z)}$$

and hence there is a simple pole at z_k with residue N. Thus the residue theorem (8.5) gives

Theorem 9.1 Argument Principle
Let $f : D \to \mathbb{C}$ be a non-constant analytic function and Γ a cycle in D that is homologous to 0 in D. Suppose that f does not take the value w on $[\Gamma]$. Then

$$n(f \circ \Gamma; w) = \sum_{z : f(z) = w} \deg(f; z) n(\Gamma; z)$$

where the sum is taken over all points $z \in D$ with $f(z) = w$.

Proof:
The points where $f(z) = w$ are isolated in D and the set $[\Gamma] \cup \{z \in D : n(\Gamma; z) \neq 0\}$ is compact, so there are only a finite number of non-zero terms in the sum.

The residue theorem shows that

$$n(f \circ \Gamma; w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z) - w}\, dz = \sum_{z : f(z) = w} \mathrm{Res}(f'/(f - w); z) n(\Gamma; z) = \sum_{z : f(z) = w} \deg(f; z) n(\Gamma; z) \ .$$

□

It is usual to apply the argument principle to a cycle Γ that bounds a subdomain of D. Then the winding numbers are all 0 or 1 and we obtain:

Theorem 9.1' Argument Principle
Let $f : D \to \mathbb{C}$ be a non-constant analytic function and Γ a cycle in D that bounds a subdomain Ω of D. Suppose that f does not take the value w on $[\Gamma]$. Then

$$n(f \circ \Gamma; w) = \sum_{z \in \Omega : f(z) = w} \deg(f; z) \ .$$

The sum on the right side is the number of solutions of $f(z) = w$ in Ω, counting multiplicity. □

We can also apply this argument when f is a meromorphic function. If f has a pole of order N at z_o then

$$f(z) = w + (z - z_o)^{-N} F(z)$$

on a neighbourhood of z_o with F analytic and $F(z_o) \neq 0$. Hence

$$\frac{f'(z)}{f(z) - w} = \frac{-N}{z - z_o} + \frac{F'(z)}{F(z)}$$

and we see that $f'(z)/(f(z) - w)$ has a simple pole at z_o with residue $-N$. This proves:

Theorem 9.2 Argument Principle for meromorphic functions
Let $f : D \to \mathbb{C}$ be a non-constant meromorphic function and Γ a cycle in D that bounds a subdomain Ω of D. Suppose that f takes neither the value w nor ∞ on $[\Gamma]$. Then

$$n(f \circ \Gamma; w) = \sum_{z \in \Omega : f(z) = w} \deg(f; z) \; - \sum_{z \in \Omega : f(z) = \infty} \deg(f; z) \ .$$

□

Rouché's theorem formalises this type of argument.

Proposition 9.3 Rouché's Theorem
Let Γ be a cycle in a domain D that bounds a subdomain Ω. If $f, g : D \to \mathbb{C}$ are analytic functions with

$$|f(z) - g(z)| < |g(z)| \qquad \textit{for all} \quad z \in [\Gamma]$$

then f and g have the same number of zeros within Ω, counting multiplicity.

Proof:

The inequality shows that neither f nor g has a zero on $[\Gamma]$. We may therefore apply Proposition 4.3 to the component curves of $f \circ \Gamma$ and $g \circ \Gamma$ to obtain $n(f \circ \Gamma; 0) = n(g \circ \Gamma; 0)$. Now the argument principle (9.1') completes the proof. □

Local Mapping Theorem

We can now complete our study of the local behaviour of analytic functions.

Theorem 9.4 Local Mapping Theorem
Let $f : D \to \mathbb{C}$ be a non-constant analytic function, $z_o \in D$, $w_o = f(z_o)$ and $K = \deg(f; z_o)$. Then there are $r, s > 0$ such that, for each $w \in B(w_o, s) \setminus \{w_o\}$ there are exactly K points $z \in B(z_o, r)$ with $f(z) = w$.

Proof:

We know that there is an analytic function $F : D \to \mathbb{C}$ with $f(z) = w_o + (z - z_o)^K F(z)$ and $F(z_o) \neq 0$. Hence, we can choose $r > 0$ so that the closed disc $\overline{B(z_o, r)}$ lies within D and $F(z) \neq 0$ on $\overline{B(z_o, r)}$. Let C be the circle $\partial B(z_o, r)$. Then $[f \circ C]$ is a compact subset of $\mathbb{C}$ that does not contain w_o. Choose $s > 0$ so that $B(w_o, s)$ does not meet $[f \circ C]$.

The winding number $n(f \circ C; w)$ is constant on each component of $\mathbb{C} \setminus [f \circ C]$ and hence it is constant on $B(w_o, s)$. The argument principle shows that $n(f \circ C; w)$ is the number of solutions of $f(z) = w$ in $B(z_o, r)$, counting multiplicity. For $w = w_o$, this number is K. Therefore, there are K solutions of $f(z) = w$ in $B(z_o, r)$ for each $w \in B(w_o, s)$.

The derivative of f is $f'(z) = (z - z_o)^{K-1} (KF(z) + (z - z_o)F'(z))$, so we can choose r sufficiently small that $f'(z) \neq 0$ on $B(z_o, r) \setminus \{z_o\}$. Then $f - w$ can not have any multiple zeros in $B(z_o, r) \setminus \{z_o\}$. Hence, there are exactly K distinct solutions of $f(z) = w$ in $B(z_o, r)$ for each $w \in B(w_o, s)$ except w_o. For $w = w_o$, the only solution of $f(z) = w$ in $B(z_o, r)$ is at z_o where it has multiplicity K. □

Corollary 9.5 Open Mapping Theorem
A non-constant analytic function $f : D \to \mathbb{C}$ maps open sets in D to open sets in $\mathbb{C}$.

Proof:

If U is an open subset of D and $z_o \in U$, then we wish to prove that there is a disc about $f(z_o)$ that lies within $f(U)$. The local mapping theorem shows that we can choose $r, s > 0$ so that $B(z_o, r) \subset U$ and $B(f(z_o), s) \subset f(U)$. □

Corollary 9.6 Maximum Modulus Theorem
Let $f : D \to \mathbb{C}$ be a non-constant analytic function on a domain D. Then the modulus $|f|$ can have no local maximum on D.

Proof:

For any $z_o \in D$ the local mapping theorem (Theorem 9.4) shows that there are $r, s > 0$ with $f(B(z_o, r)) \supset B(f(z_o), s)$. This certainly implies that $|f(z)|$ can not have a local maximum at z_o. □